On the wave turbulence theory of 2D gravity water waves

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Introduction

We consider the free boundary incompressible Euler equations

$$v_t + v \cdot \nabla v = -\nabla p - g e_n, \qquad \nabla \cdot v = 0, \qquad x \in \Omega_t,$$

where g is the gravitational constant. The free surface $\Gamma_t = \{z(\alpha, t) : \alpha \in \mathbb{R}^{n-1}\}$ moves with the velocity, according to the kinematic boundary condition

$$(\partial_t z - v)|_{\Gamma_t} \qquad \text{tangent to } \Gamma_t.$$

In the presence of surface tension the pressure on the interface is given by

$$p(x, t) = \sigma\kappa(x, t), \quad x \in \Gamma_t\,,$$

where κ is the mean-curvature of Γ_t and $\sigma > 0$.

Introduction

Natural questions:

- Local regularity
- Dynamical formation of singularities
- Steady water waves
- Long-term regularity and asymptotics

Possible variants:
Periodic conditions, finite bottom, two-fluid model.

Local wellposedness

- The local regularity theory is well understood, due to the work of many authors: Nalimov (1974), Yosihara (1982), Craig (1985), Wu (1997, 1999), Beyer–Gunther (1998), Christodoulou–Lindblad (2000), Ambrose–Masmoudi (2005), Lannes (2005), Coutand–Shkoller (2007), Christianson–Hur–Staffilani (2010), Alazard–Burq–Zuily (2011), Shatah–Zeng (2008, 2011), ...

- One has local regularity if $\sigma > 0$ or if the Rayleigh–Taylor condition is satisfied. The time of existence depends on two quantities: the smoothness, say in H^4, of the interface and the fluid velocities, and the arc-chord constant of the interface.

Formation of singularities

- There are two possible scenarios that lead to singularities:
(1) loss of regularity;
(2) self-intersection of the interface.

- The "splash" singularity of Castro–Cordoba–Fefferman–Gancedo–Gomez-Serrano and Coutand–Shkoller.

- The splash singularity cannot form in the two-fluid model (Fefferman–I.–Lie and Coutand-Shkoller).

- Loss of regularity of the interface (dynamical formation of corners) is an important open problem.

Steady water waves

- There are interesting traveling waves solutions in 2 dimensions, both in the periodic case and in the Euclidean case, with or without vorticity. Such solutions can be constructed using local and global bifurcation theory.

- Many authors: Stokes (1847), Nekrasov (1921), Levi-Civita (1924), Struik (1926),..., Constantin–Strauss–Varvaruca (2016, 2021), Chen–Walsh–Wheeler (2016, 2018, 2020), Haziot–Wheeler (2021).

- Beautiful open problems, such as the stability of certain traveling waves and the existence of waves with overhanging profiles.

Long term and global regularity

- Global in time solutions for small, smooth, and localized irrotational initial data, with gravity and/or surface tension, with or without finite bottom in 2D or 3D (1D or 2D Euclidean interfaces): Wu (2009, 2011), Germain–Masmoudi–Shatah (2011, 2014), I.-Pusateri (2013, 2015), Alazard–Delort (2015), Ifrim–Tataru (2016), Wang (2017, 2018), Deng–I.–Pausader–Pusateri (2017).

- In the periodic case one can sometimes prove long-term regularity in 2D (1D interfaces) in the presence of exterior parameters: Berti–Delort (2018), Berti–Maspero–Murgante (2023).

Irrotational water waves

We consider fluid domains described as regions in the plane below the graph of function over a large box $\mathbb{T}^d_R := [\mathbf{R}/(2\pi R\mathbb{Z})]^d$, $R \geq 1$. We consider irrotational flows, i.e. $\mathrm{curl}\, \mathrm{v} = 0$, in which case the water waves system can be reduced to a system on the boundary.

More precisely let $h = h(x,t) : \mathbb{T}^d_R \times I \to \mathbf{R}$, and let

$$\Omega_t = \{(x,y) \in \mathbb{T}^d_R \times \mathbf{R} \,:\, y \leq h(x,t)\}, \quad S_t = \{(x,y) : y = h(x,t)\}.$$

Let Φ denote the velocity potential, $\nabla_{x,y}\Phi(x,y,t) = v(x,y,t)$ for $(x,y) \in \Omega_t$. If $\phi(x,t) := \Phi(x, h(x,t), t)$ is the restriction of Φ to the boundary S_t, the equations of motion reduce to a system of equations for the functions $h, \phi : \mathbb{T}^d_R \times I \to \mathbf{R}$.

