Propagation of Gaussianity and application to the incompressible Euler dynamics

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2024 Simons Collaboration on Wave Turbulence Annual Meeting

Introduction : Framework

General idea :

- Equation from fluid mechanics;
- Initial datum has independent (centred) Gaussian Fourier coefficients;
- Quantify how Fourier coefficients at ulterior time differ from independent (centred) Gaussian variables;
- Related to propagation of chaos in wave turbulence;
- Not a dispersive phenomenon.

Introduction : How?

Centred Gaussian variables satisfy the Wick formula :

$$\mathbb{E}\Big(\prod_{l=1}^{R} g_l\Big) = \sum_{\underline{\mathfrak{S}_R}} \prod_{l \in S_\sigma^+} \mathbb{E}(g_l g_{\sigma(l)})$$

where $\underline{\mathfrak{S}_R}$ is the set of involutions of $[|1, R|] := [1, R] \cap \mathbb{N}$ without fixed points and

$$S_\sigma^+ = \{l \in [|1, R|] \mid l < \sigma(l)\}.$$

Remark on the Wick formula

If R even, in $\mathbb{R}$, $g_1 = \ldots = g_R = g$ normalized, we find

$$\mathbb{E}(g^R) = \#\underline{\mathfrak{S}_R} = \frac{R!}{2^{R/2}(R/2)!}.$$

In $\mathbb{C}$, $g_1 = \ldots = g_{R/2} = g$, $g_{R/2+1} = \ldots = g_R = \bar{g}$ normalized, we find

$$\mathbb{E}(|g|^R) = (R/2)!.$$

Motivations

Derivation of kinetic equations in the Physics literature. The reasons invoked are combinatory. The more independent Gaussian variables you have, the less probable it becomes to pick two that are not independent. This is what we follow here.

Work by Deng-Hani on quantum equations : derive kinetic equations for Schrödinger equations and then deduce propagation of chaos up to kinetic timescale. Tools : oscillating integrals, dispersion.

Framework

Incompressible Euler equation :

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \nabla \cdot u = 0 \end{cases}$$

on $\mathbb{R}^d$, u with values in $\mathbb{R}^d$.
Let P be the Leray projector, that is the 0 order differential operator

$$Pu = u - \nabla \vartriangle^{-1} (\nabla \cdot u).$$

Note that $P(\nabla p) = 0$ and that $\nabla \cdot P = 0$, $P^2 = P$. The equation becomes

$$\partial_t u + P(u \cdot \nabla u) = 0.$$

This equation is invariant under the action of spatial translations, hence if the i.d. is periodic, so is the solution.

Framework, cont.

We consider the Cauchy problem :

$$\begin{cases} \partial_t u_L + P(u_L \cdot \nabla u_L) = 0 \\ u_L(t=0) = a_L := \sum_{\mathbb{Z}^d_*} \frac{e^{ikx/L}}{(2\pi L)^{d/2}} g_k a(k/L) \end{cases}$$

where a is an even, bounded, compactly supported function with real values and such that $\xi \cdot a(\xi) = 0$ for all $\xi \in \mathbb{R}^d$; where $(g_k)_k$ are centred normalised Gaussian variables such that $\mathbb{E}(g_k g_l) = \delta_{k+l}$ – which means $g_k = \overline{g_{-k}}$ but otherwise they are independent.

Invariance

The function a_L is real valued and $2\pi L$-periodic. So is u_L. About $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$, we remark that $\int_{[-\pi L, \pi L]^d} u$ is a priori conserved by the flow for periodic solutions.

The law of a_L is invariant under the action of spatial translations, so is the law of $u_L(t)$.

