Quantum Chern-Simons Theory and Resurgence

by

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The Witten-Reshetikhin-Turaev (WRT) Quantum Invariants

Recall the Witten-Reshetikhin-Turaev Quantum Invariant:

$$Z^{(r)}(M, L, \lambda) \in \mathbb{C}$$

- $r \in \{2, 3, \ldots\}$.
- M closed oriented 3-manifold
- $L \subset M$ embedded link, each component colored by representations λ of $SU(2)$

These invariants $Z^{(r)}$ are precise mathematical construction of Witten's $\mathrm{SU}(2)$ quantum Chern-Simons theory ($r = k + 2$):

$$Z^{(r)}(M, L, \lambda) \text{ " = " } \int_{A \in \mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)}} \exp\left(2\pi i k \, \mathrm{CS}(A)\right) \mathrm{tr}\, \lambda\left(\mathrm{Hol}_L(A)\right) \mathcal{D}A$$

where

- $\mathcal{A}_{\mathrm{SU}(2)}$ space of all connections in a principal $\mathrm{SU}(2)$ bundle over M
- $\mathcal{G}_{\mathrm{SU}(2)}$ group of gauge transformations in the principal bundle over M
- $\mathrm{Hol}_L(A)$ holonomy around L w.r.t. $A \in \mathcal{A}_{\mathrm{SU}(2)}$.

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- $\int \ \ldots \ \mathcal{D}A$ Still no known mathematical definition of Feynman path integrals!

Quantum Chern-Simons theory

$Z^{(r)}$ $2+1$ dimensional **Topological Quantum Field Theory**:

- 2 dim. part of $Z^{(r)}$: **Modular functor**
  · Λ_r finite label set for $Z^{(r)}$ with involution $\dagger : \Lambda_r \to \Lambda_r$

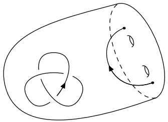
$$Z^{(r)} : \quad \left\{\begin{array}{l}\text{Category of closed oriented}\\ \text{surfaces with } \Lambda_r\text{-labeled}\\ \text{marked points}\end{array}\right\} \to \left\{\begin{array}{l}\text{Category of finite dim}\\ \text{vector spaces over } \mathbb{C}\end{array}\right\}$$

· $Z^{(r)}(\bar{\Sigma}, \lambda_1, \dots, \lambda_n) \cong Z^{(r)}(\Sigma, \lambda_1^\dagger, \dots, \lambda_n^\dagger)^*$ $\quad$ $\bar{\Sigma} = \Sigma$ with reversed orientation.

· $Z^{(r)}(\Sigma_c, \lambda) \cong \bigoplus_{\mu \in \Lambda_r} Z^{(r)}(\Sigma, \mu, \mu^\dagger, \lambda)$ $\quad$ Σ_c glue of blow up of Σ at two parked points.

- 3 dim. part of $Z^{(r)}$:

$$X : \text{compact 3–manifold}$$
$$(L, \partial L) \subseteq (X, \partial X) : \text{oriented link}$$
$$\lambda : \Lambda_r\text{-labeling of } L$$
$$Z^{(r)}(X, L, \lambda) \in Z^{(r)}(\partial X, \partial L, \partial \lambda)$$

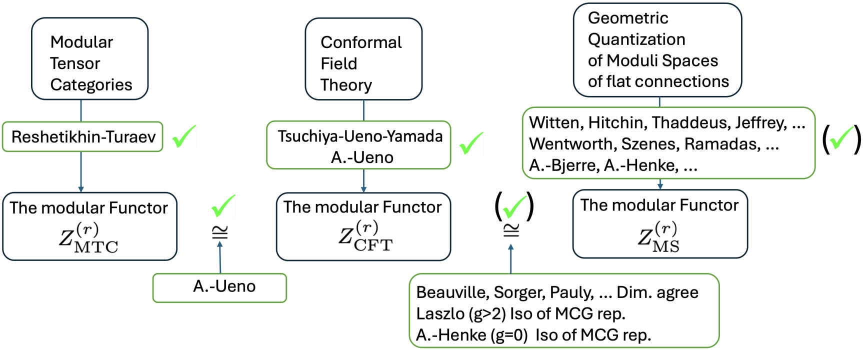


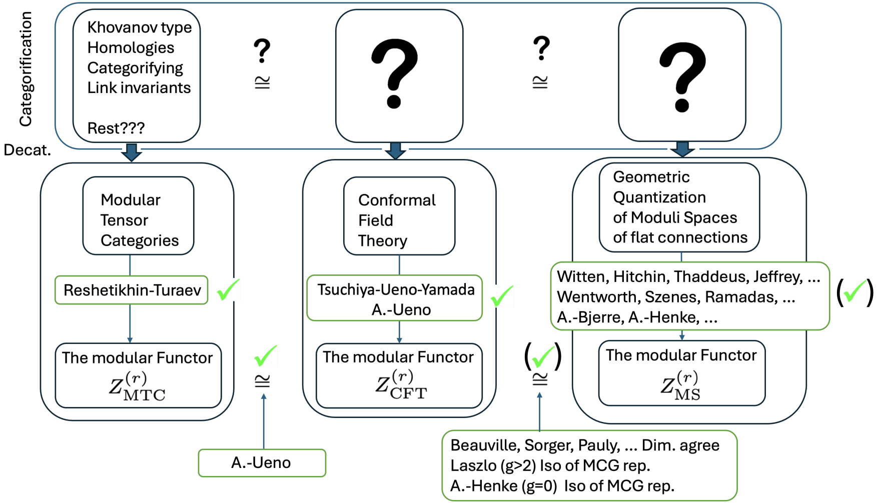
- $Z^{(r)}$ satisfies the Atiyah-Segal-Witten T.Q.F.T. axioms:

$$Z^{(r)}(X_1, L_1, \lambda_1) \circ Z^{(r)}(X_2, L_2, \lambda_2) = Z^{(r)}(X_1 \cup_\partial X_2, L_1 \cup_\partial L_2, \lambda_1 \cup_\partial \lambda_2)$$

2 dim. Modular Functor part determines entire TQFT.

The three different approaches to the WRT-TQFT (Conjectured by Witten):




Categorification
Decat.
Khovanov type Homologies Categorifying Link invariants
Rest???
?
≅
?
?
≅
?
Modular Tensor Categories
Reshetikhin-Turaev
The modular Functor $Z^{(r)}_{\mathrm{MTC}}$
≅
A.-Ueno
Conformal Field Theory
Tsuchiya-Ueno-Yamada
A.-Ueno
The modular Functor $Z^{(r)}_{\mathrm{CFT}}$
( )
≅
Geometric Quantization of Moduli Spaces of flat connections
Witten, Hitchin, Thaddeus, Jeffrey, ...
Wentworth, Szenes, Ramadas, ...
A.-Bjerre, A.-Henke, ...
( )
The modular Functor $Z^{(r)}_{\mathrm{MS}}$
Beauville, Sorger, Pauly, ... Dim. agree
Laszlo (g>2) Iso of MCG rep.
A.-Henke (g=0) Iso of MCG rep.

Picard-Lefshetz theory, resurgence and TQFT

Resurgence in TQFT: In a number of different publications **Witten, Garoufalidis, Gukov-Marino-Putrov and Gukov-Pei-Putrov-Vafa, Garoufalidis-Gu-Marino-Wheller** have proposed/argued one can in **lots of examples** apply the theory of Resurgence to the partition function of $\mathrm{SU}(2)$ Chern-Simons theory ($r = k + 2$)

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by thinking of

$$\mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)} \subset \mathcal{A}_{\mathrm{SL}(2,\mathbb{C})}/\mathcal{G}_{\mathrm{SL}(2,\mathbb{C})}.$$

as a real "middle dimensional cycle" and decomposing it into

"middle dimensional Picard-Lefshetz thimbles"

in this path integral setting.

