Formulation and simulation of a linear dynamics leading to a loss of regularity

Generation of Fractional Fields and application to Fluid Turbulence

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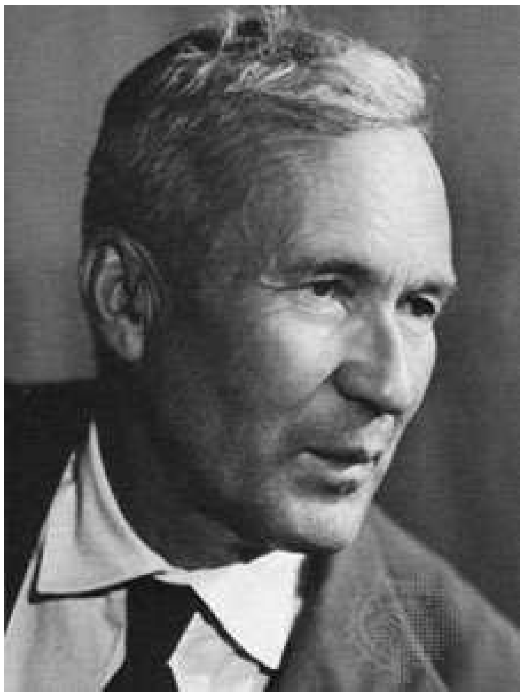
The Navier-Stokes equations

In three-dimensional space, consider the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, where $\boldsymbol{u}^\nu = (u_1, u_2, u_3)$, $\boldsymbol{x} \in \mathbb{R}^3$ and say $t > 0$. Given a (large-scale smooth, divergence-free forcing) $\boldsymbol{f}$, it is solution of

$$\frac{\partial \boldsymbol{u}^\nu}{\partial t} + (\boldsymbol{u}^\nu \cdot \boldsymbol{\nabla})\boldsymbol{u}^\nu = -\frac{1}{\rho}\boldsymbol{\nabla} p^\nu + \nu \Delta \boldsymbol{u}^\nu + \boldsymbol{f} \text{ and } \boldsymbol{\nabla} \cdot \boldsymbol{u}^\nu = 0,$$

where p is the pressure field, and ν the kinematic viscosity.



Kolmogorov 1903-1987

"I became interested in turbulent liquid and gas flows at the end of the thirties. From the very beginning it was clear that the theory of random functions of many variables (random fields), whose development only started at that time, must be the underlying mathematical technique. Moreover, I soon understood that there was little hope of developing a pure, closed theory, and because of the absence of such a theory the investigation must be based on hypotheses obtained by processing experimental data."

The Navier-Stokes equations

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Consider then a forcing of the form, zero-average, Gaussian,

$$\mathbb{E}\left[f_i(t_1,x_1)f_j(t_2,x_2)\right] = F(t_1-t_2)\int_{\mathbb{R}^3} e^{2i\pi k\cdot(x_1-x_2)}\,|\widehat{\varphi}(|k|)|^2\,\overbrace{P_{ij}(k)}^{\text{Leray}}\,dk$$

Take for instance $F(t)=\delta_0(t)$ to simplify the discussion, but $F(t)=e^{-|t|/T}$ would be more realistic, and more importantly take $\widehat{\varphi}(|k|) = \mathbb{1}_{1/L\le|k|\le 2/L}$, L being known as the *integral* length scale.

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where p is the pressure field, and ν the kinematic viscosity.

Then, it is <u>observed</u> that the system reaches a statistically stationary regime, in which

$$\lim_{\nu\to 0}\lim_{t\to\infty} \mathbb{E}|\boldsymbol{u}^\nu|^2 = \sigma^2 < \infty$$