Irrotational water waves

In Eulerian formulation the system is

$$\partial_t h = G(h)\phi,$$

$$\partial_t \phi = -gh + \sigma \mathrm{div}\left[\frac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\right] - \frac{1}{2}|\nabla\phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla\phi)^2}{2(1+|\nabla h|^2)}.$$

where

$$G(h) := \sqrt{1+|\nabla h|^2}\mathcal{N}(h),$$

and $\mathcal{N}(h)$ is the Dirichlet-Neumann map associated to the domain Ω_t. The system admits the conserved Hamiltonian

$$\mathcal{H}(h,\phi) := \frac{1}{2}\int_{\mathbb{T}_R^d} G(h)\phi\cdot\phi\, dx + \frac{g}{2}\int_{\mathbb{T}_R^d} h^2\, dx + \sigma\int_{\mathbb{T}_R^d} \frac{|\nabla h|^2}{1+\sqrt{1+|\nabla h|^2}}\, dx,$$

which is the sum of the kinetic energy (the L^2 norm of the velocity field) and the potential energy due to gravity and surface tension (the Zakharov formulation of the water waves system).

Wave turbulence

Our work is motivated by the outstanding problem of understanding rigorously the wave turbulence theory of water waves. This is a classical problem in Mathematical Physics, going back to work of Hasselmann and Zakharov in 1960's, as a key component of the *Hasselman model of climate variability*.

The wave turbulence problem has received intense attention in recent years, mainly in the context of semilinear dispersive models such as Schrödinger or KdV equations (Buckmaster–Germain–Hani–Shatah, Deng–Hani, Collot–Germain, Tran-Staffilani). The focus in this work has been the rigorous derivation of the *wave kinetic equation* (WKE). At the analytical level, this amounts to understanding precisely the dynamics of solutions corresponding to randomized initial data, in the large box and weakly nonlinear limit, and on long time intervals.

For the cubic NLS

$$(i\partial_t + \Delta)u = |u|^2 u$$

on $\mathbb{T}_L^d$. The kinetic theory seeks to give the effective dynamics of frequency amplitudes $\mathbb{E}|\widehat{u}(t,k)|^2$ where

$$u(t,x) = \frac{1}{(2\pi L)^d} \sum_{k \in (\mathbb{Z}/L)^d} \widehat{u}(t,k) e^{ik\cdot x}$$

and the averaging happens over a random initial data distribution,

$$\widehat{u}(0,k) := \lambda L^{d/2} \sqrt{\varphi(k)} \eta(k,\omega),$$

where η denotes a collection of i.i.d random variables. The kinetic time scale is defined as $T_{\mathrm{kin}} := \lambda^{-4}$. Then, for any $\tau \in [0,\delta]$

$$\lim_{L\to\infty} \sup_{k\in(\mathbb{Z}/L)^d} \left| \mathbb{E}|\widehat{u}(\tau \cdot T_{\mathrm{kin}}, k)|^2 - n(\tau,k) \right| = 0$$

where n satisfies a kinetic equation (the resonant equation associated to the original NLS).

Wave turbulence

The main idea is to expand

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}(uu\overline{u})(s)\,ds = \ldots.$$

Our goal is to initiate a similar investigation for irrotational water waves models. There is a fundamental difference between water waves, which are quasilinear evolutions, and semilinear models: the solution cannot be constructed using the Duhamel formula, due to unavoidable loss of derivatives.

In particular, the solution cannot be described using iterations of the Duhamel formula (Feynman trees), which is the central idea of the analysis of semilinear models.

To address this fundamental difficulty we propose a new mechanism, based on a combination of two main ingredients.

Wave turbulence

(1) *Energy estimates:* We prove first estimates on the energy increment of the highest order derivatives of the solution. These are *deterministic* estimates, in the sense that they hold for all solutions. The key point is to show that the increment is controlled by an L^∞-based norm of the solution (not an L^2-based norm), which can then be exploited effectively in the analysis of solutions with random initial data.