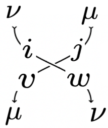
The Reshetikhin-Turaev formula for the colored Jones polynomial via Quantum R-matrices

Recall $r \in \{2,3,\ldots\}$. Quantum integers are $[l] = \frac{\sin(\frac{l}{\pi r})}{\sin(\frac{1}{\pi r})}$ and $[l]! = \prod_{i=1}^{l}[i]$.
Let $\mathcal{L} = \frac{1}{2}\mathbb{Z}_+ \cup \{0\}$ and $\Lambda_r = \{\lambda \in \mathcal{L} \mid 0 < 2\lambda + 1 < r\}$.
For $\lambda \in \Lambda_r$ let $V^{(\lambda)}$ be a complex v.s. of dim. $2\lambda+1$, basis $e_i^{(\lambda)}$ indexed by

$$i \in B_\lambda = \{-\lambda, -\lambda+1, \ldots, \lambda\}.$$

For $\nu, \mu \in \mathcal{L}$ consider the braiding isomorphism

$$C^{\pm}(\nu,\mu) : V^{(\nu)} \otimes V^{(\mu)} \to V^{(\mu)} \otimes V^{(\nu)},$$



Given by

$$C^{\pm}(\nu,\mu)\left(e_i^{(\nu)} \otimes e_j^{(\mu)}\right) = \sum_{v,w} C(\nu,\mu)_{i;j}^{v;w} e_v^{(\mu)} \otimes e_w^{(\nu)},$$

$$C^{+}(\nu,\mu)_{i;j}^{v;w} = \delta_{i+j}^{v+w} \frac{[\nu+w]![\mu-v]!}{[w-i]![\nu+i]![\mu-j]!} e^{\frac{i\pi}{2r}(4ij-2(w-i)(i-j)-(w-i)(w-i+1))}$$

$$\mathcal{C}_+ : \quad \begin{matrix} w \geq i, \\ j \geq v \end{matrix}$$

$$C^{-}(\nu,\mu)_{i;j}^{v;w} = \delta_{i+j}^{v+w} \frac{[\mu+v]![\nu-w]!}{[v-j]![\mu+j]![\nu-i]!} e^{-\frac{i\pi}{2r}(4ij-2(v-j)(j-i)-(v-j)(v-j+1))}$$

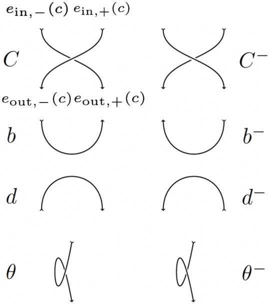
$$\mathcal{C}_- : \quad \begin{matrix} v \geq j, \\ i \geq w \end{matrix}$$

The Reshetikhin-Turaev formula for the colored Jones polynomial via Quantum R-matrices

Let $E, C, \cup_-, \cap_-$ and K are the set of edges, crossings, cups, caps and kinks of a link diagram D_L of a framed link L and define

$$\mathcal{I} = \{j : E \mapsto \frac{1}{2}\mathbb{Z} \mid \ j(e) \in B_{\lambda_e} \ \forall e \in E, \ \ \mathcal{C}_+, \mathcal{C}_- \ \text{ satisfied } \forall \text{c} \in \text{C}\}.$$

The Jones polynomial of the framed link L colored by $\lambda \in \Lambda_r^{\pi_0(L)}$ is given by



$$J_\lambda^{(r)}(L) = J_\lambda(L)(e^{2\pi i/r}) =$$

$$\sum_{j \in \mathcal{I}} \left(\prod_{c \in C} C^{\epsilon(c)}(\lambda_{l(c)}, \lambda_{r(c)})^{j(e_{\text{in},-}(c));j(e_{\text{in},+}(c))}_{j(e_{\text{out},-}(c));j(e_{\text{out},+}(c))} \prod_{u \in \cup_-} e^{-\frac{2\pi i}{r} j(u)} \prod_{n \in \cap_-} e^{\frac{2\pi i}{r} j(n)} \prod_{k \in K} e^{\frac{\pi i}{r}\epsilon(k)(2\lambda(k)+1)(\lambda(k)+1)} \right)$$

The WRT quantum invariant via quantum R-matrices

For $\lambda \in \Lambda_r^{\pi_0(L)}$ and $\sigma(L) =$ signature of the framed link L, we set

$$[\lambda] = \left(\prod_{\ell \in \pi_0(L)} [\lambda_\ell] \right), \quad \star(L) = e^{\pi i \frac{3(r-2)}{r} \sigma(L)} \left(\sqrt{\frac{2}{r}} \sin(\frac{\pi}{r}) \right)^{|\pi_0(L)|+1}$$

Theorem (Reshetikhin and Turaev)

$$Z^{(r)}(M) = \ \star(L) \sum_{\lambda \in \Lambda_r^{\pi_0(L)}} [\lambda] J_\lambda^{(r)}(L).$$

is an invariant of $M = M_L$, the oriented closed three manifold obtained by doing surgery on the framed link L in S^3.

Similarly, if $L = L_s \cup L_c$ and if we choose $\lambda \in \Lambda_r^{\pi_0(L_c)}$ then

$$Z^{(r)}(M_{L_s}, L_c, \lambda) = \ \star(L_s) \sum_{\mu \in \Lambda_r^{\pi_0(L_s)}} [\lambda] J_{(\mu,\lambda)}^{(r)}(L)$$

where M_{L_s} is obtained by surgery on L_s.
From now on we simply write

$$\star = \star(L)$$

Faddeev's quantum dilogarithm

Recall Faddeev's quantum dilogarithm S_γ with parameter γ given by

$$S_\gamma(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+i\epsilon} \frac{e^{zy}}{\sinh(\pi y)\sinh(\gamma y) y}\, \mathrm{d}\, y\right).$$

for $|\mathrm{Re}(z)| < \gamma + \pi$, and $\gamma \in (0,1)$.

- For $\mathrm{Re}(\gamma) > 0$, S_γ is a meromorphic function and $(1+e^{iz})S_\gamma(z+\gamma) = S_\gamma(z-\gamma)$

$$S_{\pi \mathsf{b}^2}(-2\pi i \mathsf{b} z) = \frac{(wq, q^2)_\infty}{(\tilde{w}\tilde{q}, \tilde{q}^2)_\infty}, \quad \begin{matrix} w = e^{2\pi \mathsf{b} x}, q = e^{i\pi \mathsf{b}^2} \\ \tilde{w} = e^{\frac{2\pi x}{\mathsf{b}}}, \tilde{q} = e^{-\frac{i\pi}{\mathsf{b}^2}}, \end{matrix}, \quad \mathrm{Im}(\mathsf{b}^2) > 0$$

Introduce the following normalized version of the Faddeev's quantum dilogarithm

$$\tilde{S}_\gamma(z) = \frac{S_\gamma(\pi - (2z+1)\gamma)}{S_\gamma(\pi - \gamma)}.$$