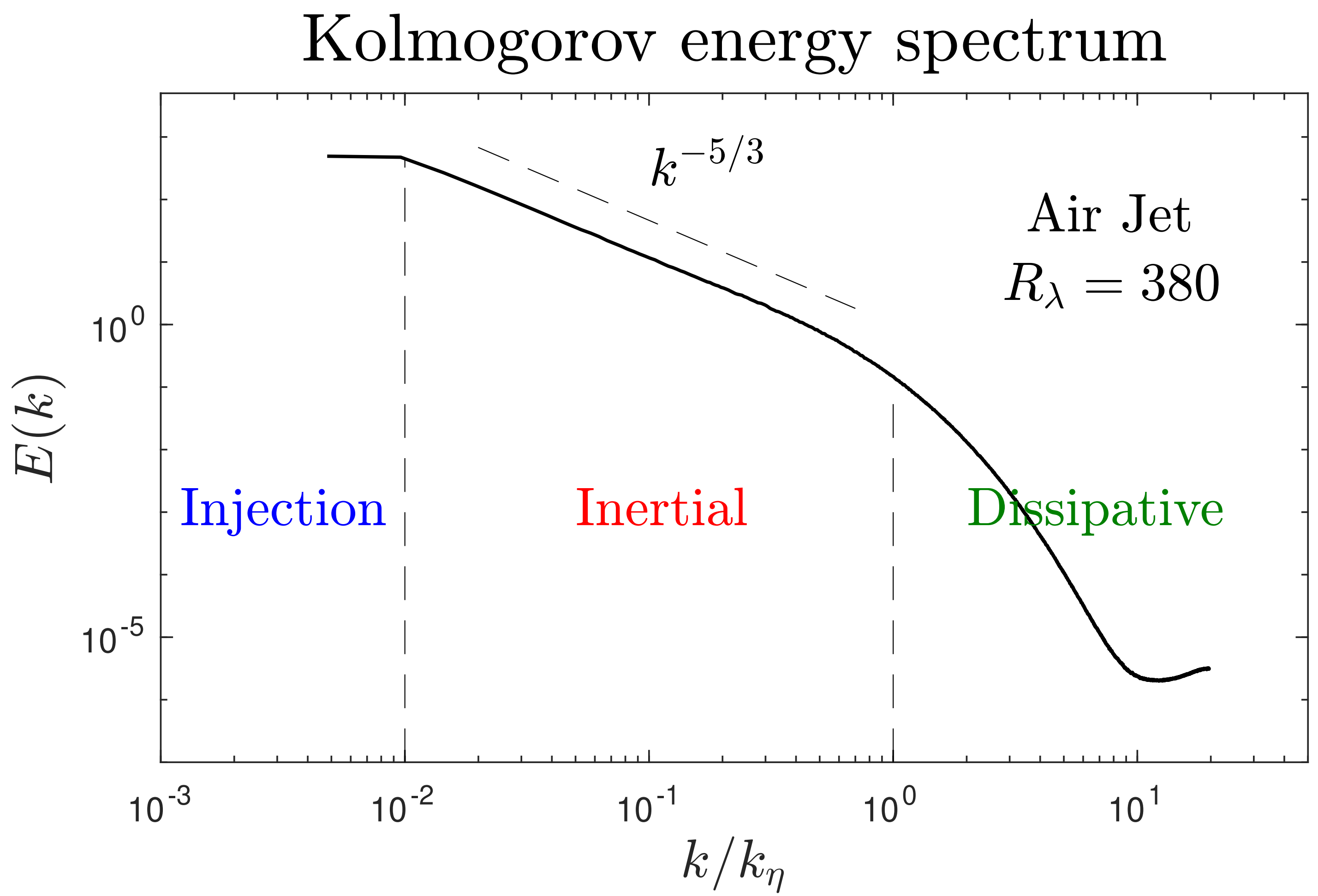
In other words, velocity variance is uniformly bounded with respect to viscosity.

$\Rightarrow$ To do so, the fluid will loose its regular nature

Two-point statistical structure of turbulence

Define the energy spectrum (Fourier transform of the correlation) as

$$E(k) = \int e^{-2i\pi k\ell} \langle u(x)u(x+\ell)\rangle \, d\ell$$



Fractional Gaussian Fields

Can we give a probabilistic representation of these observed velocity fluctuations?

$$u_H(x) = \int_{y\in\mathbb{R}} P_H(x-y)W(dy)$$

- $W(dy)$ a (distributional) Gaussian white measure
- P_H the fractional operator (Fourier multiplier) of Hurst $H \in]0,1[$

$$P_H(x) = \int e^{2i\pi kx}|k|_{1/L}^{-H-1/2}dk.$$

It is a perfect representation of these former observations using the particular value $H = 1/3$:

- It is a zero-average and finite-variance **statistically** homogeneous field.
- Moreover, (obviously)

  $$E(k) = |k|_{1/L}^{-2H-1},$$

  and corresponds to Kolmogorov's power spectral density for $H = 1/3$.
- easy generalization to a dissipative range (with proper phenomenology for ε) and to three-dimensional divergence-free fields.

Building a simple (linear) dynamics reproducing this phenomenology?

