

A VARIANT OF INSTANTON FLOER HOMOLOGY

New Structures in Low Dimensional Topology Annual Meeting

Flatiron Institute

March 29, 2024



UNDER CONSTRUCTION

Instanton Floer Homology

Choose Y a 3-manifold (maybe containing)
a knot or link K , if so construct orbifold Y_K .

Choose a principal $SU(2)$ or $SO(3)$
bundle over Y , $P \rightarrow Y$ (orbi bundle if nec).

A_P = Space of orbifold connections in P . An affine space.

$\tilde{g} = \text{Aut}_{\text{orb}}(P)$ acts on A_P . ($\tilde{g} \simeq \text{Map}(Y, \text{SU}(2))$)

$$T_A A_P = \Omega^1(Y, \text{ad } P).$$

A_P = Space of orbifold connections in P . An affine space.

$\tilde{g} = \text{Aut}_{\text{orb}}(P)$ acts on A_P . ($\tilde{g} \simeq \text{Map}(Y, \text{SU}(2))$)

$$T_A A_P = \Omega^1(Y, \text{ad } P). \quad \Theta_A: T_A A_P \rightarrow \mathbb{R}$$

$$\Theta_A(b) = \int_Y \text{tr}(F_A b)$$

A_P = Space of orbifold connections in P . An affine space.

$\tilde{g} = \text{Aut}_{\text{orb}}(P)$ acts on A_P . ($\tilde{g} \cong \text{Map}(Y, \text{SU}(2))$)

$$T_A A_P = \Omega^1(Y, \text{ad } P). \quad \Theta_A: T_A A_P \rightarrow \mathbb{R}$$

$$\Theta_A(b) = \int_Y \text{tr}(F_A b) \quad \left\} \tilde{g} \text{ invariant}$$

$d\Theta_A \equiv 0 \implies$ CS function (multivalued)

A_P = Space of orbifold connections in P . An affine space.

$\tilde{g} = \text{Aut}_{\text{orb}}(P)$ acts on A_P . ($\tilde{g} \cong \text{Map}(Y, \text{SU}(2))$)

$$T_A A_P = \Omega^1(Y, \text{ad} P). \quad \Theta_A: T_A A_P \rightarrow \mathbb{R}$$

$$\Theta_A(b) = \int_Y \text{tr}(F_A b) \quad \left. \begin{array}{l} \nearrow \\ \searrow \end{array} \right\} \hat{g} \text{ invariant}$$

$d\Theta_A \equiv 0 \implies$ CS function (multivalued)

$$\nabla_A CS = *F_A. \iff (\text{Gued flow} \iff \text{ACD \& ME}).$$

$$\text{Crit}(CS)/g \iff R(Y)_g = \{ \rho: \pi_1(Y) \rightarrow \text{SU}(2) \mid \tilde{Y} \times_P \text{SU}(2) \cong P^3 / S^1 = \text{sub}_{2/2} \}$$

A_P = Space of orbifold connections in P . An affine space.

$\tilde{g} = \text{Aut}_{\text{orb}}(P)$ acts on A_P . ($\tilde{g} \cong \text{Map}(Y, \text{SU}(2))$)

$$\Theta_A(b) = \int_Y \text{tr}(F_A \wedge b) = \int_Y \text{tr}(*F_A \wedge *b)$$

$$\nabla_A CS = *F_A.$$

$$\nabla_A CS = 0 \Leftrightarrow F_A = 0$$

$$R^0(Y)_P = \{ \rho: \pi_1(Y) \rightarrow \text{SU}(2) \mid \tilde{Y} \times_P \text{SU}(2) \cong P \}$$

$$\text{Crit}(CS)/\tilde{g} = R^0(Y)_P / \text{conj} \doteq R(Y)_P$$

Note

$$\text{SO}(3) = \text{SU}(2) / \mathbb{Z}(\text{SU}(2))$$

acts here

Would like to do Morse Homology for CS

3 issues.

1. CS is multivalued (cf. Novikov-Morse Thm)
2. General issue with compactness & gluing
3. $\mathcal{G}$ does not act freely !

Would like to do Morse Homology for CS

3 issues.

1. CS is multivalued (cf. Novikov-Morse Thm)
CS is "positively monotone". No need for Novikov rays ✓
2. General issue with compactness & gluing
3. $\mathcal{G}$ does not act freely !

Would like to do Morse Homology for CS

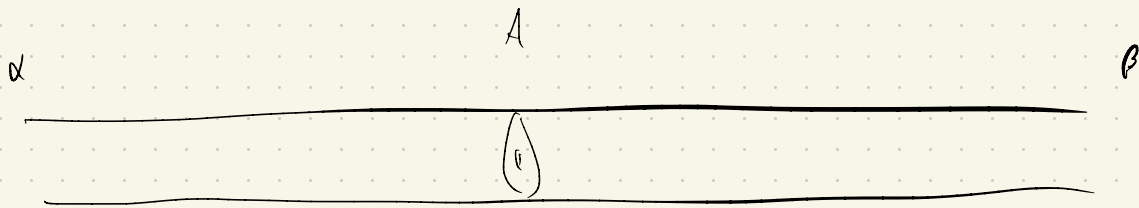
3 issues.

