Generalizations of Rasmussen's invariant

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Outline of the talk

1. Review of Khovanov homology and Rasmussen's invariant for knots in S^3;

2. Extensions to links in other specific 3-manifolds: $S^1 \times D^2$ (**Grigsby-Licata-Wehrli**) , $\#^r S^1 \times S^2$ (**Marengon-M.-Sarkar-Willis**), $\mathbb{RP}^3$ (**M.-Willis, Chen**);

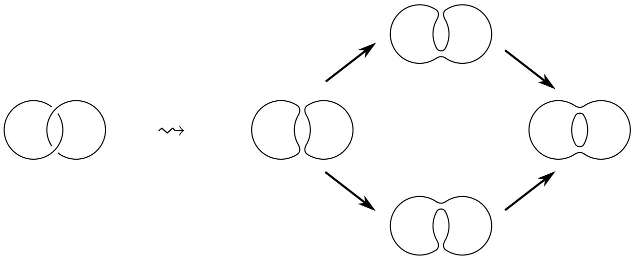
3. A general invariant for links in the boundary of any compact 4-manifold, from skein lasagna modules (cf. **Morrison-Walker-Wedrich**, **Ren-Willis**); detection of exotic smooth structures on some compact 4-manifolds with boundary.

Khovanov homology

For links $K \subset S^3$, **Khovanov** (1999) defined a homology theory

$$Kh(K) = \bigoplus_{i,j} Kh_{i,j}(K).$$

Its construction involves taking all possible "resolutions" of a link diagram, associating a two-dimensional vector space V to each circle in a resolution, and defining a chain complex using an algebraically-defined differential d:



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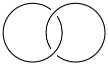
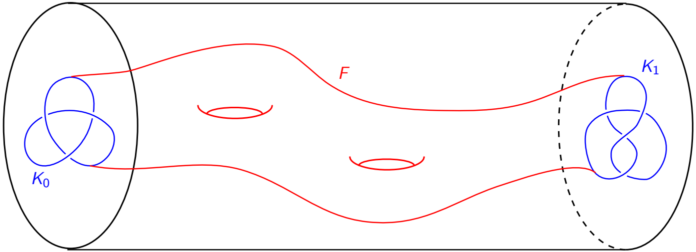


$$\rightsquigarrow \quad \begin{tikzcd} & V\{1\} \arrow[dr, "d"] & \\ V^{\otimes 2} \arrow[ur, "d"] \arrow[dr, "d"'] & & V^{\otimes 2}\{2\} \\ & V\{1\} \arrow[ur, "d"'] & \end{tikzcd}$$

The Euler characteristic of Khovanov homology is the Jones polynomial:

$$\sum_{i,j}(-1)^i q^j \dim Kh_{i,j}(K) = J_K(q).$$

A link cobordism $F \subset S^3 \times [0,1]$ from K_0 to K_1 induces a map on Khovanov homology (up to sign): $Kh(F) : Kh(K_0) \to Kh(K_1)$.



Rasmussen's invariant

Lee (2002) defined a deformation of the Khovanov complex, inducing a spectral sequence from $Kh(K)$ to a simpler *Lee homology* $LH(K)$:

$$\text{rk } LH(K) = 2^{\ell},$$

where ℓ is the number of components of K. In fact, $LH(K)$ is generated by cycles $\mathfrak{s}_o$, one for each orientation o of K.

From this, **Rasmussen** (2004) extracted a numerical knot invariant denoted s (generalized to links by **Beliakova-Wehrli** and **Pardon**):

$$s(K) = \frac{\mathrm{q}([\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}]) + \mathrm{q}([\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}])}{2}.$$

This gives bounds on the genus of link cobordisms in $I \times S^3$. For knots, we have a lower bound for the slice genus:

$$|s(K)|/2 \leq g_s(K) = \min\{\text{genus}(\Sigma) \mid \Sigma \text{ oriented}, \Sigma \subset B^4, \partial\Sigma = \Sigma \cap S^3 = K\}.$$

Applications

1. **Rasmussen** (2004) gave a new proof of the following:

Theorem (Milnor conjecture, cf. Kronheimer-Mrowka, 1993)

The slice genus of the torus knot $T_{p,q}$ is $(p-1)(q-1)/2$.

2. One can use s to show the existence of topologically slice knots that are not smoothly slice, which gives a new proof (without gauge theory) of the existence of exotic smooth structures on $\mathbb{R}^4$.