In d-dimensional space, consider a (scalar) velocity field $u^\nu(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^d$ and say $t > 0$.
Given a (large-scale smooth) f, can we build a very simple (linear) dynamics of the form

$$\frac{\partial u^\nu}{\partial t} + \mathcal{L}u^\nu = \nu \Delta u^\nu + \boldsymbol{f},$$

which

- somehow transfer energy from the large towards the small scales thanks to a linear mechanism given by the operator $\mathcal{L}$
- solution $u^\nu(t, x)$ is statistically homogeneous at any time t, and reaches a statistically stationary regime at a given finite finite viscosity ν
- dissipates in a very efficient manner such that

$$\lim_{\nu \to 0} \lim_{t \to \infty} \mathbb{E}|\boldsymbol{u}^\nu|^2 = \sigma^2 < \infty.$$

  For $\mathcal{L} = 0$ (the stochastic heat equation), straightforward calculations show that $\lim_{t\to\infty} \mathbb{E}|\boldsymbol{u}^\nu|^2 \propto \nu^{-1}$.
- moreover develops a spatial structure similar to a fractional Gaussian field

The underlying idea is a **transport** mechanism in **Fourier** space

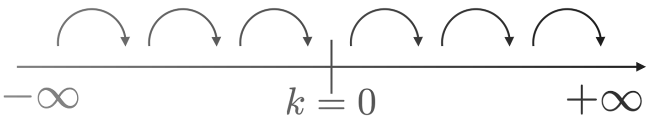
The Fourier transform: once and for all

$$\widehat{u}(t,k) = \mathcal{F}[u(t,.)](k) = \int_{\mathbb{R}^d} e^{-2i\pi k\cdot x} u(t,x)dx$$

Warmup: Generation of a white noise from smooth ingredients

Complex 1d White noise

$$\partial_t \widehat{u}(t,k) + \partial_k \widehat{u}(t,k) = \widehat{f}(t,k)$$

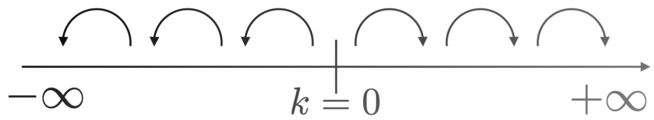


$$\partial_t u(t,x) - ix\, u(t,x) = f(t,x)$$

G.B. Apolinário, L. Chevillard, J.-C. Mourrat

Real 1d White noise

$$\begin{cases} \partial_t \widehat{u}(t,k) + \partial_{|k|} \widehat{u}(t,k) = \widehat{f}(t,k) \\ \widehat{u}(t,|k|=0) = 0 \end{cases}$$



$$\begin{cases} \partial_t u(t,x) + x\mathcal{H}[u](t,x) = f(t,x) \\ \int_{\mathbb{R}} u = 0 \end{cases}$$

- $\mathcal{L} = -ix\cdot$ is an homogeneous operator of degree 0 that lies in the class pinpointed by Colin de Verdière and Saint-Raymond
- Asymptotic behavior of the solution:

$$\lim_{t\to\infty} \mathbb{E}\left[u(t,x)u(t,y)\right] = c\delta(x-y) + \Psi(x-y)$$

- with G. Apolinàrio, G. Beck, I. Gallagher, R. Grande, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, arXiv:2301.00780 (2023)

Generalization to a Real Multidimensional Fractional setup

Consider the following transport equation in Fourier space:

$$\begin{cases} \partial_t \widehat{u}(t,k) + \mathrm{div}_k \left(\frac{ck}{|k|}\, \widehat{u}(t,k) \right) + c\dfrac{H+\frac{1}{2}}{|k|}\, \widehat{u}(t,k) = \widehat{f}(t,k) & t>0, k\in\mathbb{R}^d, |k|>\kappa>0, \\ \widehat{u}(t,k) = 0 & t>0, k\in\mathbb{R}^d, |k|\le\kappa, \\ \widehat{u}(0,k) = 0, & \end{cases}$$

with parameters $H \in]0,1[$ and κ and c positive and fixed, and the source f satisfies

$$\mathbb{E}[f(t,x)f(s,y)] = \delta_{t-s}\, C_f(x-y),$$

where C_f is smooth and satisfies aforementioned assumptions.

- Asymptotic behavior of the solution:

$$\lim_{t\to\infty} \mathbb{E}\left[u(t,x)u(t,y)\right] = C(H,d)K_H(x-y) + \Psi_H(x-y),$$