We are going to set $\gamma = \pi/r$, then the function $\tilde{S}_\gamma$ has poles at $z = -1, -2, \ldots$ and zeros at $z = r, r+1, \ldots$. In particular, we observe that

$$z \mapsto \frac{1}{\tilde{S}_\gamma(-z^2)}$$

is 1 in zero and zero on all other integers.
Further we have that

$$[m]! = \frac{e^{\frac{i\pi}{2r}m(m+1)}}{(2i\sin(\frac{\pi}{r}))^m}\tilde{S}_\gamma(m)$$

Meromorphic extension of the quantum R-matrix

Definition

$$R^{\pm} \in \mathcal{M}(\mathbb{C}^2 \times \mathbb{C}^4)$$

$$\begin{aligned} R^+(y_1, y_2, z_1, z_2, z_3, z_4) &= \frac{\tilde{S}_\gamma(y_1 + z_4)\tilde{S}_\gamma(y_2 - z_3)}{\tilde{S}_\gamma(z_4 - z_1)\tilde{S}_\gamma(y_1 + z_1)\tilde{S}_\gamma(y_2 - z_2)} \times \\ &\quad \frac{e^{\frac{i\pi}{2r} Q_+(y_1, y_2, z_1, z_2, z_3, z_4)}}{\tilde{S}_\gamma(-(z_1 + z_2 - z_3 - z_4)^2)} \\ R^-(y_1, y_2, z_1, z_2, z_3, z_4) &= \frac{\tilde{S}_\gamma(y_2 + z_3)\tilde{S}_\gamma(y_1 - z_4)}{\tilde{S}_\gamma(z_3 - z_2)\tilde{S}_\gamma(y_2 + z_2)\tilde{S}_\gamma(y_1 - z_1)} \times \\ &\quad \frac{e^{\frac{i\pi}{2r} Q_-(y_1, y_2, z_1, z_2, z_3, z_4)}}{\tilde{S}_\gamma(-(z_1 + z_2 - z_3 - z_4)^2)} \end{aligned}$$

Here $Q_\pm$ are explicit quadratic fn's of $y_1, y_2, z_1, z_2, z_3, z_4$.

Lemma

$$C^{\pm}(\nu, \mu)_{i;j}^{v;w} = R^{\pm}(\nu, \mu, i, j, v, w)$$

Definition

Let $J(D_L)_r \in \mathcal{M}(\mathbb{C}^C \times \mathbb{C}^E)$ be given by

$$J(D_L)_r(y,z) = \prod_{c\in C} R^{\epsilon(c)}(y_{l(c)}, y_{r(c)}, z_{e_{\mathrm{in},-}(c)}, z_{e_{\mathrm{in},+}(c)}, z_{e_{\mathrm{out},-}(c)}, z_{e_{\mathrm{out},+}(c)}) \prod_{e\in E} \cot(\pi(z_e + y_e))$$

Let Γ_λ be a contour in the complex plan close to the real axis, which encircles $\{-\lambda, -\lambda+1, \ldots, \lambda-1, \lambda\}$.

- $\Gamma_{D_L} = \prod_{e\in E} \Gamma_{\lambda(e)} \subset \mathbb{C}^E$.

Theorem (A. & Mistegaard)

$$J_\lambda^{(r)}(L) = \int_{z\in\Gamma_{D_L}} J(D_L)_r(\lambda, z) \bigwedge_{e\in E} dz_e$$

- $\tilde{\Gamma}_r \subset \mathbb{C}$ is a contour in $\mathbb{C}$ close to the real axis, which encircles $\{1, \ldots, r\}$.

Theorem (A. & Mistegaard)

$$Z^{(r)}(M_L) = \star \int_{y\in\tilde{\Gamma}_r^{\pi_0(L)}} J_y^{(r)}(D_L) \prod_{\gamma\in\pi_0(L)} \frac{\sin(\frac{(2y_\gamma+1)\pi}{r})}{\sin(\frac{\pi}{r})} \cot(2\pi y_\gamma) \bigwedge_{\gamma\in\pi_0(L)} dy_\gamma$$

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Finite dimensional model

We have just derived the following finite dimensional model

$$Z^{(r)}(M_L) = \star \int_{(y,z)\in \tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L}} Z(D_L)_r(y,z) \bigwedge_{\gamma\in\pi_0(L)} dy_\gamma \bigwedge_{e\in E} dz_e$$

where

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$$\tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L} \subset \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E$$

is a middle dimensional cycle in this finite dimensional affine space.

Recall that

$$\star = e^{\pi i \frac{3(r-2)}{r}\sigma(L)} \left(\sqrt{\frac{2}{r}}\sin(\frac{\pi}{r})\right)^{|\pi_0(L)|+1}$$

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Recall that

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Question: In what sense might

$$J(D_L)_r, Z(D_L)_r \in \mathcal{M}(\mathbb{C}^{\pi_0(L)}\times\mathbb{C}^E)$$

be an invariant of L and M_L respectively?

Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating $Z^{(r)}$ to classical Chern-Simons theory.

$$Z^{(r)}(M) \sim_{r\to\infty} \sum_{\theta\in \mathrm{CS}_{\mathrm{SU}(2)}} \exp(2\pi i r\theta) r^{d_\theta} b_\theta (1 + Z_\theta(r^{-1})).$$

where $Z_\theta(r^{-1}) = c_\theta^{(1)} r^{-1} + c_\theta^{(2)} r^{-2} + \ldots$.

Here $\sim_{r\to\infty}$ means

$$\left| Z^{(r)}(M) - \sum_{\theta\in \mathrm{CS}_{\mathrm{SU}(2)}} \exp(2\pi i r\theta) r^{d_\theta} b_\theta (1 + Z_\theta^L(r^{-1})) \right| \leq C_L r^{d-L-1} \quad \forall r \in \{2, 3 \ldots\},$$

where C_L are some constants and

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Conjectures about $Z^{(r)}$ and resurgence as a Rosetta stone

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2. The volume conjecture: relating $Z^{(r)}$ to hyperbolic geometry.
   Kashaev's original volume conjecture (in the MM formulation):

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log\left(\frac{|J_\lambda(K)(e^{2\pi i/\lambda})|}{|J_\lambda(U)(e^{2\pi i/\lambda})|}\right) = \frac{1}{2\pi}\mathrm{Vol}(S^3 - K)$$

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1. The asymptotic expansion conjecture: relating $Z^{(r)}$ to classical Chern-Simons theory.
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3. Integrality and categorification of $Z^{(r)}$: The GPPV invariant of a 3-mfd. M (defined as a string theory BPS index) with a spin^c structure a is an essential integer power series
$$\hat{\mathrm{Z}}_a(M;q) \in 2^{-c} q^{\Delta_a} \mathbb{Z}[[q]],$$
where $\Delta_a \in \mathbb{Q}$, $c \in \mathbb{Z}_+$. Conjecture: There exists a homology theory $\mathrm{H}_{\bullet,\bullet}(M;a)$ s.t.
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The asymptotic expansion of the TQFT $Z^{(r)}$, the volume conjecture and $\hat{\mathrm{Z}}_a(M;q)$ are **connected via resurgence.**

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Resummation of divergent power series

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- **Divergent series**: In mathematical physics divergent series are common and many examples comes from path integrals.
- **Problem**: Given a divergent power series

  $$\varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]].$$

  we want to construct a holomorphic function ($D =$ domain in $\mathbb{C}$ such that $\infty \in \bar{D}$)

  $$\widehat{\varphi} \in \mathcal{O}(D)$$

  having φ as an asymptotic expansion, i.e. $\forall m \in \mathbb{N}$

  $$\widehat{\varphi}(z) = \varphi_0 z^{-1} + \cdots + \varphi_m z^{-m-1} + O(z^{-m-2}).$$

- **Borel-Laplace resummation** is (sometimes) a solution to this problem.