This implies that for all $\xi_1, \dots, \xi_R$,

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_L(t, \xi_l) \Big) \neq 0 \Rightarrow \sum_l \xi_l = 0.$$

Fourier coefficients

Fourier coefficients are normalized as

$$\hat{u}_L(t,\xi) = \frac{1}{(2\pi L)^{d/2}} \int_{[-\pi L,\pi L]^d} u_L(x) e^{-i\xi x} dx$$

such that

$$\hat{u}_L(0,\xi) = g_{L\xi} a(\xi).$$

Goal : estimate, for a given R (at least locally in time),

$$\sup_{(\xi_l)_l,(j_l)_l,t} \left| \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_L^{(j_l)}(t,\xi_l) \Big) - \sum_{\sigma \in \underline{\mathfrak{S}_R}} \prod_{l \in S_\sigma^+} \mathbb{E}\Big(\hat{u}_L^{(j_l)}(t,\xi_l) \hat{u}_L^{(j_{\sigma(l)})}(t,\xi_{\sigma(l)}) \Big) \right|$$

and prove that it goes to 0 as $L \to \infty$.

Issue : a_L is a Gaussian so any norm of a_L might be quite big and thus the time of existence of the flow may be very small.

Results : notations

We define the sequence $(u_{L,n})_n$ in the following way

$$u_{L,0} = a_L, \quad u_{L,n+1} = -\sum_{n_1+n_2=n} \int_0^t P(u_{L,n_1}(\tau) \cdot \nabla u_{L,n_2}(\tau)) d\tau.$$

We have $\sum u_{L,n} = u_L$ (if the series converges).

We set

$$u_{L,\leq N} = \sum_{n=0}^{N} u_{L,n}.$$

This last function is said to be a quasi-solution.

Result on quasi-solutions

Theorem [dS 22] For all $R \in \mathbb{N}^*$, all $t \in \mathbb{R}$ and all $N \in \mathbb{N}$, there exists $C = C(a, R, N, t)$ such that for all L, all $(j_l)_{1\leq l\leq R}$, all $(\xi_l)_{1\leq l\leq R}$, we have

$$\left|\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_{L,\leq N}^{(j_l)}(t,\xi_l)\Big) - \sum_{\sigma\in\underline{\mathfrak{S}_R}} \prod_{l\in S_\sigma^+} \mathbb{E}\Big(\hat{u}_{L,\leq N}^{(j_l)}(t,\xi_l)\hat{u}_{L,\leq N}^{(j_{\sigma(l)})}(t,\xi_{\sigma(l)})\Big)\right| \leq \frac{C}{L^{d/2}}.$$

Remarks

- one can prove a finer estimate which depends on the parity of R and the algebra of the $(\xi_l)_l$,
- the constant C is explicit,
- it depends badly on N, as in (NR)!.

How to get a result on the full solution

Because the above constant behaves badly in N, to get a result on the full solution one must get the convergence of the sequence $u_{L,\leq N}$ in a classical/deterministic way.

Problem :

- $\|a_L\|_{L^2(L\mathbb{T}^d)} \sim L^{d/2}$ (idem Sobolev norms),
- $\|a_L\|_{L^p(L\mathbb{T}^d)} \sim (\sqrt{p})L^{d/p}$,
- $\|a_L\|_{L^\infty} \sim \sqrt{\ln L}$.

The idea for the last estimate is that there are circa L^d independent Gaussian variables forming a_L.

Result on the full solution

We change the framework and set

$$\underline{a_L} = \varepsilon(L)a_L$$

with $\varepsilon(L) = O(\frac{1}{\sqrt{\ln L}})$.
Theorem Let $\theta > 0$. There exists $c(\theta) > 0$ such that for all L, there exists a set $\mathcal{E}_{L,\theta}$ such that

$$\mathbb{P}(\mathcal{E}_{L,\theta}) \geq 1 - e^{-c\varepsilon(L)^{-2}} \to 1$$

and such that the flow of the Euler equation is well-defined on $[-\theta, \theta]$ when the i.d. is taken in $\mathcal{E}_{L,\theta}$.

Result on the full solution, cont.