3. **Plamenevskaya** (2004) used s to bound the self-linking number of transverse links.

4. **Piccirillo** (2018) found knots whose shake genus is different from the slice genus.

5. **Piccirillo** (2018) proved that the Conway knot is not slice.

Extensions of Khovanov homology

1. **Asaeda-Przytycki-Sikora** (2004): for links in I-bundles over surfaces; in particular, in $S^1 \times D^2$ (annular Khovanov homology) and in $\mathbb{RP}^2 \tilde{\times} I = \mathbb{RP}^3 \setminus \{*\}$ (but only with $\mathbb{Z}/2$ coefficients);
2. **Manturov** (2005): for virtual links;
3. **Rozansky** (2010): for links in $S^1 \times S^2$;
4. **Willis** (2018): for links in $\#^n(S^1 \times S^2)$;
5. **Manturov** (2006) and **Gabrovšek** (2018): for links in $\mathbb{RP}^3$, with $\mathbb{Z}$ coefficients;
6. **Morrison-Walker-Wedrich** (2019): for links in any $Y^3 = \partial X^4$, we have the *skein lasagna module* $\mathcal{S}_0(X; L)$. When $X = B^4$, we recover Kh.

Extensions of Rasmussen's invariant

1. **Mackaay-Turner-Vaz** (2005): from the Bar-Natan deformation of Kh (over a field characteristic p, for any prime p);
2. **Lobb** (2010): from $\mathfrak{sl}(N)$ link homology;
3. **Lipshitz-Sarkar** (2012), **Sarkar-Scaduto-Stoffregen** (2019): using the Steenrod squares from (even and odd) Khovanov stable homotopy;
4. **Sano-Sato** (2022), **Schütz** (2022), **Dunfield-Lipshitz-Schütz** (2024), **Lewark** (2024): from the deformations of (even and odd) Kh over $\mathbb{Z}$;
5. **Dye-Kaestner-Kauffman** (2014): for virtual knots.

In this talk, we will focus on generalizations of s to links in 3-manifolds other than S^3.

Extensions of the Rasmussen invariant: in $S^1 \times D^2$

Let $A = S^1 \times I$ be the annulus. Khovanov homology for annular links $(K \subset A \times I \cong S^1 \times D^2)$ has the structure of a triply graded abelian group.

In 2016, **Grigsby-Licata-Wehrli** computed the homology of its Lee deformation, and defined a family of Rasmussen-type invariants

$$d_t(K) \in \mathbb{R}, \ \ t \in [0, 1],$$

such that $d_t = d_{2-t}$, $d_0 = d_2 = s$, and d_t is piecewise linear. We denote by m_t its right-hand slope.

The invariants d_t give bounds on the genus of knot cobordisms in $I \times S^1 \times D^2$.

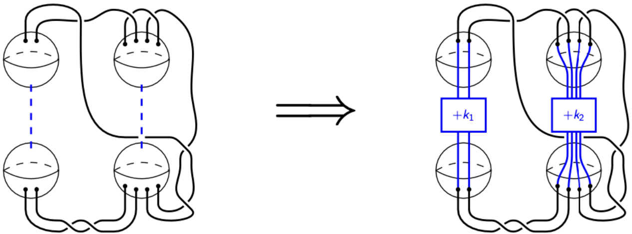
Furthermore, for closures $\hat{\sigma}$ of braids $\sigma \in B_n$, they showed:

- If σ is quasi-positive, then $m_t(\hat{\sigma}) = n$ for all $t \in [0, 1)$.
- If $m_t(\hat{\sigma}) = n$ for some $t \in [0, 1)$, then σ is right-veering.

Extensions of the Rasmussen invariant: in $\#^r S^1 \times S^2$

Rozansky (2010) defined Kh for links in $S^1 \times S^2$. This was generalized to $\#^r(S^1 \times S^2)$ by **Willis** (2018). We assume that $[K] = 0 \in H_1(M; \mathbb{Z})$.

We represent a link $K \subset \#^r(S^1 \times S^2)$ by a diagram D as on the left. From here we get a link $D(\vec{k}) \subset S^3$ by inserting $\vec{k} = (k_1, \dots, k_r)$ full twists:



Then take the limit of $Kh(D(\vec{k}))$ as $k_i \to \infty$, after suitable grading shifts.

Extensions of the Rasmussen invariant: in $\#^r S^1 \times S^2$

Marengon-M.-Sarkar-Willis (2019) computed the Lee deformation of this theory, and defined an invariant $s(K)$ for null-homologous $K \subset \#^r S^1 \times S^2$. This gives bounds on the genus of link cobordisms in $I \times \#^r S^1 \times S^2$.