The Borel transform $\mathcal{B}$

Definition

The Borel transform

$$\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$$

is the $\mathbb{C}$-linear extension of

$$\mathcal{B}\left(z^{-\alpha-1}\right) = \frac{\zeta^\alpha}{\Gamma(\alpha+1)} = \frac{\zeta^\alpha}{\alpha!}.$$

Here Γ is the Gamma function, which for $\mathrm{Re}(x) > 0$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}\,t.$$

The Borel transform $\mathcal{B}$ as the inverse of the Laplace transform $\mathcal{L}$

- **The Laplace transform $\mathcal{L}$:** Let $\gamma \subset \mathbb{C}$ be an oriented countour. Let g be a holomorphic function. Define

$$\mathcal{L}_\gamma(g)(z) = \int_\gamma \exp(-z \cdot \zeta) g(\zeta) \, \mathrm{d}\,\zeta.$$

  when ever this integral converges say absolutely.

Proposition

For all $m \in \mathbb{N}$ one has

$$\mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(z^{-m-1}) = z^{-m-1},$$
$$\mathcal{B} \circ \mathcal{L}_{\mathbb{R}_+}(\zeta^m) = \zeta^m.$$

Borel-Laplace resummation $\varphi \mapsto \mathcal{L} \circ \mathcal{B}(\varphi)$

Proposition

Let $\varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$. *Assume* $\mathcal{B}(\varphi)(\zeta)$ *extends to an analytic function of appropriate bound along* $\gamma(\theta) = \exp(i\theta)\mathbb{R}_+$. *Consider*

$$\widehat{\varphi}(z) \overset{\text{def.}}{=} \mathcal{L}_{\gamma(\theta)} \circ \mathcal{B}(\varphi)(z) = \int_{\gamma(\theta)} e^{-\zeta z} \mathcal{B}(\varphi)(\zeta) \, \mathrm{d}\,\zeta.$$

Borel-Laplace resummation $\varphi \mapsto \mathcal{L} \circ \mathcal{B}(\varphi)$

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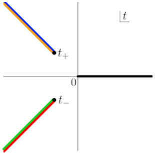
The function $\widehat{\varphi}(z)$ *is analytic on* Re$(z\exp(i\theta)) > 0$ *and has* $\varphi(z)$ *as Poincaré asymptotic expansion*

$$\widehat{\varphi}(z) \sim_{|z|\to\infty} \varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]].$$

- **Picard-Lefshetz theory and resurgence:** Let $f \in \mathcal{O}(X^d)$, let $\omega \in \Omega^d_{\mathrm{Hol}}(X^d)$. Let $\lambda \in \mathbb{C}^*$. Let Δ_φ, $\varphi = \arg(\lambda))$ be a PL-thimble emanating from a critical point z_c of f and consider

$$\mathrm{I}_{\Delta_\varphi}(\lambda) = \int\limits_{\Delta_\varphi} \exp(-\lambda f(z))\, \omega(z)$$



Think of the t-plane as the set of values of f with a discrete set of critical values (here $0, t_+, t_-$) with curves γ emanating from them, along which the exponential factor is decaying.

Illustration of a Picard-Lefschetz thimble $\Delta(\sigma, \gamma)$

Below we illustrate a Picard-Lefshetz thimble $\Delta_\varphi(\sigma)$ foliated by vanishing cycles $\sigma(t) \in \mathrm{H}_*(f^{-1}(t), \mathbb{Z})$ which are parallel (w.r.t. the Gauss-Manin connection) along a curve $\gamma \subset \mathrm{Im}(f)$

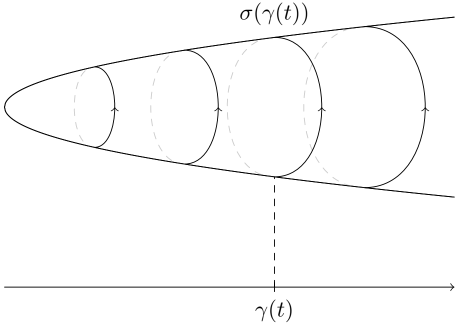


Figure: Thimble $\Delta_\varphi(\sigma)$ in $d = 2$.

Picard-Lefshetz theory and resurgence

Picard-Lefshetz theory and resurgence: Let $f \in \mathcal{O}(X^d)$, let $\omega \in \Omega^d_{\mathrm{Hol}}(X^d)$, let Δ_φ be a PL-thimble emanating from a critical point z_c and consider

$$\mathrm{I}_{\Delta_\varphi}(\lambda) = \int_{\Delta_\varphi} \exp(-\lambda f(z))\, \omega(z).$$

Let $\tilde{\mathrm{I}}_{\Delta_\varphi} \in \xi^{-1}\mathbb{C}[[\xi^{-1}]]$ be such that we have the asymptotic expansion

$$\mathrm{I}_{\Delta_\varphi}(\lambda) \sim \exp(-\lambda f(z_{\Delta_\varphi}))\lambda^{d_{\Delta_\varphi}}(1 + \tilde{\mathrm{I}}_{\Delta_\varphi}(\lambda^{-1})).$$

The aim of Écalle's theory of resurgence is to decode the information contained in the divergent series

$$\tilde{\mathrm{I}}_{\Delta_\varphi} \in \xi^{-1}\mathbb{C}[[\xi^{-1}]]$$

and determine the properties of the Borel resummation of them as analystic multi-value functions and in turn to recover from them the actual integrals $\mathrm{I}_{\Delta_\varphi}(\lambda)$.

Picard-Lefshetz theory and resurgence

In particular, when we turn the argument φ of the expansion parameter λ around, then the γ's emanating from some critical z_c point will hit some other critical point z'_c, say at φ_0 and at this point the corresponding $\mathrm{I}_{\Delta_\varphi}$ jumps to

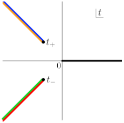
$$\mathrm{I}_{\Delta_{\varphi_0+\epsilon}} = \mathrm{I}_{\Delta_{\varphi_0-\epsilon}} + n(z_c, z'_c)\, \mathrm{I}_{\Delta'_{\varphi_0}},$$

hence we see that

$\mathrm{I}_{\Delta'_{\varphi_0}}$ resurges with a multiplicity factor $n(z_c, z'_c)$ in $\mathrm{I}_{\Delta_{\varphi_0}}$.

Écalle has developed his Alien calculus precisely to determine these jumping or wall crossing phenomenon directly from the divergent series $\tilde{\mathrm{I}}_{\Delta_\varphi}$.

There are new reformulations of parts of Écalle's Alien calculus, such as
- **Analytic Wall Crossing Structure** by Kontsevich-Soibelman
- Ongoing work of A. and Kontsevich on **Cycle Systems and Analytic Realizations**.



Picard-Lefshetz theory, resurgence and TQFT

Resurgence in TQFT: In a number of different publications **Witten, Garoufalidis, Gukov-Marino-Putrov and Gukov-Pei-Putrov-Vafa, Garoufalidis-Gu-Marino-Wheller** have proposed/argued one can in **lots of examples** apply the theory of Resurgence to the partition function of $\mathrm{SU}(2)$ Chern-Simons theory ($r = k + 2$)

$$Z^{(r)}(M) \text{ "} = \text{" } \int_{A \in \mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)}} \exp\left(2\pi i k \mathrm{CS}(\mathrm{A})\right) \mathcal{D}A$$

by thinking of

$$\mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)} \subset \mathcal{A}_{\mathrm{SL}(2,\mathbb{C})}/\mathcal{G}_{\mathrm{SL}(2,\mathbb{C})}.$$

as a real "middle dimensional cycle" and decomposing it into

"middle dimensional Picard-Lefshetz thimbles"

in this path integral setting.