(2) *Propagation of randomness and pointwise bounds:* The second step consists in proving propagation of randomness (in a suitable form), and using Khintchine-type arguments to derive optimal pointwise bounds on randomized solutions in a large box. For this one can use the iterated Duhamel formula; the point is that losses of derivatives are allowed at this stage, thanks to the accompanying estimate on the high order energy.

Our first main theorem provides a quintic a priori estimate on the increment of energy of solutions of the gravity water waves system in 2D ($d = 1$, $\sigma = 0$).

Main Theorem 1. Assume that $N_0 \geq 7$, $N_1 \in [4, N_0 - 2]$, $T_1 \leq T_2 \in \mathbb{R}$, and $(h, \phi) \in C([T_1, T_2] : H^{N_0} \times \dot{H}^{N_0,1/2})$ is a solution of the system

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -h - \dfrac{1}{2}|\nabla_x \phi|^2 + \dfrac{(G(h)\phi + \nabla_x h \cdot \nabla_x \phi)^2}{2(1 + |\nabla_x h|^2)}. \end{cases}$$

Moreover, assume that

$$\|h(t)\|_{H^{N_0}} + \||\partial_x|^{1/2}\omega(t)\|_{H^{N_0}} \leq A,$$
$$\|h(t)\|_{\dot{W}^{N_1,0}} + \||\partial_x|^{1/2}\omega(t)\|_{\dot{W}^{N_1,0}} \leq \varepsilon(t),$$

for any $t \in [T_1, T_2]$, where $\omega = \phi - T_B h$ is the "good unknown", $A \in (0, \infty)$, and $\varepsilon : [T_1, T_2] \to [0, \infty)$ is a continuous function satisfying $\varepsilon(t) \ll 1$.

Quintic energy estimates

Then

$$\big\|(h+i|\partial_x|^{1/2}\omega)(t_2)\big\|^2_{H^{N_0}} \lesssim \big\|(h+i|\partial_x|^{1/2}\omega)(t_1)\big\|^2_{H^{N_0}} + A^2 \int_{t_1}^{t_2} [\varepsilon(s)]^3\, ds,$$

for any $t_1 \leq t_2 \in [T_1, T_2]$.

Remarks and main ideas:

(1) *The quintic increment*. The point is that the energy increment is quintic, that is, the increment involves A^2 and the third power $[\varepsilon(s)]^3$ of the L^∞-based norm. For applications, in particular in the random data setting, it is important to distinguish these two quantities and allow A (the high Sobolev energy) to be large, while only $\varepsilon(t)$ (the controlling L^∞ norm) is small.
"Quartic" energy inequalities have been proved earlier, in both 1 and 2 dimensions for gravity waves. The key ingredient in these results is the absence of quadratic resonances.

(2) *Resonances.* At the linearized level, the water waves system is of the form

$$(\partial_t + i|\partial_x|^{1/2})U = \mathcal{N}(U, \overline{U}),$$

where $U = h + i|\partial_x|^{1/2}\phi$ and $\mathcal{N}(U, \overline{U})$ is a quasilinear nonlinearity. The main obstruction to proving a quintic energy inequality is the presence of quadratic and cubic resonances in the system.

There are no non-trivial quadratic resonances (or resonant three-waves interactions), i.e. there are no numbers $\xi_1, \xi_2, \xi_3 \in \mathbb{R} \setminus \{0\}$ solving

$$\sigma_1|\xi_1|^{1/2} + \sigma_2|\xi_2|^{1/2} + \sigma_3|\xi_3|^{1/2} = 0, \qquad \xi_1 + \xi_2 + \xi_3 = 0,$$

where $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$. This algebraic fact implies that the quadratic nonlinearities can be (at least formally) eliminated by normal forms. One can avoid losses of derivatives in normal form procedures and obtain a "quartic" energy inequality.