Theorem What is more, for all $R \in \mathbb{N}^*$, there exists $\theta_0 = \theta_0(R, a)$ and $C = C(a, R)$ such that for all $t \in [-\theta_0, \theta_0]$, for all L, all $(j_l)_{1 \leq l \leq R}$, all $(\xi_l)_{1 \leq l \leq R}$, we have

$$\left| \mathbb{E}\left(\mathbf{1}_{\mathcal{E}_{L,\theta_0}} \prod_{l=1}^{R} \underline{\hat{u}_L}^{(j_l)}(t, \xi_l) \right) - \sum_{\sigma \in \underline{\mathfrak{S}_R}} \prod_{l \in S_\sigma^+} \mathbb{E}\left(\mathbf{1}_{\mathcal{E}_{L,\theta_0}} \underline{\hat{u}_L}^{(j_l)}(t, \xi_l) \underline{\hat{u}_L}^{(j_{\sigma(l)})}(t, \xi_{\sigma(l)}) \right) \right| \leq \frac{C\varepsilon(L)^R}{L^{d/2}}.$$

Remark : If $\varepsilon(L) = o(\frac{1}{\sqrt{\ln L}})$ then the result is global.

Time rescaling

- $T = \varepsilon^{-1}\sqrt{\ln L}$ is ok;
- One has no counter effect to large number of couples due to oscillating integrals to use an Ansatz of type $u_{L,\leq|\ln\varepsilon|}$ as in Deng-Hani, no $T = \varepsilon^{-1}$ (even less $T = \varepsilon^{-2}$);
- Ansatz $u_{L,\leq N}$ with fixed N might allow to go up to $\varepsilon^{-1+\eta}$ which would be nice if $\varepsilon = (\ln L)^{-\beta}$ but it is not clear how it articulates with the hypercontractivity lemmas one might have to use in order to bound the source term.

Ideas of proof

The proof of the first theorem is completely combinatory.

Example : $R = 6$. We have

$$u_{L,1}(t) = -\int_0^t P(a_L \cdot \nabla a_L) d\tau = -tP(a_L \cdot \nabla a_L),$$

and

$$\hat{u}_{L,1}(t,\xi) = \frac{t}{L^{d/2}} \sum_{\xi_1+\xi_2=\xi} \psi(\xi_1,\xi_2) g_{L\xi_1} g_{L\xi_2}.$$

We deduce

$$\mathbb{E}\Big(\prod_{l=1}^{6} \hat{u}_{L,1}^{(1)}(t,\xi_l)\Big) = \frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi^{(1)}(\xi_{l,1},\xi_{l,2}) \mathbb{E}\Big(\prod_l g_{L\xi_{l,1}} g_{L\xi_{l,2}}\Big).$$

A priori, we sum on 6 parameters, which makes the sum of order L^{3d} but the $(g_k)_k$ satisfy the Wick formula...

Orbits

We have that

$$\mathbb{E}\Big(\prod_{l} g_{L\xi_{l,1}} g_{L\xi_{l,2}}\Big) = \underline{\sum_{\mathfrak{S}(6,2)}} \prod_{(l,j)\in S_\sigma^+} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma(l,j)}}) \leq 6!$$

and if it is not null then there exists an involution σ without fixed points of $[|1,6|] \times \{1,2\}$ such that for all (l,j), we have $\xi_{l,j} = -\xi_{\sigma(l,j)}$. Note that since $\xi_l \neq 0$, we have $\sigma(l,1) \neq (l,2)$.

We set for $\mathcal{I} \subseteq [|1,6|]$,

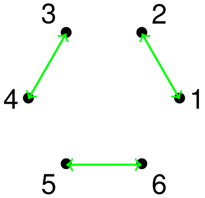
$$\tilde{\sigma}(\mathcal{I}) = \{l \in [|1,6|] \mid \exists l' \in \mathcal{I}, j, j',\ \sigma(l',j') = (l,j)\}$$

and for $l \in [|1,6|]$,

$$o(l) = \bigcup_{n} \tilde{\sigma}^n(\{l\}).$$

We call $o(l)$ the orbit of l. The orbits of each element form a partition of $[|1,6|]$. Each orbit has at least 2 elements. There are at most $R/2 = 3$ orbits.