Finite dimensional model

We have just derived the following finite dimensional model

$$Z^{(r)}(M_L) = \star \int_{(y,z)\in\tilde{\Gamma}_r^{\pi_0(L)}\times\Gamma_{D_L}} Z(D_L)(y,z) \bigwedge_{\gamma\in\pi_0(L)} dy_\gamma \bigwedge_{e\in E} dz_e$$

where

$$Z(D_L)(y,z) = J(D_L)(y,z) \prod_{\gamma\in\pi_0(L)} \frac{\sin(\frac{(2y_\gamma+1)\pi}{r})}{\sin(\frac{\pi}{r})} \cot(2\pi y_\gamma)$$

$$\tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L} \subset \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E$$

is a middle dimensional cycle in this finite dimensional affine space.

Recall that

$$\star = e^{\pi i \frac{3(r-2)}{r}\sigma(L)} \left(\sqrt{\frac{2}{r}}\sin(\frac{\pi}{r})\right)^{|\pi_0(L)|+1}$$

Conjectures about $Z^{(r)}$ and resurgence as a Rosetta stone

Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating $Z^{(r)}$ to classical Chern-Simons theory.

$$Z^{(r)}(M) \sim_{r\to\infty} \sum_{\theta\in \mathrm{CS}_{\mathrm{SU}(2)}} \exp(2\pi i r\theta) r^{d_\theta} b_\theta (1 + Z_\theta(r^{-1})).$$

where $Z_\theta(r^{-1}) = c_\theta^{(1)} r^{-1} + c_\theta^{(2)} r^{-2} + \dots$.

Here $\sim_{r\to\infty}$ means

$$\left| Z^{(r)}(M) - \sum_{\theta\in \mathrm{CS}_{\mathrm{SU}(2)}} \exp(2\pi i r\theta) r^{d_\theta} b_\theta (1 + Z^L_\theta(r^{-1})) \right| \leq C_L r^{d-L-1} \ \ \forall r \in \{2,3\dots\},$$

where C_L are some constants and

$$d = \max\{d_\theta\}, \quad Z_\theta^L(r^{-1}) = \sum_{\ell=1}^{L} c_\theta^{(\ell)} r^{-\ell}$$

The Resurgence conjecture as a Rosetta stone

The Resurgence Conjecture:
(Garoufalidis; Gukov, Marino, Putrov; Gukov, Pei, Putrov, Vafa; A. & Mistegaard)

1. Borel Resummability: The series Z_θ are Borel resummable, $\mathcal{B}(Z_\theta)$ endless continuable.
2. Generalised Volume Conjecture: The set of poles $\Omega(\theta)$ of the meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies that
$$-2\pi i\, \mathrm{CS}_{\mathbb{C}}(M) = \cup_\theta \left(\Omega(\theta) - 2\pi i\theta\right) \mod \mathbb{Z}.$$
3. Wall Crossing: The meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies and are in part determined by a Wall crossing structure.
4. The $\hat{Z}$-Conjecture: The $\hat{\mathrm{Z}}_a$ GPPV invariants can be obtained from the functions $\mathcal{B}(Z_\theta)$ by a finite Laplace type transform.
5. AK-TQFT: $\theta_{\mathrm{c}} =$ CS of conjugate representation: $\mathcal{L} \circ \mathcal{B}(Z_{\theta_{\mathrm{c}}}) = Z^{AK}$.
6. The Radial Limit Conjecture: The WRT invariant $Z^{(r)}$ can be (re)-obtained from $\hat{\mathrm{Z}}_0$ as a limit $Z^{(r)} = \frac{1}{\sqrt{r}} \lim_{q \to e^{2\pi i/r}} \hat{\mathrm{Z}}_0(q)$.

Here Z^{AK} = Andersen-Kashaev TQFT.

Topological invariance and the definition of $\hat{Z}_a(Y;q)$

Definition (Gukov,Pei,Putrov,Vafa)

Let Γ be a plumbing graph. Then there is an explicit definition of

$$\Delta_a \in \mathbb{Q},\ c \in \mathbb{Z}_+,\ \hat{Z}_a(\Gamma;q) \in 2^{-c}q^{\Delta_a}\mathbb{Z}[[q]],$$

for each spin$_c$-structure a on $\mathbf{Y}_\Gamma$.

Theorem (Gukov, Manolescu)

If $\mathbf{Y}_\Gamma = \mathbf{Y}'_\Gamma$ (e.g. Γ and Γ' are related by Neumann moves) then

$$\hat{Z}_a(\Gamma;q) = \hat{Z}_a(\Gamma';q).$$

The AK-TQFT

Theorem (Andersen & Kashaev)

For any hyperbolic knot K the AK-TQFT is determined by a **Jones function**

$$J_{K,\hbar} \in S(\mathbb{R})$$

parametrised by $\hbar \in \mathbb{R}_+$, which is topological invariant of the knot.

Conjecture (AK-Volume conjecture)

The hyperbolic volume of $S^3 - K$ is recovered as the following limit

$$\lim_{\hbar \to 0} 2\pi\hbar \log |J_{K,\hbar}(0)| = -\mathit{vol}(S^3 - K).$$

Examples: $(\Phi_{\mathsf{b}}(z) = S_{\pi \mathsf{b}^2}(-2\pi i \mathsf{b} z)$ and $\hbar = (\mathsf{b} + \mathsf{b}^{-1})^{-2})$

$$J_{4_1,\hbar}(x) = \int_{\mathbb{R}-i\epsilon} \frac{\Phi_{\mathsf{b}}(x-y)}{\Phi_{\mathsf{b}}(y)} e^{2\pi i x(2y-x)}\, \mathrm{d}y$$

$$J_{5_2,\hbar}(x) = e^{-\mathrm{i}\pi/3} \int_{\mathbb{R}-i\epsilon} \frac{e^{\mathrm{i}\pi(z-x)(z+x)}}{\Phi_{\mathsf{b}}(z+x)\Phi_{\mathsf{b}}(z-x)\Phi_{\mathsf{b}}(z)}\, \mathrm{d}z.$$

Theorem (Andersen & Kashaev)

The AK-Volume Conjecture holds for these two knots.