1. CS is multivalued (cf. Novikov-Morse Thm)
CS is "positively monotone". No need for Novikov rings ✓
2. General issue with compactness & gluing
Uhlenbeck, Taubes, Donaldson etc ✓
3. $\mathcal{G}$ does not act freely !

Would like to do Morse Homology for CS

3 issues.

1. CS is multivalued (cf. Novikov-Morse Thm)
CS is "positively monotone". No need for Novikov rays ✓
2. General issue with compactness & gluing
Uhlenbeck, Taubes, Donaldson etc ✓
3. $\mathcal{G}$ does not act freely ! Turns out to be delicate

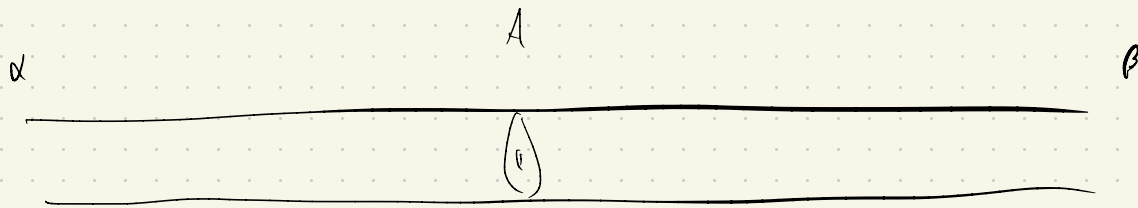


$$\mathbb{R} \times Y$$

differentiable
 $\Leftarrow$

$$M_Y(\alpha, \beta)$$

$$= \{ [A] \mid F_A = - * F_A, \lim_{t \rightarrow +\infty} [A] = \beta, \lim_{t \rightarrow -\infty} [A] = \alpha \} \quad \mathbb{R}$$

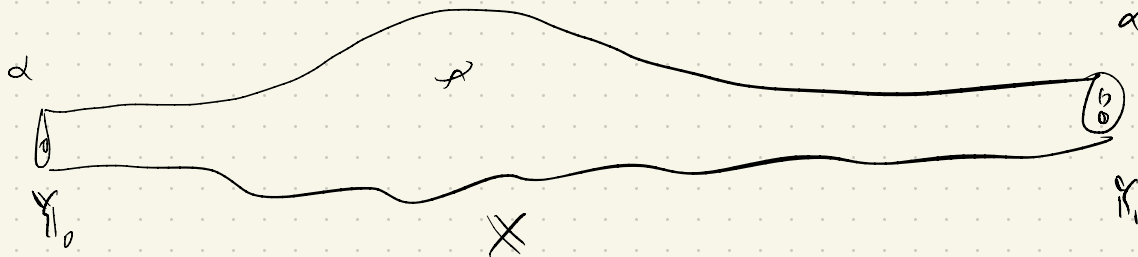


$$\mathbb{R} \times Y$$

differentiable $\Leftarrow$

$$M_Y(\alpha, \beta) = \{ [A] \mid F_A = - * F_A, \lim_{t \rightarrow +\infty} \tau(A) = \beta, \lim_{t \rightarrow -\infty} \tau(A) = \alpha \} \quad \mathbb{R}$$

maps $\Leftarrow$



$$M_X(\alpha, \beta) = \{ [A] \mid F_A = - * F_A, \lim_{t \rightarrow +\infty} \tau(A) = \beta, \lim_{t \rightarrow -\infty} \tau(A) = \alpha \}$$

3. g does not act freely ! Turns out to be delicate

- Floer : $\rightarrow$ admissible bundles - No flat reducible connections

$\rightarrow$ Homology spheres. Ignore the trivial rep! $OG I(Y)$

- Donaldson - Braam - Austin - Eismen-Miller - Daemi - Scaduto Various constructions analogous to motivated by Equivariant Homology
- A cheat. (Kronheimer-M) Make all Y admissible by summing with an "Atom".

3. G does not act freely ! Turns out to be delicate

• Floer : $\rightarrow$ admissible bundles - No flat reduced connections

$\rightarrow$ homology spheres. Ignore the trivial rep! $OG I(Y)$

3. g does not act freely ! Turns out to be delicate

• Floer : $\rightarrow$ admissible bundles - No flat reducible connections

$\rightarrow$ homology spheres. Ignore the trivial rep! $OG I(Y)$

• Donaldson - Braam - Austin - Eismen-Miller - Daemi - Scaduto
Various constructions analogous to motivated by Equivariant Homology

Brief notational aside. Introduce black board bold for

$$\mathbb{Y} = (\mathbb{Y}, K, P)$$

3 manifold $\nearrow$ Knot $\nwarrow$ orbifold bundle over $\mathbb{Y}_K$.

Brief notational aside. Introduce black board bold for

$$\mathbb{Y} = (Y, K, P)$$

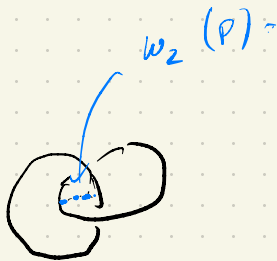
3 manifold $\nearrow$ Knot $\nearrow$ orbifold bundle over Y_K .