Moreover, denote by $g_{S^1 \times B^3}(K)$ the minimum genus of a surface $\Sigma \subset \natural^r(S^1 \times B^3)$ with $\partial\Sigma = L$. We define $g_{B^2 \times S^2}(K)$ similarly, but using surfaces $\Sigma \subset \natural^r(B^2 \times S^2)$. If K is a knot, we have:

$$\boxed{s(K)/2 \leq g_{B^2 \times S^2}(K), \quad -s(-K)/2 \leq g_{S^1 \times B^3}(L).}$$

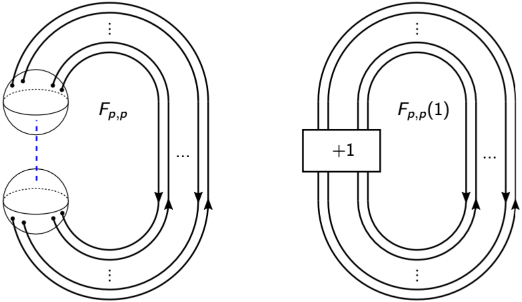
In a roundabout way, $s(K)$ for $K \subset \#^r S^1 \times S^2$ can also be used to say something new about the s-invariant of knots in S^3. This is based on the following fact: if D is a diagram of K with n_+ positive crossings, then

$$s(K) = s(D(\vec{k}))$$

where $\vec{k} = (k, \cdots, k)$ with $k \geq \left\lceil \frac{n_+ + 2}{2} \right\rceil$.

Extensions of the Rasmussen invariant: in $\#^r S^1 \times S^2$

In particular, consider the null-homologous link $F_{p,p} \subset S^1 \times S^2$, which is the union of $2p$ fibers $S^1 \times \{x_i\}$. It has a diagram with 0 crossings, so its s invariant is the same as that of the $(2p, 2p)$ balanced torus link $F_{p,p}(1) \subset S^3$:



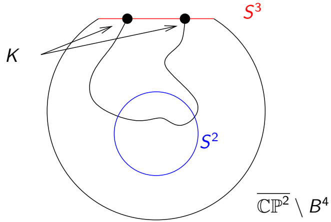
Using Hochschild homology, we computed $s(F_{p,p}(1)) = s(F_{p,p}) = 1 - 2p$.

A genus bound in $\#^r\overline{\mathbb{CP}^2}$

Theorem (MMSW, 2019)

If a knot $K \subset S^3$ bounds a null-homologous surface $\Sigma \subset \#^r\overline{\mathbb{CP}^2} \setminus B^4$, then $s(K)/2 \leq g(\Sigma)$.

Sketch of proof: A surface $\Sigma \subset \overline{\mathbb{CP}^2} \setminus B^4$ intersects $S^2 = \overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$ in p positive and p negative points. This gives a cobordism $C \subset S^3 \times [0,1]$ between K and the torus link $F_{p,p}(1)$:



We then apply the genus bounds for surfaces in $S^3 \times [0,1]$. The case of $\#^r\overline{\mathbb{CP}^2} \setminus B^4$ is similar.

A more general genus bound in $\#^r\overline{\mathbb{CP}^2}$

Ren (2023) gave a more elementary proof of the formula $s(F_{p,p}(1)) = 1 - 2p$, and in fact computed s for all torus links $T(n, m)$, equipped with any orientation (where p strands are oriented one way and q the other way):

$$s(T(n,m)_{p,q}) = \left(\tfrac{n}{\gcd(n,m)}|p-q| - 1\right)\left(\tfrac{m}{\gcd(n,m)}|p-q| - 1\right) - 2\min(p,q).$$

A corollary is the general adjunction inequality:

Theorem (Ren, 2023)

If a knot $K \subset S^3$ bounds a surface $\Sigma \subset \#^r\overline{\mathbb{CP}^2} \setminus B^4$, then

$$s(K) \leq 1 - \chi(\Sigma) - [\Sigma]^2 - |[\Sigma]|.$$

This is similar to the adjunction inequality for the τ invariant in Heegaard Floer homology, previously proved by **Ozsváth** and **Szabó** (2003).

New obstructions to k-sliceness

The trace embedding lemma says that: a knot $K \subset S^3$ is slice $\iff$ the trace of 0-surgery $X_0(K)$ embeds in S^4. Here, $\partial X_0(K) = S^3_0(K)$.