Examples of involutions, 1



$$\sigma_1(1,1)=(2,2),\qquad \sigma_1(3,1)=(4,2),\qquad \sigma_1(5,1)=(6,2),$$
$$\sigma_1(2,1)=(1,2),\qquad \sigma_1(4,1)=(3,2),\qquad \sigma_1(6,1)=(5,2).$$

$$\boxed{\xi_{1,1}}+\xi_{2,2}=0,\qquad \boxed{\xi_{3,1}}+\xi_{4,2}=0,\qquad \boxed{\xi_{5,1}}+\xi_{6,2}=0,$$
$$\xi_{1,2}+\xi_{2,1}=0,\qquad \xi_{3,2}+\xi_{4,1}=0,\qquad \xi_{5,2}+\xi_{6,1}=0.$$

We deduce by summing the equalities, since $\xi_l=\xi_{l,1}+\xi_{l,2}$, that

$$\xi_1+\xi_2=0,\quad \xi_3+\xi_4=0,\quad \xi_5+\xi_6=0.$$

Example 1, cont.

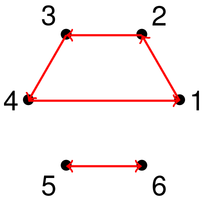
What is more, only three parameters $\xi_{1,1}$, $\xi_{3,1}$ and $\xi_{5,1}$ determine all the $\xi_{l,j}$. We deduce

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1},\xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_1}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_1(l,j)}})$$

is a sum on three parameters in $\frac{1}{L}\mathbb{Z}^d_*$, is null if $\xi_1+\xi_2\neq 0$ or $\xi_3+\xi_4\neq 0$ or $\xi_5+\xi_6\neq 0$ and otherwise

$$\frac{t^6}{L^{3d}} \sum_{\xi_{l,1}+\xi_{l,2}=\xi_l} \prod_l \psi(\xi_{l,1},\xi_{l,2}) \prod_{(l,j)\in S^+_{\sigma_1}} \mathbb{E}(g_{L\xi_{l,j}} g_{L\xi_{\sigma_1(l,j)}}) \lesssim 1.$$

Examples of involutions, 2

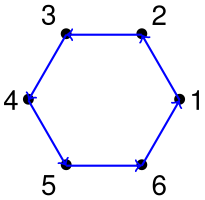


$$\sigma_2(1,1) = (2,2),$$
$$\sigma_2(2,1) = (3,2), \qquad \sigma_2(5,1) = (6,2),$$
$$\sigma_2(3,1) = (4,2), \qquad \sigma_2(6,1) = (5,2).$$
$$\sigma_2(4,1) = (1,2),$$

$$\boxed{\xi_{1,1}} + \xi_{2,2} = 0,$$
$$\xi_{2,1} + \xi_{3,2} = 0, \qquad \boxed{\xi_{5,1}} + \xi_{6,2} = 0,$$
$$\xi_{3,1} + \xi_{4,2} = 0, \qquad \xi_{5,2} + \xi_{6,1} = 0.$$
$$\xi_{4,1} + \xi_{1,2} = 0,$$

$$\Rightarrow \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \quad \xi_5 + \xi_6 = 0.$$

Examples of involutions, 3



$$\begin{aligned}
\sigma_3(1,1) &= (2,2), \\
\sigma_3(2,1) &= (3,2), \\
\sigma_3(3,1) &= (4,2), \\
\sigma_3(4,1) &= (5,2), \\
\sigma_3(5,1) &= (6,2), \\
\sigma_3(6,1) &= (1,2).
\end{aligned}
\qquad
\begin{aligned}
\boxed{\xi_{1,1}} + \xi_{2,2} &= 0, \\
\xi_{2,1} + \xi_{3,2} &= 0, \\
\xi_{3,1} + \xi_{4,2} &= 0, \\
\xi_{4,1} + \xi_{5,2} &= 0, \\
\xi_{5,1} + \xi_{6,2} &= 0, \\
\xi_{6,1} + \xi_{1,2} &= 0.
\end{aligned}$$

We deduce by summing the equalities, since $\xi_l = \xi_{l,1} + \xi_{l,2}$, that

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0.$$

What do we learn

The contribution to an involution is given by the number of its orbits. For example, the sum on σ is on 3 paramaters, this yields a sum of order 1 (in L). For the same reasons, the other involutions contribute as $\frac{1}{L^d}$ and $\frac{1}{L^{2d}}$.