Defⁿ: A is an **atom** if $\exists!$ irreducible orbifold

$$\pi_1^{\text{orb}}(A) \rightarrow \text{SU}(2)$$

My favorite

$$A = (S^3, \text{Hept link})$$



Key property of atoms.

based rep (not mod conj)

KP

$$R(Y * A) = R^0(Y)$$

$$\begin{array}{ccccc} \pi_1(Y * A) & = & \pi_1(Y) * \pi_1(A) \\ \downarrow \rho_1 & & \downarrow \rho_1 & & \downarrow \rho_2 \\ \text{SU}(2) & \cong & \text{SU}(2) & & \text{SU}(2) \end{array}$$

but ρ_2 is unique
up to conjugacy.

$$\Rightarrow R^0(Y * A) = R^0(Y) \times \text{SU}(2) / \pm I$$

$$R(Y * A) = R^0(Y)$$

mod out by
conjugacy

Note: $\text{Stab}(p) \subset \text{SU}(2)$ can be $\pm I$, $U(1)$, $\text{SU}(2)$

$\Rightarrow$ Orbits of p are $\underbrace{\text{SU}(2)/\pm I = \mathbb{RP}^3}_{\text{Im}(p) \not\subset U(1)}$, $\text{SU}(2)/U(1) = S^2$, $\text{SU}(2)/\text{SU}(2) = \text{pt.}$
 $\text{Im}(p) \subset U(1)$ $\text{Im}(p) \subset U(1)$ $\text{Im}(p) \subset \pm I$
 $\not\subset \pm I$

$$\mathbb{Z}_2\text{-red} \Rightarrow \text{Hom}(\pi, (Y), \mathbb{Z}_2) = H^1(Y, \mathbb{Z}_2)$$

$$U_1\text{-red} \Rightarrow \text{Hom}(\pi, (Y), U_1) = \text{tor } H^2(Y, \mathbb{Z})$$

Thus if $R(Y)$ consists of isolated points
 $b^1(Y; \mathbb{Z}_2)$

$$R^0(Y) =$$

$$t^2(Y) = (|\text{tor } H^2(Y, \mathbb{Z})| - b^1(Y, \mathbb{Z}_2)) \text{ copies of } S^2$$

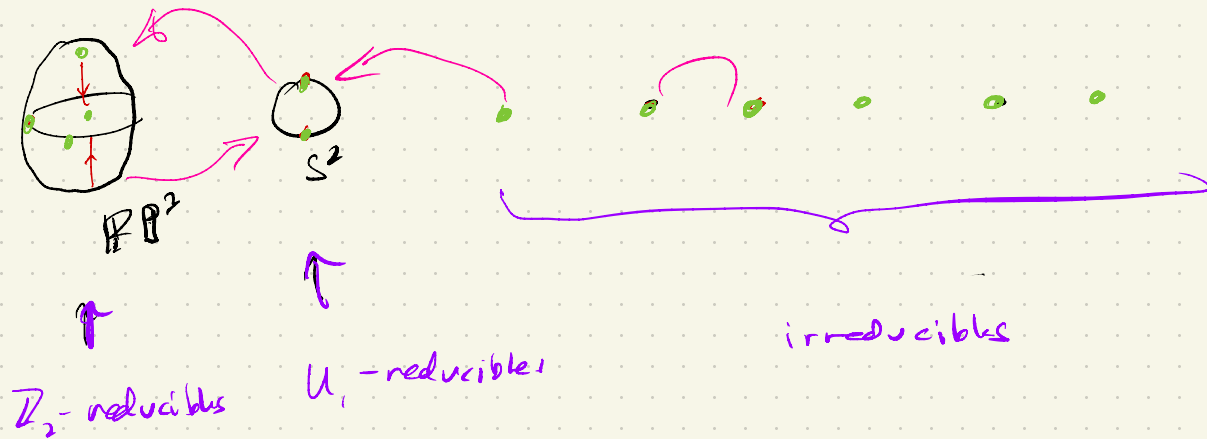
for each irrep

points

S^2

$\mathbb{RP}^3$

- ← critical points after a perturbation.



Differentials decompose into " ε " energy along
 $\mathbb{R}P^2$'s and S^2
 and

Energy bounded below between components

$$Y = (Y, k, p)$$

$$Y^\# = Y \# A \quad (\text{needs a choice of ball } B \hookrightarrow Y) \\ (\text{a based 3-manifold})$$

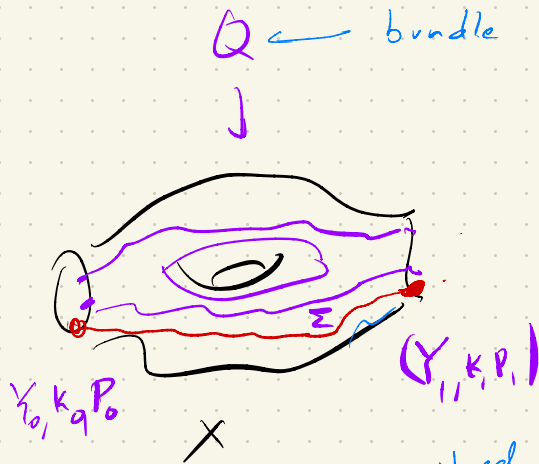
$I(Y^\#)$ is always defined

$$I^*(Y) \doteq I(Y^\#) \quad (\text{framed instanton homology})$$

$$X = (X, \Sigma, Q)$$

$$X^\# = X \overset{\Sigma}{\text{glued to}} I \times A$$

$$I^\#(X) = I(X^\#), \quad I^\#(Y_0) \rightarrow I^\#(Y_1)$$



Need a
path of balls
"a rigged
4 manifold"



Uhlenbeck, Taubes, Donaldson, Floer ...