where $C(H,d)$ a factor that depends on H and d, and

$$K_H(x) = \mathcal{F}^{-1}\left(|k|^{-2H-d} \mathbb{1}_{|k|>\kappa} \right)$$

Simulation of the Real Multidimensional Fractional setup

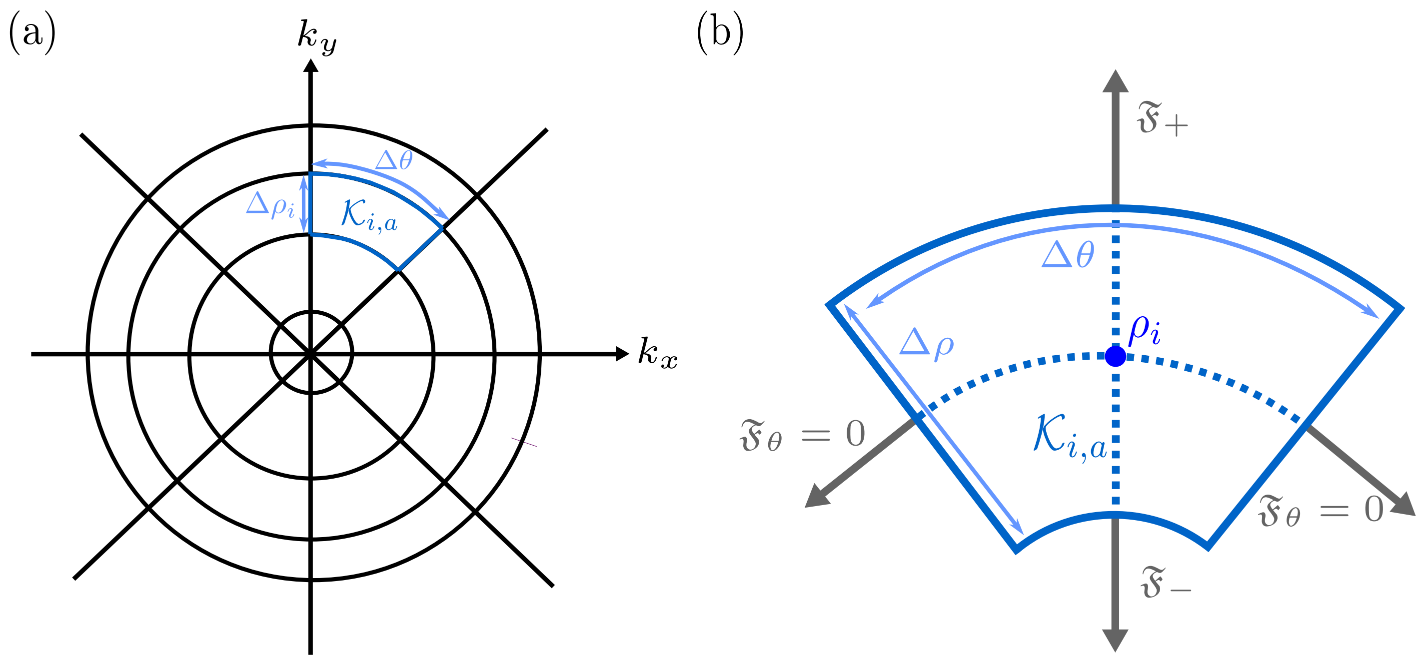
Consider the following transport equation in Fourier space:

$$\begin{cases} \partial_t \widehat{u} + \mathsf{div}_k \left(\frac{ck}{|k|}\, \widehat{u} \right) + c\dfrac{H+\frac{1}{2}}{|k|}\, \widehat{u} = \widehat{f} - \nu(2\pi)^2|k|^2\widehat{u} & t>0, k\in\mathbb{R}^d, |k|>\kappa>0, \\ \widehat{u}(t,k) = 0 & t>0, k\in\mathbb{R}^d, |k|\leq\kappa, \\ \widehat{u}(0,k) = 0, & \end{cases}$$

- It can be further shown that the solution $u(t,x)$ is statistically homogenous at any time, and as a consequence, $\widehat{u}(t,k)$ has the regularity of the white noise.
- preliminary simulations using pseudo-spectral methods were unable to give proper (i.e. theoretically expected) representation of the solution
- this justifies and motivates a simulation using (spectral) finite volumes
- with G. Beck, C.-E. Brehier, R. Grande, W. Ruffenach, Phys. Rev. Research 6, 033048 (2024).

$$\hat{u}_{\mathcal{K}}(t) = \frac{1}{|\mathcal{K}|}\int_{\mathcal{K}} \hat{u}(t,k)dk$$

Spectral finite-volume method: *Geometry* of the cell



such that, by Gauss's theorem, noticing that only radial fluxes contribute in this spherical geometry

$$\int_{\mathcal{K}} \text{div}_k \left(\frac{k}{|k|} \hat{u} \right) = \int_{\partial\mathcal{K}} \frac{k \cdot n}{|k|} \hat{u}(t,k) dk = \mathfrak{F}_+ + \mathfrak{F}_-$$

$$\overset{\text{upwind}}{\approx} |\partial_+ \mathcal{K}_{i,a}| \hat{u}_{i,a}(t) - |\partial_- \mathcal{K}_{i,a}| \hat{u}_{i-1,a}(t)$$

and finally $$\frac{1}{|\mathcal{K}_{i,a}|} \int_{\mathcal{K}} \text{div}_k \left(\frac{k}{|k|} \hat{u} \right) \approx \overbrace{\frac{\hat{u}_{i,a} - \hat{u}_{i-1,a}}{h_i}}^{\text{advection}} + \underbrace{d_i \hat{u}_{i,a}}_{\text{numerical dissipation}}$$