To improve the quartic estimate we need to look at "cubic resonances" (or resonant 4-waves interactions) that is, solutions of the system

$$\sum_{j=1}^{4} \sigma_j |\xi_j|^{1/2} = 0, \qquad \sum_{j=1}^{4} \xi_j = 0, \tag{1}$$

for $\xi_j \in \mathbf{R} \setminus \{0\}$ and $\sigma_j \in \{-1, 1\}$, $j = 1, \ldots 4$. This equation has a set of obvious solutions of the form $(\xi, \eta, -\xi, -\eta)$ with $\sigma_1 = \sigma_2 = 1$, $\sigma_3 = \sigma_4 = -1$, together with their natural permutations; these are the so-called "trivial resonances" and cannot be avoided. However, due to the Hamiltonian/reversible nature of the equations, these resonance give an integrable contribution and do not appear in the energy increment.

Quintic energy estimates

A simple calculation shows that (1) has another two parameter family of resonances, the so called Benjamin-Feir resonances, which for the case $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (+, -, +, -)$ are given by

$$\mathcal{R}_{BF} := \bigcup_{\lambda \in \mathbf{R} \setminus \{0\},\, b \in [0,\infty)} \Big\{ \xi_1 = \lambda b^2,\, \xi_2 = \lambda (b+1)^2, \\ \xi_3 = -\lambda (b^2 + b + 1)^2,\, \xi_4 = \lambda (b+1)^2 b^2 \Big\},$$

up to the natural symmetries. These resonances are a potential obstruction to proving quintic energy inequalities. The key point is that the contribution of these resonances to the energy increment also vanishes. This is due to the fact that the symbol of the cubic interactions vanishes on the set above.

This algebraic fact was already known (Dyachenko–Zakharov (1994)). Our main contribution is to use it to prove a rigorous quintic energy inequality, without derivatives losses.

The quintic energy inequality can be used for deterministic initial data with small Sobolev norm, to prove new extended lifespan results. More precisely we have:

Main Theorem 2 Let $N_0 = 7$ and assume that $(h_0, \phi_0) \in H^{N_0+1/2}(\mathbb{T}_R) \times H^{N_0+1/2}(\mathbb{T}_R)$ is initial data satisfying the assumptions

$$\| \langle \partial_x \rangle^{1/2} h_0 \|_{H^{N_0}} + \| |\partial_x|^{1/2} \phi_0 \|_{H^{N_0}} \leq \rho \leq 1.$$

Then there is a unique solution $(h, \phi) \in C([0, T_{R,\varepsilon}] : H^{N_0} \times \dot{H}^{N_0,1/2})$ of the gravity water waves system on $\mathbb{T}_R \times [0, T_{R,\varepsilon}]$, with initial data $(h(0), \phi(0)) = (h_0, \phi_0)$, where

$$T_{R,\varepsilon} \gtrsim \rho^{-3} \min \left(\rho^{-3}, R^{3/4} \right).$$

Moreover, letting $\omega = \phi - T_B h$ denote the good variable as before, the solution satisfies the bounds

$$\sup_{t \in [0, T_{R,\varepsilon}]} \left\| (h + i|\partial_x|^{1/2} \omega)(t) \right\|_{H^{N_0}} \lesssim \rho.$$

Remarks and main ideas:
(1) In the case of a fixed-size torus $R \approx 1$, our theorem gives a time of existence of order $O(\rho^{-3})$, matching the earlier results of Berti–Feola–Pusateri and Wu. This result is probably optimal for general Sobolev data due to the presence of resonant 5-wave interactions.

If $R \gtrsim \rho^{-4}$ or in the Euclidean case $R = \infty$, we obtain a time of existence $T_{R,\rho} \approx \rho^{-6}$. This improves substantially on all the previous results on 2D water waves with Sobolev initial data.

Recall that in the Euclidean case $R = \infty$, for localized initial data in a weighted Sobolev space, one can actually prove global regularity results, relying on dispersion and pointwise decay. However, localization is not compatible with randomized initial data, so these results are not applicable to the problem of understanding wave turbulence.