But this is a cheat:

Doesn't recover Donaldson invts of closed 4-mflds.

Behaves like $\widetilde{HF}$ is Heegaard Floer World

Beside we should have $HF^\pm$ or $\widetilde{HF}$

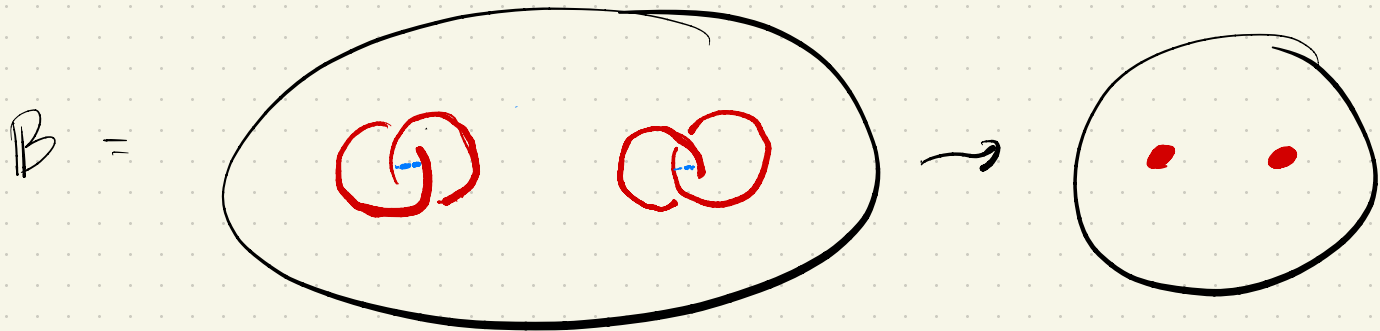
(cf Eismann-Miller's work gives such an extension,
(Daemi))

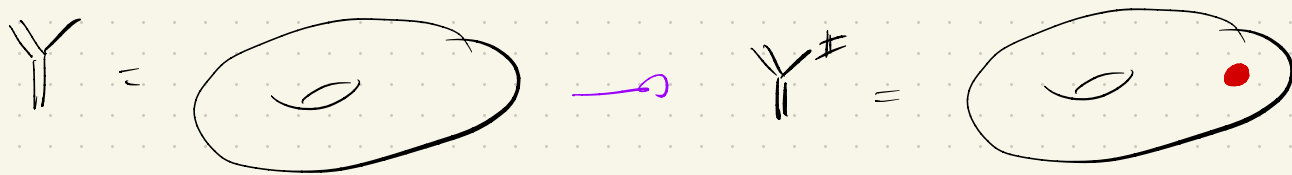
There is a natural upgrade of this structure.

Consider $B = A \# A$

its representation variety is by $\boxed{KP}$

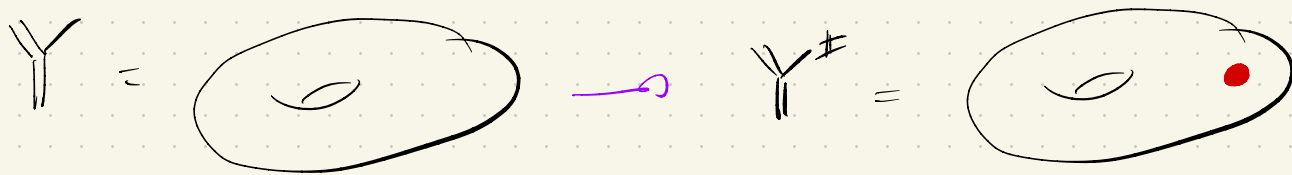
$$R(B) = \mathrm{SU}(2)/\mathbb{Z}(\mathrm{SU}(2)) = \mathrm{SO}(3).$$