Spectral finite-volume method: *Time Splitting*

The time evolution is split in two dynamics:

$$\partial_t \hat{u}^{sd}_{i,a}(t) + D_i \hat{u}^{sd}_{i,a}(t) = \hat{f}_{i,a}(t) \qquad \text{and} \qquad \partial_t \hat{u}^{ad}_{i,a}(t) + c\frac{\hat{u}^{ad}_{i,a}(t) - \hat{u}^{ad}_{i-1,a}(t)}{h_i} = 0,$$

- The stochastic dynamics of Ornstein-Uhlenbeck type is solved according to an exponential integrator (exact in distribution)
- The advection dynamics is solved with an explicit Euler scheme that requires to work with a **unit** Courant number to prevent from spurious numerical dissipation

With proper initial and boundary conditions, a Lie-Trotter splitting scheme gives as a fully-discrete scheme

$$\begin{cases} \hat{u}^{n+\frac{1}{2}}_{i,a} = e^{-\Delta t D_i}\, \hat{u}^{n}_{i,a} + \varrho_i\, \hat{\gamma}^{n}_{i,a} \\ \hat{u}^{n+1}_{i,a} = \hat{u}^{n+\frac{1}{2}}_{i-1,a}, \end{cases}$$

where $\hat{\gamma}^{n}_{i,a}$ are independent standard complex Gaussian random variables (up to Hermitian symmetry).

A quick word on the respective Inverse Fourier transform

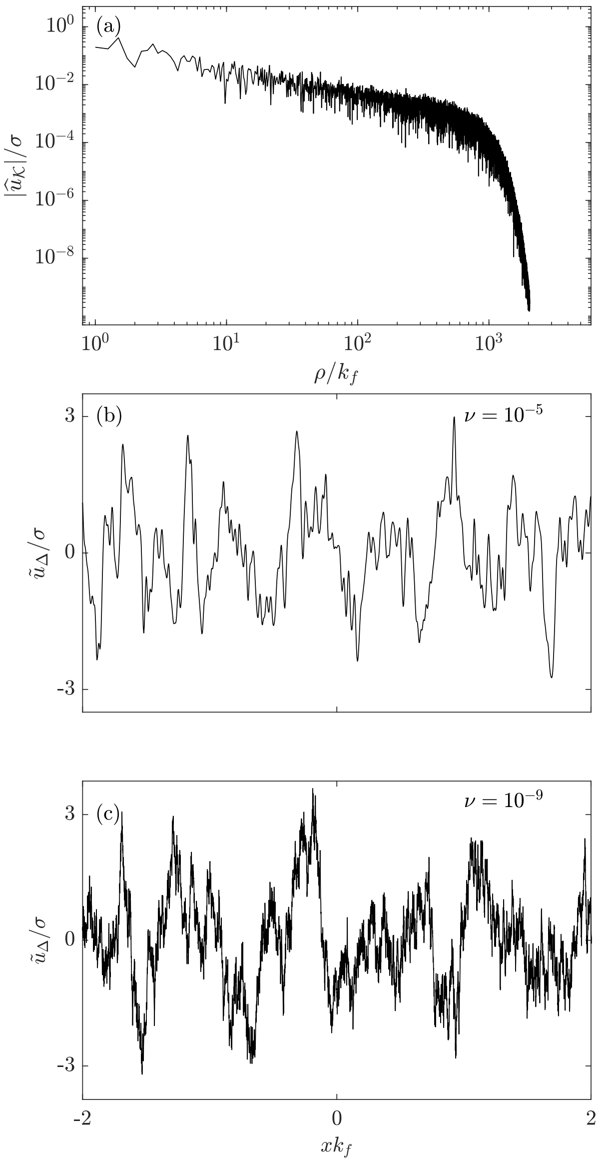
Of tremendous importance is the obtention of fields in the **physical space** $u(t, x)$