However, the involutions fix conditions on the $(\xi_l)_l$. Namely, if $\xi_1 + \xi_2 = 0$, $\xi_3 + \xi_4 = 0$, $\xi_5 + \xi_6 = 0$ then the green, red and blue involutions will contribute to the expectation. However, if we have only $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ and $\xi_5 + \xi_6 = 0$, then only the red and blue involutions will contribute.

More precisely

Given $(\xi_1, \dots, \xi_R)$. Consider all the partitions of $[|1, R|]$:

$$[|1,R|] = \bigsqcup_{i \in \mathcal{I}} o_i$$

such that for all $i \in \mathcal{I}$,

$$\sum_{l \in o_i} \xi_l = 0.$$

One has maximal cardinal $\#\mathcal{I}$. Then

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}^{(j_l)}_{L,\leq N}(t,\xi_l)\Big) = O(L^{d(\#\mathcal{I} - \frac{R}{2})}).$$

As we have already seen $\#\mathcal{I} \leq \frac{R}{2}$ and in case of equality the leading order is

$$\sum_{\sigma \in \underline{\mathfrak{S}_R}} \prod_{l \in S_\sigma^+} \mathbb{E}\Big(\hat{u}^{(j_l)}_{L,\leq N}(t,\xi_l)\hat{u}^{(j_{\sigma(l)})}_{L,\leq N}(t,\xi_{\sigma(l)})\Big)$$

Idea of proof, full solution

For these types of equations, a good way to solve the problem is to perform a fixed point for analytic data/regularized problem and extend the result to Sobolev spaces by exploiting the conservation laws of the equation (compactness/bootstrap arguments)

Here, we solve the problem for Gevrey-type analytic data that is adapted to the L^∞ norms and exploit the Cauchy-Kowalevskaia theorem.

Functional framework

Traditionally (Benzoni-Gavage–Coulombel–Tzvetkov) on the torus

$$\|u\|_\rho = \sum_{k\in\mathbb{Z}^d_*} |\hat{u}(k)| e^{\rho|k|}, \qquad \sqrt{\sum_{k\in\mathbb{Z}^d_*} |\hat{u}(k)|^2 e^{\rho|k|}}.$$

Pass badly to $L \to \infty$.

We define

$$\|u\|_\rho = \sum_{n\in\mathbb{Z}^d} e^{\rho|n|} \|u_n\|_{L^\infty}$$

where u_n is u but localised in frequencies around $n \in \mathbb{Z}^d$ and such that $\sum u_n = u$.

Take $\rho_0 > 0$, we have

$$\mathbb{P}(\|\underline{a_L}\|_{\rho_0} \geq A) \leq e^{-c(\rho_0)A^2 \varepsilon(L)^{-2}}.$$

Functional framework, cont.

We (Caflish, '90) define for $\beta \in (0,1)$, $\theta > 0$, $\theta(\rho) = \theta(\rho_0 - \rho)$,

$$\|u\|_{\rho_0,\beta,\theta} = \sup_{\rho\in(0,\rho_0)} \sup_{t\in[0,\theta(\rho))} \Big(\|u(t)\|_\rho + \|\nabla u(t)\|_\rho (\theta(\rho) - t)^\beta \Big).$$

We have bilinear estimates

$$\Big\| \int_0^t P(u(\tau)\cdot\nabla v(\tau))d\tau \Big\|_{\rho_0,\beta,\theta} \lesssim C(\theta)\|u\|_{\rho_0,\beta,\theta}\|v\|_{\rho_0,\beta,\theta}.$$

Conclusion

There exists $A(\theta)$ such that on

$$\mathcal{E}_{L,\theta} = \{\|\underline{a_L}\|_{\rho_0} \leq A(\theta)\},$$

we have

$$\|\underline{u_{L,n}}\|_{\rho_0,\beta,\theta} \leq 2^{-n} A(\theta).$$

Since we know the dependence of the constant in N in the first theorem, it remains to optimize.

Thank you for your attention!