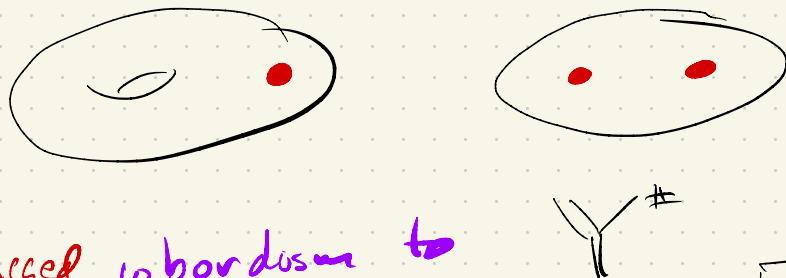


Now consider $Y^\# \sqcup \mathbb{B}$

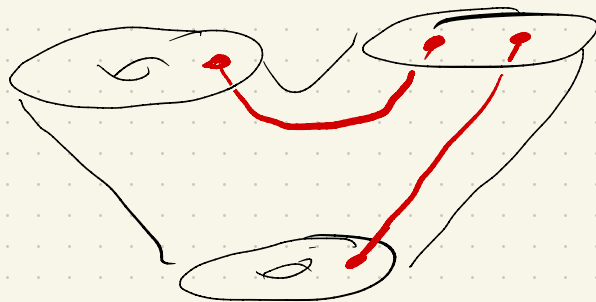




Now consider $Y^\# \sqcup B$

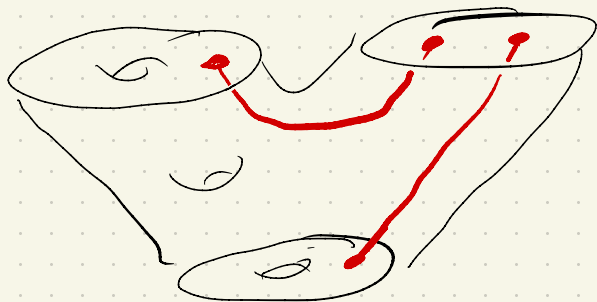


Natural riggle cobordism to



$= X'$

X' is $Y^\# \times [0,1]$
minus a 4-ball
hitting $B^3 \times \{0,1\}$



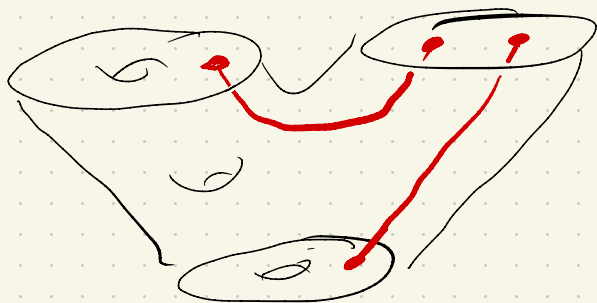
induces

$$CI(Y^\#) \otimes CI(\beta)$$

$$\downarrow m_2$$

$$CI(Y^\#).$$

$$m_2(\alpha \otimes \beta) = \sum_{\beta \in \text{Cnt}(Y^\#)} \# M_{X^{(1)}}^0(\alpha \times \beta, \beta) \beta \leftarrow \begin{array}{l} 0 \text{ dim'l} \\ \text{moduli spaces} \\ \text{on } X' \end{array}$$



induces

$$CI(Y^\#) \otimes CI(B)$$

$$\downarrow m_2$$

$$CI(Y^\#).$$

$$m_2(\alpha \otimes \rho) = \sum_{\beta \in \text{Cnt}(Y^\#)} \# M_{X^{(1)}}^0(\alpha \times \rho, \beta) \beta \leftarrow \begin{array}{l} \text{0 dim'l} \\ \text{moduli spaces} \\ \text{on } X' \end{array}$$

Note m_2 also has a small energy contribution.

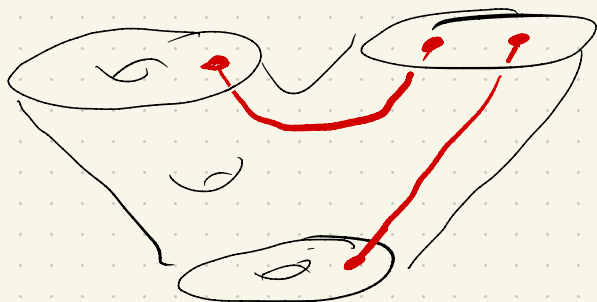
$$\begin{array}{lcl} R(X') & \simeq & R(Y^\#) \times R(B) \\ & \simeq & R^0(Y) \times \text{SO}(3) \end{array}$$

not projection

$R(Y^\#)$

mult

$R(Y^\#) \times R(B)$



induces

$$CI(Y^*) \otimes CI(B)$$

$$\downarrow m_2$$

$$CI(Y^*)$$

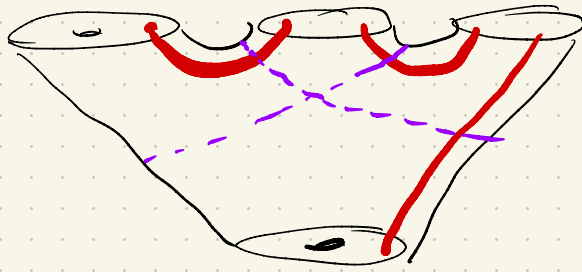
$$m_2(\alpha \otimes \beta) = \sum_{\beta \in \text{Cat}(Y^*)} \# M_{X^{(1)}}^0(\alpha \times \beta, \beta) \beta \leftarrow \begin{array}{l} \text{0 dim'l} \\ \text{moduli spaces} \\ \text{on } X' \end{array}$$

boundary of 1 dim'l moduli space on $X^{(1)}$ gives

$$m_2(\partial_Y \alpha \otimes \beta) + m_2(\alpha \otimes \partial \beta) \neq \partial m_2(\alpha \otimes \beta) = 0$$

i.e. m_2 is a chain map.