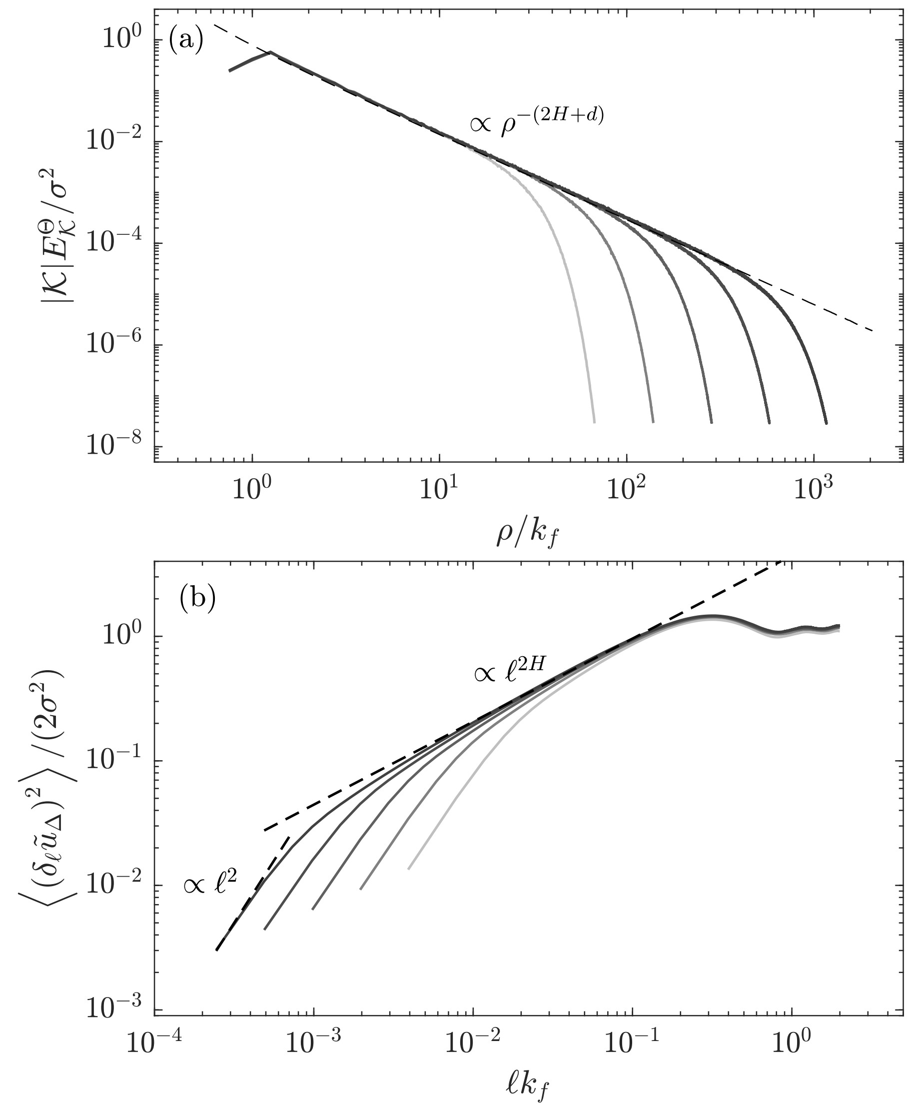
Define in any dimension

$$\widetilde{u}_{\Delta}(t, x) = \sum_{n=1}^{N} \sum_{a \in \mathcal{A} \subset \mathbb{S}^{d-1}} e^{2i\pi x \cdot k_n} \hat{u}_{\mathcal{K}_{n,a}}(t) |\mathcal{K}_{n,a}|$$

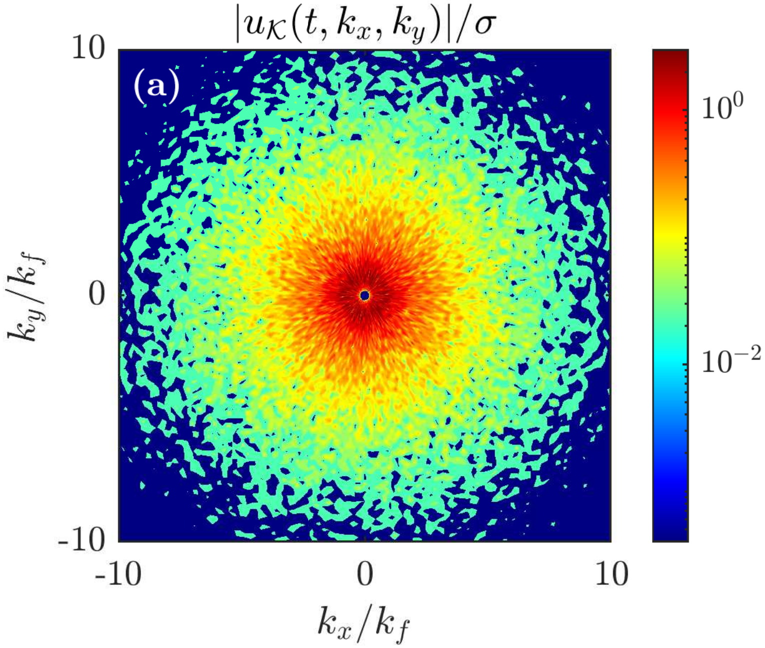
- The relationship to the continuous field $u(t, x)$ not obvious when $d \geq 2$
- For $d = 1$, it corresponds to a truncation of the Fourier series of an approximation of $u(t, x)$, which coincides with a DFT (up to the Nyquist mode)
- For $d \geq 2$, it is less clear what is $\widetilde{u}_{\Delta}(t, x)$