This was used by **Piccirillo** (2018) to show that the Conway knot C is not slice, even though $s(C) = 0$, by finding a knot C' with $s(C') \neq 0$ and $X_0(C) = X_0(C')$.

In principle, one could attempt to find exotic S^4 (resp. exotic $\#^r \overline{\mathbb{CP}^2}$) by exhibiting knots $K, K' \subset S^3$ with $S^3_0(K) = S^3_0(K')$ such that K is slice (resp. k-slice in $\#^r \overline{\mathbb{CP}^2}$) and K' is not.

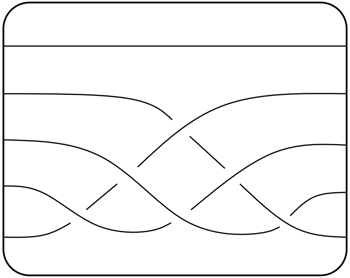
M.-Piccirillo (2021) studied a family of such knots, and found some examples of pairs (K, K') where K' is not slice, and K was of unknown sliceness. However, **Nakamura** (2022) showed that those examples of K are not slice (and, in fact, not 0-slice in $\#^r \overline{\mathbb{CP}^2}$), by using the null-homologous adjunction inequality in $\#^r \overline{\mathbb{CP}^2}$.

Using Ren's general adjunction inequality, **Qin** (2023) found new obstructions to knots being k-slice in $\#^r \overline{\mathbb{CP}^2}$.

Extensions of the Rasmussen invariant: in $\mathbb{RP}^3$

We distinguish between **class-0 links** and **class-1 links**, according to their homology class in $H_1(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}/2$. In particular, we have the *class-0 unknot* $U_0 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$ and the *class-1 unknot* $U_1 = \mathbb{RP}^1 \subset \mathbb{RP}^3$.

Note that $\mathbb{RP}^3 \setminus *$ is an I-bundle over $\mathbb{RP}^2$. We represent links through their projections to $\mathbb{RP}^2$. The antipodal points on the boundary of the disk are identified to form $\mathbb{RP}^2$. For example, this is a class-1 knot K_1:



Extensions of the Rasmussen invariant: in $\mathbb{RP}^3$

Khovanov homology for links in $\mathbb{RP}^3$ was defined by **Asaeda-Przytycki-Sikora** (2004) over $\mathbb{Z}/2$, and by **Manturov** (2006) and **Gabrovšek** (2018) over $\mathbb{Z}$.

M.-Willis (2023) construct a Lee deformation, and use it to define a Rasmussen-type invariant $s(K)$ for $K \subset \mathbb{RP}^3$. This bounds the genus of (oriented) link cobordisms in $I \times \mathbb{RP}^3$.

In particular, if $K \subset \mathbb{RP}^3$ is a class-α knot, $\alpha \in \{0, 1\}$, we define the *slice genus* $g_s(K)$ to be the minimal genus of a compact, oriented cobordism $\Sigma \subset I \times \mathbb{RP}^3$ from K to the class-α unknot U_α.

Theorem (M.-Willis, 2023)

If $K \subset \mathbb{RP}^3$ is a knot, then $|s(K)|/2 \leq g_s(K)$.

Extensions of the Rasmussen invariant: in $\mathbb{RP}^3$

Theorem (M.-Willis, 2023)

There exist knots K_0, K_1 in $\mathbb{RP}^3 = S^3/\tau$ that are not concordant (they do not co-bound an annulus in $I \times \mathbb{RP}^3$]), but such that their lifts to S^3 are concordant (co-bound an annulus in $I \times S^3$).

Idea of proof: Consider the lifts $\widetilde{K}$ of K and $\widetilde{K}'$ of $K' = (-K)\#\widetilde{K}$, where $K \subset \mathbb{RP}^3$ is any class-1 knot such that $s(\widetilde{K}) \neq 2s(K)$. Then $s(K') = s(-K) + s(\widetilde{K}) = -s(K) + s(\widetilde{K}) \neq s(K)$, but $\widetilde{K}$ and $\widetilde{K}' = (-\widetilde{K})\#\widetilde{K}\#\widetilde{K}$ are concordant.

An example is the knot K_1 with lift $\widetilde{K}_1 = 12n403$, which has $s(K_1) = 0$ and $s(\widetilde{K}_1) = 2$.