$X_2 =$



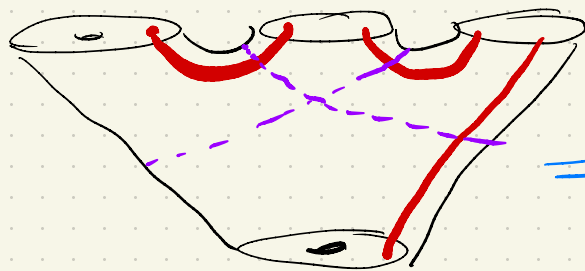
$\Rightarrow$ 1 parameter family of metrics

Two uses: ① Define chain map

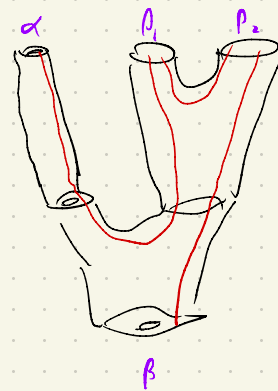
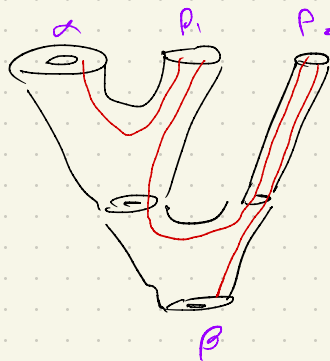
$$m_3: CI(Y^\#) \otimes CI(B) \otimes CI(B) \rightarrow CI(Y^\#)$$

using 0-dim'l moduli spaces on X_2 with 1 parameter

$$m_3(\alpha \otimes p_1 \otimes p_2) = \sum_{\beta} \# (M_{X_1, G}^{\circ}(\alpha \times p_1 \times p_2, \beta)) \beta$$



ends of
 $\Rightarrow$ 1-parameter
 families



$$m_2(m_2(\alpha \otimes p_1) \otimes p_2) - m_2(\alpha \otimes m_2(p_1 \otimes p_2)) =$$

$$m_3(\partial(\alpha \otimes p_1 \otimes p_2)) - \partial m_3(\alpha \otimes p_1 \otimes p_2)$$

A^∞ relations. CF

Stasheff - May - May - Greenlees Thibaut Mazuir

Gantra

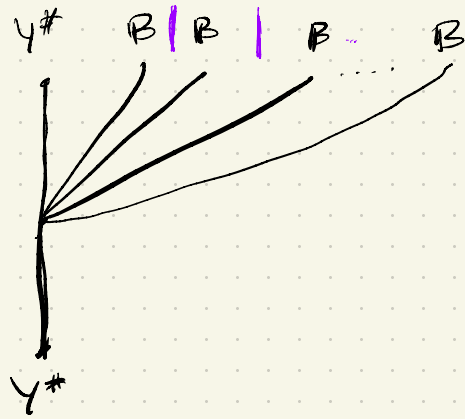
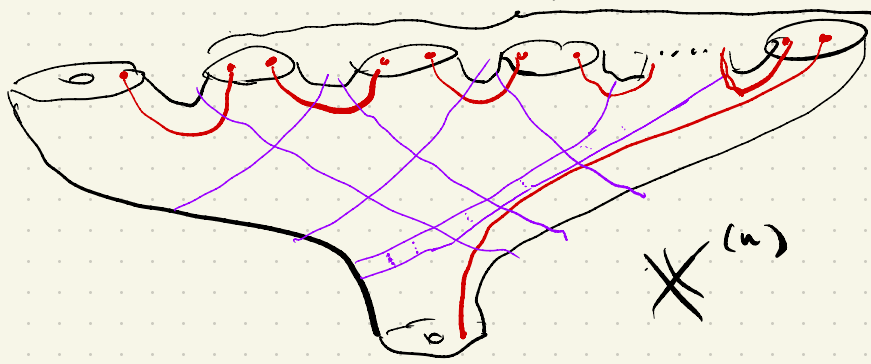
Gantra - Pardon-Skende

Francesco L n.

Medricks - Lipshitz -

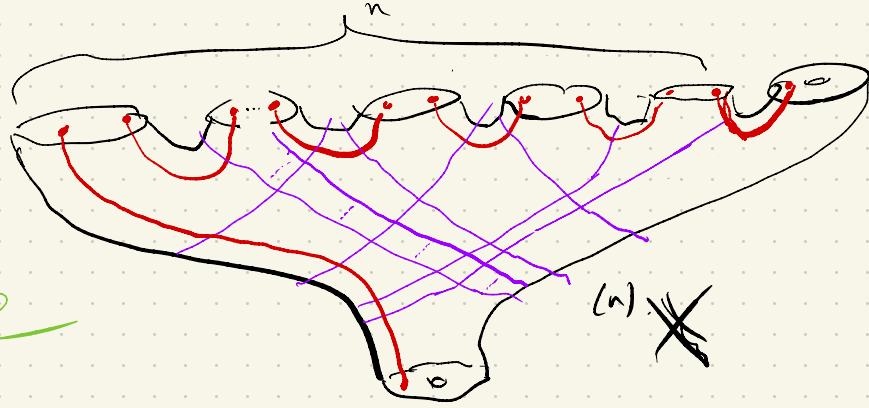
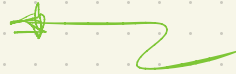
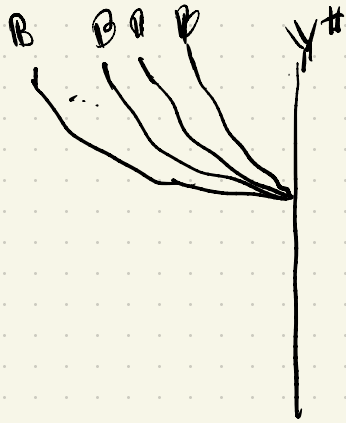
More generally

(1)