The AK-TQFT

Let $z = e^{2\pi bx}$, $q = e^{2\pi i\mathsf{b}^2}$, $\tilde{q} = e^{-2\pi i\mathsf{b}^{-2}}$ and $\tilde{z} = e^{2\pi\mathsf{b}^{-1}x}$.
Introduce

$$g(z,q) = h(z,q)^{-1}H^{+}(z,q)$$
$$G(z,q) = h(z,q)H^{-}(z,q)$$

where

$$h(z,q) = \frac{(q,q)_\infty(-zq^{1/2},q)_\infty^2(-z^{-1}q^{1/2},q)_\infty^2}{(zq,q)_\infty}$$

$$H^{+}(z,q) = \sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}z^{2m+1}}{(q,q)_m(zq,q)_m}$$

$$H^{-}(z,q) = \sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}z^{m+1}}{(q,q)_m(z^{-1}q,q)_m(1-z)}$$

$$\boxed{J_{4_1,\hbar}(x) = \frac{q^{\frac{1}{12}}\tilde{q}^{-\frac{1}{12}}}{2\pi\mathsf{b}}g(z,q)G(\tilde{z},\tilde{q}) - \frac{q^{-\frac{1}{12}}\tilde{q}^{\frac{1}{12}}}{2\pi\mathsf{b}^{-1}}G(z,q)g(\tilde{z},\tilde{q})}$$

Results: (Joint with William Elbaek Mistegaard)

Resurgence Analysis of WRT-Invariants of SF-manifolds

For a Seifert homology sphere $M = \Sigma(p_1, ..., p_n)$ we prove:

1. A decomposition and idenfitication

$$\pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \cup \mathcal{M}(\mathrm{SU}(2))) \cong \pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{C}))) \cong \mathrm{CS}_{\mathbb{C}}(X).$$

2. $\hat{\mathrm{Z}}_0(q)$ is a resummation of the Ohtsuki series Z_0

$$\hat{\mathrm{Z}}_0(q) = \frac{1}{\sqrt{\tau}} \mathcal{L} \circ \mathcal{B}(\mathrm{Z}_0)(1/\tau), \qquad (q = \exp 2\pi i \tau).$$

3. A full asymptotic expansion of $\hat{\mathrm{Z}}_0(q)$ for τ near $1/(r-2)$ implying

$$Z^{(r)}(M) = \frac{1}{\sqrt{k}} \lim_{q \to e^{2\pi i/(r-2)}} \hat{\mathrm{Z}}_0(q).$$

4. An identification of the poles Ω of the Borel transform $\mathcal{B}(\mathrm{Z}_0)$

$$-2\pi i \, \mathrm{CS}^*_{\mathbb{C}}(X) + 2\pi i \mathbb{Z} = \Omega.$$

Our work builds on work of Lawrence and Rozansky on $Z^{(r)}(X)$ and is inspired by work of Gukov, Marino and Putrov.

Hyperbolic surgeries on the figure 8 knot

We now present in more detail the analysis leading to the above results for the hyperbolic 3-manifolds

$$M(\mathbf{4}_1(a/b)) = M_{a/b}$$

with surgery link giving by the figure eight knot with surgery data a/b.

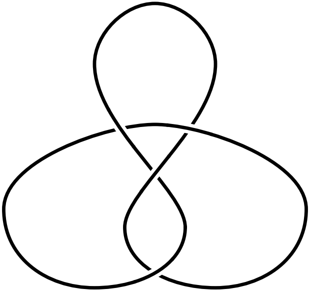


Figure: Figure eight knot $\mathbf{4}_1$

Contour integral formula for the WRT-Invariant

Choose $c, d \in \mathbb{Z}$ with $ad - cb = 1$. Define

$$\chi_{n,r}(x,y) = \sin\left(\frac{\pi}{b}(x - nd)\right) e^{2\pi i r\left(\frac{dn^2}{b} + \frac{a}{4b}x^2 - \frac{n}{b}x - xy\right)}$$
$$\times \frac{S_\gamma\left(-\pi + 2\pi(x-y)\right)}{S_\gamma\left(-\pi + 2\pi(x+y)\right)} \cot(\pi r x)\tan(\pi r y).$$

and let

$$\Omega_r^{a,b}(a,y) = \sum_{n \in \mathbb{Z}/|b|\mathbb{Z}} \chi_{n,r}(x,y)$$

Joint with Hansen we proved

$$Z^{(r)}(M_{a/b}) = c_1 r e^{\frac{2\pi i}{r} c_2} \int_{C_1(r)\times C_2(r)} \Omega_r^{a,b}(a,y)\ \mathrm{d}\, y\, \mathrm{d}\, x$$

where $c_1, c_2 \in \mathbb{C}^*$ and $C_1(r)$ is a simple closed contour which encircles the set $\{m/r : m = 1, 2, ..., r-1\}$, and $C_2(r)$ is a simple closed contour encircling $\{(m + 1/2)/r : m = 0, 1, ..., r-1\}$.

Semi-classical expansion of the quantum dilog

The semiclassical asymptotics of S_γ is given by Euler's dilogarithm: For $\mathrm{Re}(z) < \pi$ we have

$$S_\gamma(z) = \exp\left(\frac{r}{2\pi i}\,\mathrm{Li}_2\left(-e^{iz}\right) + I_\gamma(z)\right),$$
$$|I_\gamma(z)| = O(\gamma)$$

This leads us to the following phase functions indexed by $\alpha, \beta \in \{0, 1\}$ and $n \in \mathbb{Z}/|b|\mathbb{Z}$

$$\Phi_n^{\alpha,\beta}(x,y) = \frac{\mathrm{Li}_2(e^{2\pi i(x+y)}) - \mathrm{Li}_2(e^{2\pi i(x-y)})}{4\pi^2} - \frac{dn^2}{b} + (-\frac{a}{4b}x + \frac{n}{b} + y + \alpha + \beta)x + y(\alpha - \beta).$$

Identification of classical complex Chern-Simons values

Theorem (A. & Hansen)

There exists a surjection

$$(x, y) \mapsto [\rho_{x,y}]$$

from the set of critical points (x, y) *of the phase functions* $\Phi_n^{\alpha,\beta}$ *with* $x \notin \mathbb{Z}$ *onto* $\mathcal{M}^*(M_{a/b}, \mathrm{SL}(2, \mathbb{C}))$. *Moreover, we have that*

$$\Phi_n^{\alpha,\beta}(x, y) = \mathrm{S}_{\mathrm{CS}}([\rho_{x,y}]) \mod \mathbb{Z}.$$

In the variables

$$v = e^{\pi i x}, \quad w = e^{2\pi i y}$$

the critical point equations are

$$v^{-a} = \left(\frac{w - v^2}{1 - v^2 w}\right)^b, \quad (1 - v^2 w)(w - v^2) = v^2 w.$$

There are finitely many solutions $\{(v_\theta, w_\theta)\}$, $\theta \in \mathrm{CS}_{\mathbb{C}}(\mathrm{M}_{\mathrm{a/b}})$.

A resurgence Theorem

Based on generalizations of resurgence results for Laplace integrals due to Malgrange and Pham we prove

Theorem (A. & Mistegaard)

There exists power series

$$\{\mathrm{Z}_\theta(x)\}_{\theta\in\mathrm{CS}} \subset \mathbb{C}[[x^{-1}]]$$

giving a full asymptotic expansion

$$Z^{(r)}(M_{a/b}) \sim_{r\to\infty} r \sum_{\theta\in\mathrm{CS}} e^{2\pi i r\theta}\,\mathrm{Z}_\theta(r).$$

Each Borel transform $\mathcal{B}(\mathrm{Z}_\theta)$ is resurgent with singularities $\Omega(\theta)$ s.t.