(2) Our proof of Theorem 2 is based on what we call *the Z-norm method*. This general method is essentially a bootstrap argument that combines high-order energy estimates with dispersive and decay estimates at a lower order of regularity. In our problem the key $Z = Z_{\varepsilon,R}$-norm is defined by

$$\|F\|_Z := \sup_{t_1 \le t_2 \in [0,T],\, |t_2 - t_1| \le \min(\varepsilon^{-4}, R)} \|F\|_{L^4_{[t_1,t_2]} \dot{W}_x^{N_1,0}},$$

for any function $F \in C([0, T] : H^{N_0}(\mathbb{T}_R))$, where $N_1 = N_0 - 2 = 4$. The main new ingredient is the Strichartz estimates

$$\|e^{-it|\partial_x|^{1/2}} P_k f\|_{L^4_t L^\infty_x(\mathbb{T}_R \times [-T,T])} \lesssim 2^{3k/8} \|f\|_{L^2(\mathbb{T}_R)},$$

provided that $k \in \mathbb{Z}$ and $T \le R2^{k^-/2}$.

The Strichartz estimates provide sharp $L^4_t L^\infty_x$ decay of linear solutions in Sobolev spaces and are efficient substitutes for Sobolev's embedding.

We consider the 2D irrotational gravity water wave problem

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -h - \frac{1}{2}(\partial_x \phi)^2 + \frac{(G(h)\phi + \partial_x h \cdot \partial_x \phi)^2}{2(1+(\partial_x h)^2)}. \end{cases}$$

We also denote the complex physical unknown

$$u := h + i|\partial_x|^{1/2}\phi.$$

We consider the system on the large one-dimensional torus $x \in \mathbb{T}_R$, with small random initial data of average size ϵ,

$$u(0,x) = u_{\mathrm{in}}(x) = \epsilon R^{-1/2} \sum_{k \in \mathbb{Z}/R} \sqrt{\psi_{\mathrm{in}}(k)}\eta_k(\omega) \cdot e^{ik\cdot x}. \tag{2}$$

We assume ψ is a fixed, rapidly decaying Lipschitz function on $\mathbb{R}$, and $\{\eta_k(\omega)\}$ are i.i.d. normalized complex Gaussians. Finally, we assume the *scaling law* $\epsilon = R^{-\alpha}$ where $\alpha \in (0, 3/8)$.

Random initial data: main theorem

We can now state our main result as follows.

Main Theorem 3: Fix $\alpha \in (0, 1/3)$ and $\theta > 0$, let $\epsilon = R^{-\alpha}$, and let $0 < \eta \ll 1$ be fixed depending on α and θ. Then, for R large enough, the water-waves system with randomized initial data has a smooth solution $u = u(t, x)$ on the time interval $[0, T_E]$ with probability $\geq 1 - e^{-R^\eta}$, where $T_E := \epsilon^{-8/3+\theta}$. Moreover the solution satisfies the estimate

$$\|u(t)\|_{W^{4,\infty}} \leq \epsilon^{1-(\theta/4)}, \quad \forall t \in [0, T_E].$$

The randomization of the data gives enhanced control on L^∞-type norms: for a solution with Sobolev norm of size A one can hope to prove that, with high probability, the L^∞-norm is of size $\varepsilon \approx AR^{-1/2}$, consistent with uniform distribution of mass over $\mathbb{T}_R$. The main issue is to propagate randomness (using Feynman's diagrams), and combine this with the energy estimate to construct regular solutions up to times of the order $\epsilon^{-8/3+}$.

Truncation

We construct an approximate solution w_{tr}

$$(\partial_t + i|\partial_x|^{1/2})w_{\mathrm{tr}} = \sum_{r=3}^{A} P_{\leq R_{\mathrm{tr}}}\big[\mathcal{N}_r(w_{\mathrm{tr}}, \cdots, w_{\mathrm{tr}})\big].$$

This equation is obtained by (1) performing a normal form to remove the quadratic nonlinearity, (2) eliminating the higher order nonlinearities, and (3) truncating all frequencies higher than $R_{\mathrm{tr}} \approx R^{1/M}$, from the original equation. The initial data is

$$w_{\mathrm{tr}}(0) = (w_{\mathrm{tr}})_{\mathrm{in}} = P_{\leq R_{\mathrm{tr}}} w_{\mathrm{in}}.$$