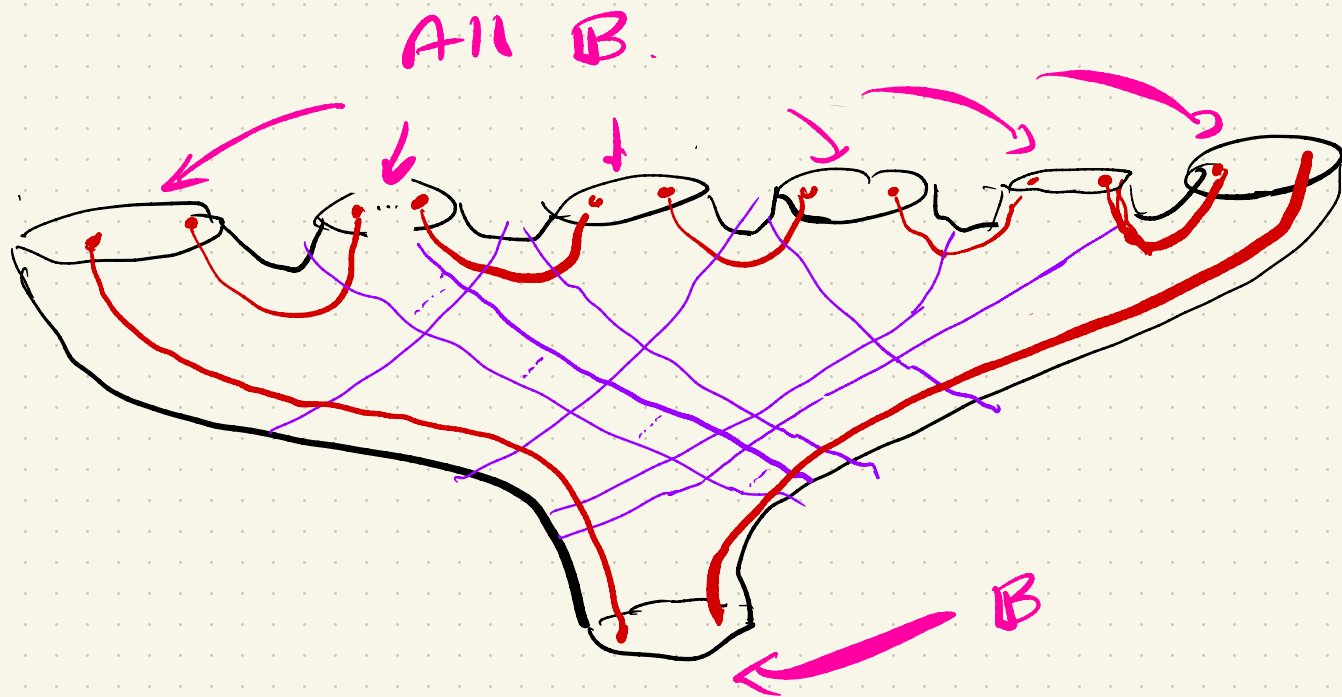


Has $(n-1)$ dim'd family of metrics parameterized
by the interior of the Stasheff associahedron $\bar{K}_{n-1}$
Natural compactification of the family is $\underline{K}_{n-1}$

Similarly there is a notion of **left** "action"



Headed to A_∞ models over $\mathcal{H}(B)$



gives

$$CI(B) \xrightarrow{d_1} CI(B) \otimes CI(B) \xrightarrow{d_2} CI(B)^{\otimes 3}$$

$\xleftarrow{d_2 = n_2}$
 $\xleftarrow{n_2 \otimes 1 - 1 \otimes n_2}$
 $\xleftarrow{etc}$

$CI(Y^\#)$ now
an A^∞ -module
over $CI(B)$.

$$CI(Y^\#) \otimes CI(B)^{\otimes n}$$

$$\vdots$$

$$CI(Y^\#) \otimes CI(B) \otimes CI(B) \rightarrow$$

$$\downarrow m_2 \otimes Id - Id \otimes m_2$$

$$CI(Y^\#) \otimes CI(B^\#) \rightarrow$$

$$\downarrow m_2$$

$$CI(Y^\#) \rightarrow$$

m_3 etc

$$\tilde{C}(Y) = \oplus CI(Y^\#) \otimes CI(B)^{\otimes n}$$

$$d = d_0 + d_1 + \dots + d_k + \dots +$$

$$d_0: CI(Y^\#) \otimes CI(B)^{\otimes n} \rightarrow$$

ordinary Floer diff

$$d_k: CI(Y^\#) \otimes CI(B)^{\otimes n}$$

$$\rightarrow CI(Y) \otimes CI(B)^{\otimes (n-k)}$$

uses sum of m_{k+1} 's.

$$d_k(x \otimes p_1 \otimes \dots \otimes p_n) = m_{k+1}(x \otimes \dots \otimes p_k) - x \otimes m_{k+1}(p_1 \otimes \dots \otimes p_{k+1}) \pm \dots$$

Get $(\tilde{\mathcal{C}}(Y), d)$ but Observation $H^*(\tilde{\mathcal{C}}(Y), d) = 0.$

There is a chain homotopy to zero of Identity.