Note: The lifts of class 1-knots in $\mathbb{RP}^3$ are *freely periodic* knots in S^3. A concordance in $I \times \mathbb{RP}^3$ is the same as a standardly equivariant concordance of the lifts in $I \times S^3$.

Extensions of the Rasmussen invariant: in $\mathbb{RP}^3$

Let DTS^2 be the disk bundle associated to the tangent bundle to S^2. Its boundary is $\mathbb{RP}^3$.

Theorem (Ren, 2023)

If $K \subset \mathbb{RP}^3$ bounds an oriented surface $\Sigma \subset DTS^2$ with $[\Sigma] = d \in H_2(DTS^2, \mathbb{RP}^3) \cong \mathbb{Z}$, then

$$g(\Sigma)/2 \geq -s(K) - \frac{d^2}{2}$$

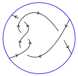
provided $d \in \{0, \pm 1, \pm 2, \pm 3\}$. (Conjecturally, for all d.)

Idea: The surface Σ gives a cobordism in $I \times \mathbb{RP}^3$ relating K to a link $T(d; p, q) \subset \mathbb{RP}^3$. The proof is based on calculating $s(T(d; p, q))$ for some values of d, p, q.

Chen (2023) studied a Bar-Natan deformation of the Khovanov complex in $\mathbb{RP}^3$ with $\mathbb{Z}/2$ coefficients. He defines an invariant $s^{BN}(K)$ for class-0 knots $K \subset \mathbb{RP}^3$.

Theorem (Chen, 2023)

Let $K \subset \mathbb{RP}^3$ be a class-0 knot, and $\Sigma \subset \mathbb{RP}^3 \times I$ be a surface with $\partial\Sigma = K \subset \mathbb{RP}^3 \times \{1\}$, such that Σ is **twisted orientable**, *i.e. its lift to $S^3 \times I$ has an orientation that is reversed under the deck transformation. Then, $|s^{BN}(K)| \leq -\chi(\Sigma)$.*

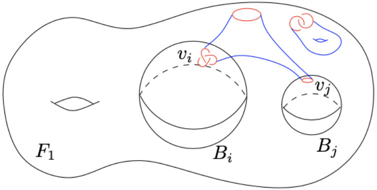
Application: The knot  has $s^{BN} = 2 \neq s$. It bounds an orientable slice surface of genus 1, but no such orientable surface that is also twisted orientable.

The skein lasagna module

Morrison-Walker-Wedrich (2019) defined an invariant of (framed, oriented) links L in any 3-manifold Y presented as the boundary of a 4-manifold X.

A *lasagna filling* $F = (\Sigma, \{(B_i, L_i, v_i)\})$ of X with boundary L consists of

- A finite collection of disjoint 4-balls B_i (called *input balls*) embedded in the interior or X;
- A framed oriented surface Σ properly embedded in $X \setminus \cup_i B_i$, meeting ∂X in L and meeting each ∂B_i in a link L_i; and
- for each i, a homogeneous label $v_i \in Kh(L_i)$.



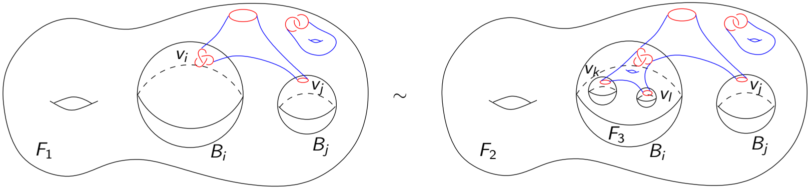
The skein lasagna module

We define

$$\mathcal{S}_0(X; L) := \mathbb{Z}\{\text{lasagna fillings } F \text{ of } X \text{ with boundary } L\}/\sim$$

where $\sim$ is the transitive and linear closure of the following relations:

- Linear combinations of lasagna fillings are set to be multilinear in the labels v_i;
- F_1 and F_2 are set to be equivalent if F_1 has an input ball B_i with label v_i, and F_2 is obtained from F_1 by replacing B_i with another lasagna filling F_3 of a 4-ball such that $v_i = Kh(F_3)$, followed by an isotopy rel ∂X (where the isotopy is allowed to move the input balls):



The skein lasagna module

Remarks:

- $\mathcal{S}_0(X; L)$ decomposes as a direct sum of $\mathcal{S}_0(X; L; \alpha)$ according to the relative homology classes α of the fillings;
- When $X = B^4$, we have $\mathcal{S}_0(X; L) = Kh(L)$ (up to a grading shift);
- When X is made of 2-handles attached to B^4 along a link K, **M.-Neithalath** (2020) showed that $\mathcal{S}_0(X; L)$ is a direct limit of the Khovanov homologies of the union of L and certain cables of K;
- For any X, this was extended to a general description of $\mathcal{S}_0(X; L)$ in terms of handle decompositions; cf. **M.-Walker-Wedrich** (2022).