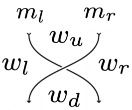
$$\bigcup_\theta (\Omega(\theta) - 2\pi i\theta) = -2\pi i\,\mathrm{CS}_{\mathbb{C}} + 2\pi i\mathbb{Z}.$$

Let $\Gamma_\theta^\hbar$ be the Lefschetz thimble at (v_θ, w_θ), then

$$\mathcal{L}\circ\mathcal{B}(Z_\theta)(\hbar) = 2\pi i c_1 \frac{e^{\hbar c_2}}{\hbar}\int_{\Gamma_s^\hbar} \Omega^{a,b}_{\frac{2\pi i}{\hbar}}(x,y)\ \mathrm{d}\,y\,\mathrm{d}\,x$$

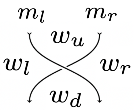
Yoon's generalization of the Cho-Murakami potential

$$\begin{aligned}
&\mathbb{W}_+(m_l, m_r, w_l, w_r, w_u, w_d)\\
&= \mathrm{Li}_2(e^{2\pi i(w_l - w_d - m_r)}) + \mathrm{Li}_2(e^{2\pi i(w_r - w_d - m_l)})\\
&- \mathrm{Li}_2(e^{2\pi i(w_u - w_r - m_r)}) - \mathrm{Li}_2(e^{2\pi i(w_u - w_l - m_l)}) - \frac{\pi^2}{6}\\
&+ \mathrm{Li}_2(e^{2\pi i(w_d - w_l + w_u - w_r)}) + 4\pi^2(w_l - w_d - m_r)(w_r - w_d - m_l),
\end{aligned}$$



and

$$\begin{aligned}
&\mathbb{W}_-(m_l, m_r, w_l, w_r, w_u, w_d)\\
&= \mathrm{Li}_2(e^{2\pi i(w_u - w_r + m_r)}) + \mathrm{Li}_2(e^{2\pi i(w_u - w_l + m_l)})\\
&- \mathrm{Li}_2(e^{2\pi i(w_l - w_d + m_r)}) - \mathrm{Li}_2(e^{2\pi i(w_r - w_d + m_l)}) + \frac{\pi^2}{6}\\
&- \mathrm{Li}_2(e^{2\pi i(w_d - w_l + w_u - w_r)}) + 4\pi^2(w_l - w_d + m_r)(w_r - w_d + m_l).
\end{aligned}$$



Yoon's generalization of the Cho-Murakami potential

• Let D_L be a link diagram, C the set of crossings and F the set of complementary regions of L.
• For each crossing c consider

$$\pi_c : \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F \to \mathbb{C}^2 \times \mathbb{C}^4.$$

We define the *pre-potential function* $\mathbb{W} \in \tilde{\mathcal{O}}(\mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F)$ by

$$\mathbb{W} = \sum_{c \in C} \mathbb{W}_{\epsilon(c)} \circ \pi_c.$$

The *Yoon-Cho-Murakami potential function* of the link L is

$$\tilde{\mathbb{W}}(y, w) = \mathbb{W}(y, w) - \sum_{f \in F} \frac{\partial \mathbb{W}}{\partial w_f}(y, w) w_f - \sum_{\ell \in \pi_0(L)} \frac{\partial \mathbb{W}}{\partial y_\ell}(y, w) y_\ell.$$

Yoon's generalization of the Cho-Murakami potential

Define

$$\begin{aligned}\mathcal{S}' \quad = \quad & \{(y,w) \in \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F \mid \frac{\partial \mathbb{W}}{\partial w_f} \in 4\pi^2\mathbb{Z}, \forall f \in F; \\ & \frac{\partial \mathbb{W}}{\partial y_\ell} \in 4\pi^2\mathbb{Z}, y_\ell \notin \mathbb{Z}, \forall \ell \in \pi_0(L)\}.\end{aligned}$$

and

$$\mathcal{M}' = \{\rho \in \hom(\pi_1(M), \mathsf{PSL}(2,\mathbb{C})) \mid \rho(\mu_\ell) \neq \pm \mathsf{Id}, \ell \in \pi_0(L)\}/\mathsf{PSL}(2,\mathbb{C}).$$

Theorem (Yoon)

There is a surjective holomorphic map $\rho : \mathcal{S}' \to \mathcal{M}'$, and we have

$$-i\,\mathrm{S}_{\mathrm{CS}_\mathbb{C}} \circ \rho = \tilde{\mathbb{W}} \text{ mod } \pi^2\mathbb{Z}.$$

Finite dimensional model

We have just derived the following finite dimensional model

$$Z^{(r)}(M_L) = \star \int_{(y,z)\in \tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L}} Z(D_L)(y,z) \bigwedge_{\gamma\in\pi_0(L)} dy_\gamma \bigwedge_{e\in E} dz_e$$

where

$$Z(D_L)(y,z) = J(D_L)(y,z) \prod_{\gamma\in\pi_0(L)} \frac{\sin(\frac{(2y_\gamma+1)\pi}{r})}{\sin(\frac{\pi}{r})} \cot(2\pi y_\gamma)$$

$$\tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L} \subset \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E$$

is a middle dimensional cycle in this finite dimensional affine space.

Recall that

$$\star = e^{\pi i \frac{3(r-2)}{r}\sigma(L)} \left(\sqrt{\frac{2}{r}} \sin(\frac{\pi}{r})\right)^{|\pi_0(L)|+1}$$

If we now apply the semiclassical asymptotics of S_γ:
For $\mathrm{Re}(z) < \pi$ we have ($\gamma = \frac{\pi}{r}$)

$$S_\gamma(z) = \exp\left(\frac{r}{2\pi i}\,\mathrm{Li}_2\left(-e^{iz}\right) + O(\gamma)\right)$$

Theorem (A. & Mistegaard)

For D_L there is a map

$$\Psi : \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F \to \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E$$

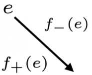
such that when one applies the above semiclassical approximation to all S_γ in $J(D_L)$ one obtains the Yoon-Cho-Murakami potential function $\tilde{\mathbb{W}}$ as the phase function, e.g.

$$J(D_L)(\Psi(y,w)) = e^{\frac{r}{2\pi i}\tilde{\mathbb{W}}(y,w) + O(\gamma)}$$

for all $(y,w) \in \Psi^{-1}(\tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L})$.

The map Ψ is the identity on the first factor and on the second factor it is given by

$$z_e = w_{f_-(e)} - w_{f_+(e)}, \quad \forall e \in E$$



Analytic conjecture

Conjecture

There exist finitely many

- *"abelian" contours* Γ_a*, and coefficients* $n_a \in \mathbb{Z}$*,* $a \in \mathrm{CS}_{\mathit{ab}}(\mathrm{M})$

and

- *Lefschetz thimbles* Γ_θ *and coefficients* $n_\theta \in \mathbb{Z}$*,* $\theta \in \mathrm{CS}_{\mathit{non\text{-}ab}}(\mathrm{M})$

such that

$$\begin{aligned} Z^{(r)}(M) &= \sum_{a \in \mathrm{CS}_{\mathit{ab}}(\mathrm{M})} n_a \int_{\Gamma_a} Z(D_L)(y,z) \bigwedge_{\gamma \in \pi_0(L)} dy_\gamma \bigwedge_{e \in E} dz_e \\ &+ \sum_{\theta \in \mathrm{CS}_{\mathit{non\text{-}ab}}(\mathrm{M})} n_\theta \int_{\Gamma_\theta} Z(D_L)(y,z) \bigwedge_{\gamma \in \pi_0(L)} dy_\gamma \bigwedge_{e \in E} dz_e \end{aligned}$$

(Work in progress on this conjecture joint with Maxim Kontsevich)

Resurgence for the WRT quantum invariants ASSUMING CONJECTURE 1

Theorem (A. & Kontsevich)

IF CONJECTURE 1 IS TRUE FOR SOME LINK L THEN:

1. *There exists power series*

$$\{\mathrm{Z}_\theta(x)\}_{\theta\in \mathrm{CS}} \subset \mathbb{C}[[x^{-1}]]$$

giving a full asymptotic expansion

$$\tau_r(M_L) \sim_{r\to\infty} r^d \sum_{\theta\in\mathrm{CS}} e^{2\pi i r\theta}\,\mathrm{Z}_\theta(r).$$