$\Rightarrow$ Not complete ~~useless~~ $\begin{matrix} 0 \\ \downarrow \end{matrix}$

$$0 \rightarrow (CI(Y^*), d) \hookrightarrow (\tilde{\mathcal{C}}(Y), d) \rightarrow (\tilde{\mathcal{C}}(Y)/CI(Y^*), d) \rightarrow 0$$

$\underbrace{\hspace{15em}}_{\approx}$

Better Idea

$$CI(Y^\#) \otimes CI(B)^{\otimes a-1} \otimes CI(A)$$

Via $CI(A)$
as a left $CI(B)$
module.

$$CI(Y^\#) \otimes CI(B) \otimes CI(A) \rightarrow$$

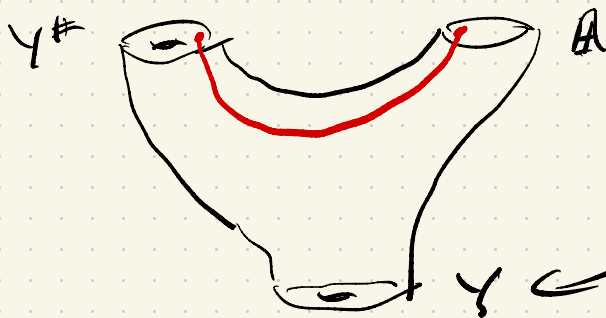
$$\downarrow m_2 \otimes Id - Id \otimes m_2$$

$$CI(Y^\#) \otimes CI(A) \rightarrow$$

$$\downarrow m_2$$

$$CI(Y) \rightarrow$$

← not necessarily defined!



not necessarily
admissible

Better Idea

$$CI(Y^\#) \otimes CI(B)^{\otimes n-1} \otimes CI(A)$$

Via $CI(A)$
as a left $CI(B)$
module.

$$CI(Y^\#) \otimes CI(B) \otimes CI(A) \rightarrow$$

$$\downarrow m_2 \otimes Id - Id \otimes m_2$$

$$CI(Y^\#) \otimes CI(A) \rightarrow$$

$$\downarrow m_2$$

$$CI(Y) \rightarrow$$

← not necessarily defined!

Construct similar

$d^v = d_0 + d_1 + \dots$ but acting on

$$CI^v(Y) = \bigoplus_{n \geq 0} CI(Y^\#) \otimes CI^\pm(B)^{\otimes n} \otimes CI^\pm(A)$$

$$HI^v(Y) = H_* (C^v(Y), d^v)$$

Compare this with Eilenberg-Moore complex computing

$$H_*(X_G) = H_*(X \times EG/G).$$

$$EG_0 \subset EG_1 \subset EG_2 \subset \dots \quad EG$$

$$X_G^0 = X_G^{(1)} \subset \dots$$

$$X_G^{(n)} / X_G^{(n-1)} \cong \Sigma^n (X \times G^n)$$

$\Rightarrow$ $\exists$ a spectral sequence: differentials are map induced by $m: H_*$

$$E_2 \cong H_*(X) \oplus H_{*-1}(X \times G) \oplus \dots \oplus H_{*-n}(X \times G^{n-1}) \oplus \dots$$

$$C^*(Y) = \bigoplus_{n \geq 0} C(I(Y^\sharp) \otimes C I^\sharp(B)^{\otimes n} \otimes C I^\sharp(A)) \quad \text{Taking } d_0$$

$$I_*(Y^\sharp) \oplus I_{*-1}(Y \perp B) \oplus I_{*-2}(Y \perp B \perp B) \oplus \dots$$

NB internal grading.

Examp $HI^v(S^3)$

Facts: ① $\exists$ a pentagon so an instance of

$CI(A)$

$\mathbb{Z}$

$CI(B)$

$\mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$

(usual Morse)

NB: Potentially $\uparrow$ $\uparrow \text{deg } 1 \text{ mod } (4)$
but it is zero.

② m_2 a Morse version of Pontryagin project.

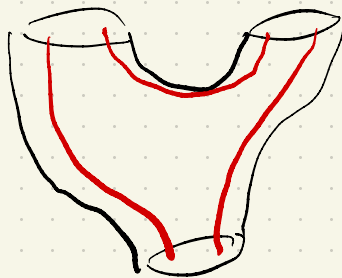
③ m_k are Morse versions A_∞ maps for $SO(2)$
and products

$CI^v(S^3)$ is a Morse version

$$\Rightarrow HI^v(S^3) \cong H_*(BSO(3); \mathbb{Z})$$

Re m_2

consider



Flat moduli space is $SO(2) \times SO(3)$ 6 dim'd
 $\swarrow \sim$
 $SO(2) \times SO(3) \xrightarrow{\text{mult}} SO(2)$

non flat moduli space is $6+4=10$ dim'd $> 9 = \dim(SO(3)^3)$

$\Rightarrow$ All pull back push forward map are zero.
(all moduli spaces are positive dim'd after perturbation)
mx argument is similar dimension counting.

More generally in high internal degree

$I^V(Y)$ is

isomorphic to
high degree part of.

$$\oplus H_{*-c(p)}(BSO(3), \mathbb{Z}) \oplus H_{*-c(p)}(BU(1), \mathbb{Z})$$

Seems like we have a functor

(based 3-manifold, rigged 4-manifolds)



$\mathbb{Z}$ -modules to ? Modules over $H^*(BSO(3); \mathbb{Z})$

Questions etc

① Write up and complete this to HI_{∞}^U .

① Prove this computer Donaldson, on an Eismann-Killing version (or not)

Extract Donaldson invariants

② Higher rank. Main structure generalizes but less clear that $HI_{SU(n)}^U(S^2) = H_n(BPU(n))$