The semilinear problem

Proposition: Suppose for some time $t \leq T_E$ we have

$$\sup_{t' \in [0,t]} \left(\|w(t')\|_{W^{8,\infty}} + \|w_{\mathrm{tr}}(t')\|_{W^{8,\infty}} \right) \leq \epsilon^{1-(\theta/4)},$$

$$\sup_{t' \in [0,t]} \|w(t')\|_{H^M} \leq \epsilon^{1-(\theta/4)} R^{1/2},$$

Then

$$\sup_{t' \in [0,t]} \|w(t') - w_{\mathrm{tr}}(t')\|_{W^{4,\infty}} \leq R^{-A/4}.$$

So it remains to construct solutions w_{tr} of the equation

$$(\partial_t + i|\partial_x|^{1/2}) w_{\mathrm{tr}} = \sum_{r=3}^{A} P_{\leq R_{\mathrm{tr}}} \left[\mathcal{N}_r(w_{\mathrm{tr}}, \cdots, w_{\mathrm{tr}}) \right].$$

on a time interval $[0, \epsilon^{-3+}]$. We would like to use the Duhamel formula and Feynman diagrams expansions, as is we were working on the cubic NLS

$$(\partial_t - i\Delta) = -iu|u|^2.$$

The semilinear problem

- The loss of derivative is R_{tr}, which is small compared to ϵ^{-1}.

The quartic and higher order nonlinearities are not worse than the cubic nonlinearity.

- The linear dispersion function is $|\xi|^{1/2}$ instead of $|\xi|^2$. This is potentially problematic when counting quasi-resonances, for example solutions of the inequality

$$\pm|\xi_1|^{1/2} \pm \ldots \pm |\xi_m|^{1/2} \in [a, b],$$

for $\xi_1, \ldots, \xi_m \in \mathbb{Z}/R$. This requires a restriction on R and $|b-a|$ that guarantees that the resulting sets of quasi-resonances are "thick", i.e. $R|b-a| \gtrsim 1$. This is related to the assumed scaling law $\epsilon = R^{-\alpha}$, $\alpha \in (0, 3/8)$, which guarantees that $T_E \ll R$.

The semilinear problem

- A significant difficulty is due to the renormalization to eliminate the trivial cubic resonances. In the cubic NLS case these resonances correspond to the cubic term

$$\frac{1}{(2\pi R)^2}\sum_{k_1\in\mathbb{Z}/R}\widehat{u}(k,t)\widehat{u}(k_1,t)\widehat{\overline{u}}(-k_1,t)$$

which can eliminated easily by conjugation with an oscillatory factor of the form e^{itC}, $C = C(\psi)$. In our case, however, we need a more complicated renormalization of the form

$$a_{\mathrm{tr}}(t,k) := \epsilon^{-1}e^{iT_E(|k|^{1/2}t+\Gamma(t,k))}\cdot\widehat{w_{\mathrm{tr}}}(T_E\cdot t,k),$$

for a suitable (deterministic) function Γ that needs to be constructed.

- Another significant difficulty is due to the randomness assumption on the data of the physical variable u. However, the semilinear analysis is performed using the normal form variable

$$w \approx u + (u + \overline{u}) * \nabla(u + \overline{u})$$

on which we don't have as good randomness information as on u. To capture this issue, we define trees with two types of branching nodes: the interaction I-branching nodes and normal form N-branching nodes.

The semilinear problem

- Another significant difficulty is due to the fact that we work in dimension 1, which leads to worse counting estimates.

The wave turbulence theory is not settled in dimension 1 even for the MMT equation with fractional dissipation and pure cubic nonlinearity. A recent result in this case was obtained by Vassilev, with a time of existence of $\epsilon^{-5/2+}$.

In our case, the final time of existence is $T_E \approx \epsilon^{-8/3+}$ due to certain obstructions involving long chains of interactions.

Conclusions

- The wave turbulence theory of irrotational water waves in 2D and 3D is wide open and substantially more difficult than the theory in the case of semilinear models.

- The Duhamel formula (or Feynman diagrams) cannot be used to describe the solution due to loss of derivative.

- The formal wave kinetic equation is probably ill-posed, again due to loss of derivative.

- We use "time of existence" as a proxy for the wave turbulence theory. This avoids some subtle issues, such as what is the canonical variable to measure (the physical variable or the normal form variable or some other variable) and what is the right definition for the size of the Fourier coefficients "$|\widehat{u}(\xi)|^2$".