2. *Each Borel transform $\mathcal{B}(\mathrm{Z}_\theta)$ is resurgent with singularities $\Omega(\theta)$ s.t.*

$$\bigcup_\theta (\Omega(\theta) - 2\pi i\theta) = -2\pi i\,\mathrm{CS}_{\mathbb{C}}(M_L) + 2\pi i\mathbb{Z}.$$

3. *We have that*

$$\mathcal{L}\circ\mathcal{B}(\mathrm{Z}_\theta) \quad = \quad \int_{\Gamma_\theta} Z(D_L)(y,z) \bigwedge_{e\in E} dz_e \bigwedge_{\ell\in\pi_0(L)} dy_\ell$$

The Resurgence conjecture as a Rosetta stone

The Resurgence Conjecture:
(Garoufalidis; Gukov, Marino, Putrov; Gukov, Pei, Putrov, Vafa; A. & Mistegaard)

1. Borel Resummability: The series Z_θ are Borel resummable, $\mathcal{B}(Z_\theta)$ endless continuable.
2. Generalised Volume Conjecture: The set of poles $\Omega(\theta)$ of the meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies that

$$-2\pi i\,\mathrm{CS}_{\mathbb{C}}(M) = \cup_\theta \left(\Omega(\theta) - 2\pi i\theta\right) \mod \mathbb{Z}.$$

3. Wall Crossing: The meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies and are in part determined by a Wall crossing structure.
4. The $\hat{Z}$-Conjecture: The $\hat{\mathrm{Z}}_a$ GPPV invariants can be obtained from the functions $\mathcal{B}(Z_\theta)$ by a finite Laplace type transform.
5. AK-TQFT: $\theta_c =$ CS of conjugate representation: $\mathcal{L} \circ \mathcal{B}(Z_{\theta_c}) = Z^{AK}$.
6. The Radial Limit Conjecture: The WRT invariant $Z^{(r)}$ can be (re)-obtained from $\hat{\mathrm{Z}}_0$ as a limit $Z^{(r)} = \frac{1}{\sqrt{r}} \lim_{q \to e^{2\pi i/r}} \hat{\mathrm{Z}}_0(q)$.

Here $Z^{AK} =$ Andersen-Kashaev TQFT.

A Seifert fibered homology 3-sphere $X = \Sigma(p_1, ..., p_n)$

Let $p_1, ..., p_n \in \mathbb{N}, n \geq 3$ be pairwise coprime and consider the Seifert fibered homology 3-sphere with $n \geq 3$ exceptional fibers

$$X = \Sigma(p_1, ..., p_n).$$

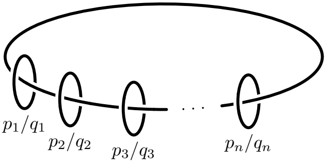


Figure: Surgery link for X.

Complex Chern-Simons theory on X

For $x \in \mathbb{Q}$ let $[x] = x \mod \mathbb{Z}$. Set $P = p_1 \cdots p_n$. We prove

Theorem (A. & Mistegaard)

The Chern-Simons action is injective on $\pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{C})))$ *and*

$$\mathrm{CS}^*_{\mathbb{C}}(X) = \left\{ \left[\frac{-m^2}{4P} \right] : \; m \in \mathbb{Z} \text{ is divisible by at most } n-3 \text{ of the } p_j\text{'s} \right\}.$$

The natural inclusion $\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \cup \mathcal{M}(\mathrm{SU}(2)) \to \mathcal{M}(\mathrm{SL}(2,\mathbb{C}))$ *induces an isomorphism on the level of* π_0

$$\pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \sqcup_{\mathcal{M}(U(1))} \mathcal{M}(\mathrm{SU}(2))) \cong \pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{C}))).$$

The Borel transform and complex Chern-Simons theory

Introduce the rational function

$$G(z) = \frac{\prod_{j=1}^{n}\left(z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}}\right)}{\left(z^P - z^{-P}\right)^{n-2}} = (-1)^n \sum_{m=1}^{\infty} \chi_m z^m \in \mathbb{Z}[[z]].$$

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Theorem (A. & Mistegaard)

1. *The Borel transform $\mathcal{B}(\mathrm{Z}_0)$ is the function*

$$\mathcal{B}(\mathrm{Z}_0)(\zeta) = \frac{4\sqrt{2\pi i P}}{\pi i \sqrt{\zeta}} G\left(\exp\left(\frac{c\sqrt{\zeta}}{P}\right)\right).$$

2. *Let Ω be the set of poles of $\mathcal{B}(\mathrm{Z}_0)$. Then*

$$\frac{i}{2\pi}\Omega = \mathrm{CS}^{*}_{\mathbb{C}}(X) + \mathbb{Z}.$$

The resurgence formula $\hat{Z}_0 = \mathcal{L} \circ \mathcal{B}(Z_0)$

Theorem (A. & Mistegaard)

Set $q = \exp(2\pi i \tau), \tau \in \mathbb{H}$. *We have*

$$\hat{Z}_0(q) = \frac{1}{\sqrt{\tau}} \int_\Gamma \exp\left(-\frac{\xi}{\tau}\right) \mathcal{B}(Z_0)(\xi) \, \mathrm{d}\,\xi = \sum_{m=1}^{\infty} \chi_m q^{\frac{m^2}{4P}}.$$

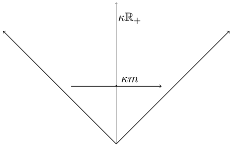

Figure: The integration contour Γ.

The asymptotic expansion of $\hat{\mathrm{Z}}_0$

For small $t > 0$ set $q_{k,t} = \exp\left(\frac{2\pi i}{k - i\frac{2Pt}{\pi}}\right) \in \mathfrak{h}$.

Theorem (A. & Mistegaard)

For each $\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)$ $\exists$ a polynomial in k of degree at most $n-3$ with coefficients in formal power series without constant terms

$$\check{\mathrm{Z}}_\theta(k,t) \in t \cdot \mathbb{Q}[\pi i, k][[t]]$$

giving an asymptotic expansion for small t and fixed even k

$$\hat{\mathrm{Z}}_0(X; q_{k,t}) \underset{t \to 0}{\sim} Z^{(r)}(X) + \sum_{\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)} e^{2\pi i k \theta} \check{\mathrm{Z}}_\theta(k,t).$$

In particular the radial limit conjecture holds for X.

A resurgence formula for $Z^{(r)}$

Our proof of the asymptotic expansion is based on the following resurgence lemma where Ω is the set of poles of $\mathcal{B}(\mathrm{Z}_0)$

Lemma (A. & Mistegaard)

$$\hat{\mathrm{Z}}_0(q) = \mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\mathrm{Z}_0)\left(\frac{1}{\tau}\right) + \sum_{\omega \in \Omega} \mathit{Res}_{y=\omega}(e^{-y/\tau}\mathcal{B}(\mathrm{Z}_0)(y)).$$

As a corollary of this and the radial limit theorem, we obtain the following resurgence formula for the WRT quantum invariant

Corollary (A. & Mistegaard)

$$Z^{(r)} = \mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\mathrm{Z}_0)(k) + \sum_{\omega \in \Omega} \mathit{Res}_{y=\omega}(e^{-ky}\mathcal{B}(\mathrm{Z}_0)(y)).$$