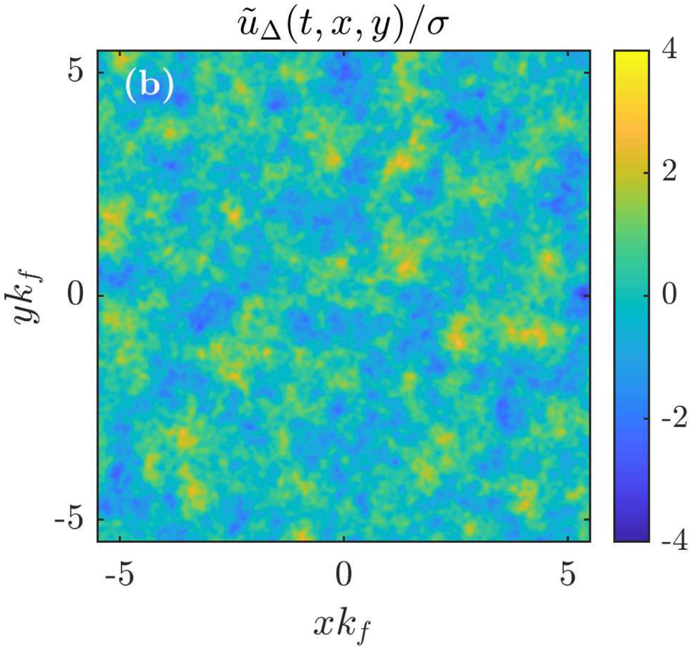
Spectral finite-volume method: Results in space dimension 1