This allowed for computations of $\mathcal{S}_0(X) := \mathcal{S}_0(X; \emptyset)$ for manifolds such as $S^1 \times S^3$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, disk bundles over S^2; e.g., we have $\mathcal{S}_0(\mathbb{CP}^2) = 0$ but $\mathcal{S}_0(\overline{\mathbb{CP}^2}) \neq 0$.

Further computations were done by **Sullivan-Zhang** (2024) and **Ren-Willis** (2024); e.g. $\mathcal{S}_0(S^2 \times S^2) = 0$.

Extensions of the Rasmussen invariant: in any $Y^3 = \partial X^4$

Morrison-Walker-Wedrich (2024) and **Ren-Willis** (2023) computed the Lee deformation of $\mathcal{S}_0(X; L)$. Ren and Willis also identified the generators of Lee homology, and used these to define invariants

$$s(X; L; \alpha) \in \mathbb{Z} \cup \{-\infty\}.$$

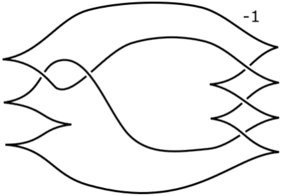
When $X = B^4$, we recover the Rasmussen invariant up to a sign and shift: $s(B^4; L) = -s(-L) - w(L) + 1$.

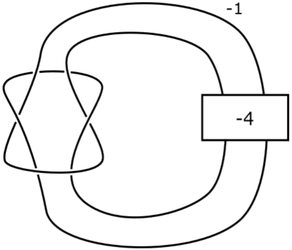
Theorem (Ren-Willis, 2024)

If $\Sigma \subset X$ is smoothly embedded with $\partial\Sigma = L$, then $2g(\Sigma) \geq s(X; L; [\Sigma]) + [\Sigma]^2 - \#L$.

In general, there is no algorithm for computing $s(X; L; \alpha)$. Furthermore, $s(X; L; \alpha)$ can be $-\infty$ (e.g. for $X = S^2 \times S^2$ or $\mathbb{CP}^2$ and $L = \emptyset$), in which case we get no genus bounds.

Nevertheless, using the formula for $\mathcal{S}_0$ of 2-handlebodies X, Ren and Willis compute $s(X;\alpha) = s(X,\emptyset;\alpha)$ in many cases. In particular, consider the traces of -1 surgeries on the knots $K_1 = -5_2$ and $K_2 = P(3,-3,-8)$:





we have $s(X_1;1) = 3$ and $s(X_2;1) = 1$. Hence X_1 and X_2 are not diffeomorphic, even though they are homeomorphic.

Detection of exotic smooth structures

Akbulut (1991) was the first to show that X_1 and X_2 is an exotic pair, using gauge theory.

Ren and Willis give the first analysis-free proof. Here are the main ideas in their computation:

- For $X_1 = X_{-1}(K_1)$, they use that $K_1 = -5_2$ is a positive knot with $s = 2$. For such knots, they prove that $s(X_n(K); 1) = s(K) - n$ by calculating Kh of cables of K in the highest homological degree, generalizing an argument of **Stošić** (2006).
- For $X_2 = X_{-1}(K_2)$, they use that $K_2 = P(3, -3, -8)$ is slice, i.e. concordant to the unknot. Hence, the cables of K_2 are concordant to the cables of the unknot, which implies that $s(X_{-1}(K_2); 1) = s(\overline{\mathbb{CP}^2}; 1) = 1$.

Open problems

1. Construct Khovanov homology and the Rasmussen invariant for links in other 3-manifolds (e.g. lens spaces beyond S^3 and $\mathbb{RP}^3$) in a more elementary way; i.e., without skein lasagna modules.
2. Does the adjunction inequality for the s-invariant hold for surfaces in any negative-definite 4-manifold, instead of just $\#^r \overline{\mathbb{CP}^2}$?
3. Is there a (class-1) slice knot in $\mathbb{RP}^3$ whose lift to S^3 is not slice?
4. Can skein lasagna modules detect exotic smooth structures on some closed 4-manifolds?
5. Can skein lasagna modules detect any new exotic smooth structures on 4-manifolds?