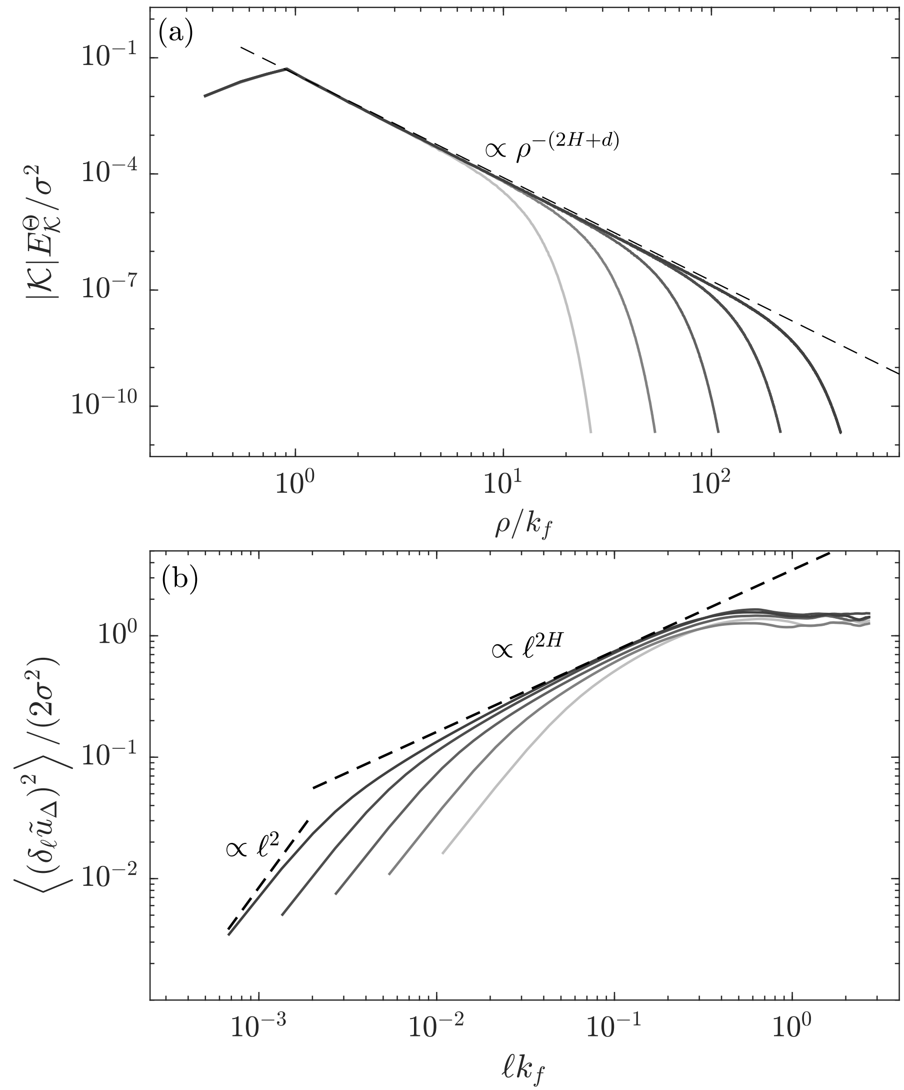




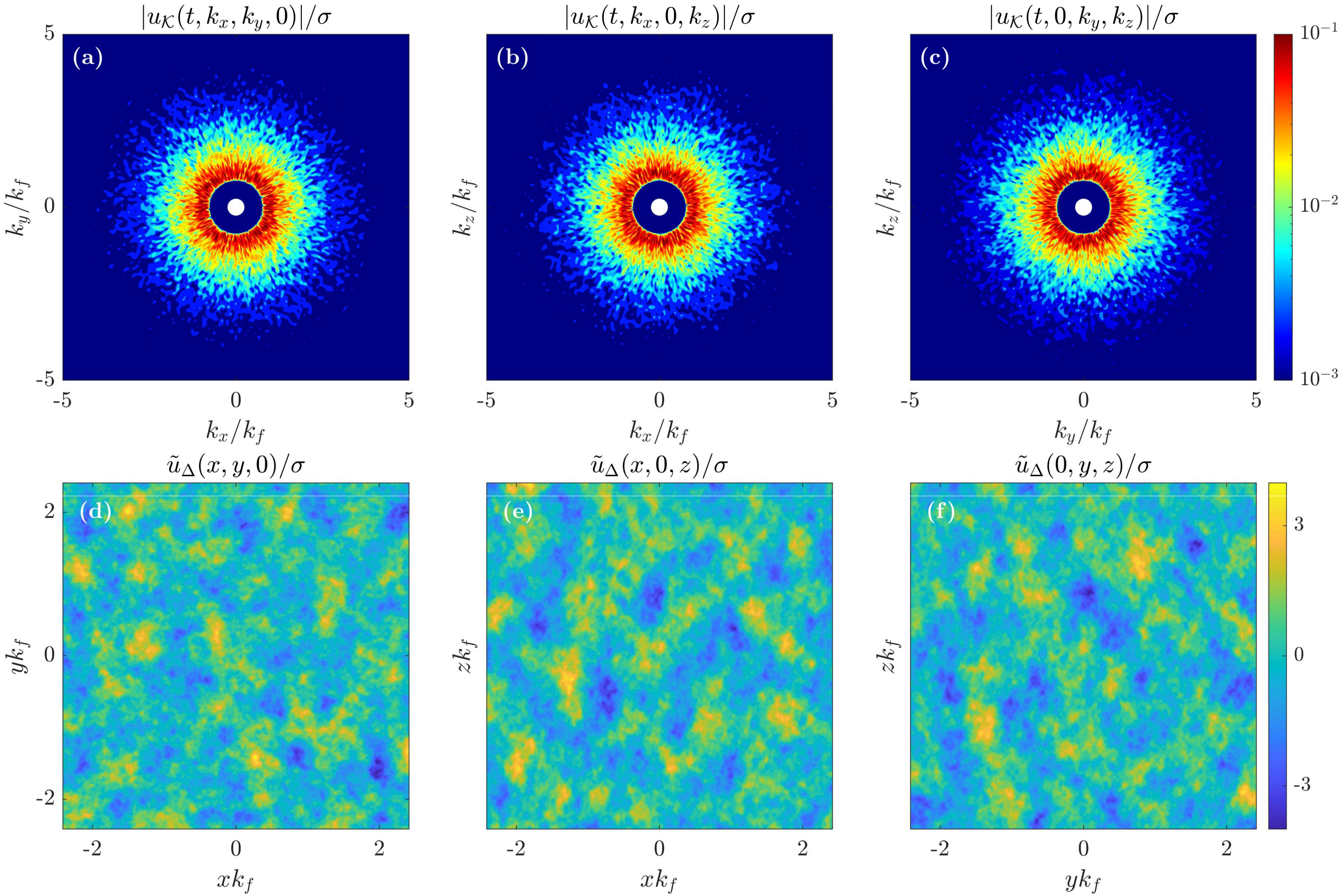
Spectral finite-volume method: Results in space dimension 2







Spectral finite-volume method: Results in space dimension 3



Conclusions Perspectives

- A meaningful and consistent numerical approach.
- What about convergence and Courant numbers smaller than unity?
- A more efficient inverse Fourier transform?
- Including intermittency (multifractality)?
- Towards more realistic fluid mechanics situations (incompressible vector fields, including inverse cascades for 2d turbulence and the effect of rotation in 3d, etc.)

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- Thank you for your attention