The Hilbert Sixth Problem: Particles and Waves

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Hilbert's Sixth Problem

- Hilbert's Sixth Problem: **Mathematical Treatment of the Axioms of Physics**

- In original list of problems in Hilbert's ICM lecture 1900, expanded later in *Bull. Amer. Math. Soc.* (1902)

D. Hilbert

Scope of the problem

This topic is extremely broad, and may contain most of modern mathematical physics if widely interpreted.

Hilbert later mentioned some specific problems in the 1902 *Bulletin* article. The most prominent one, still unsolved, pertains to Boltzmann's kinetic theory:

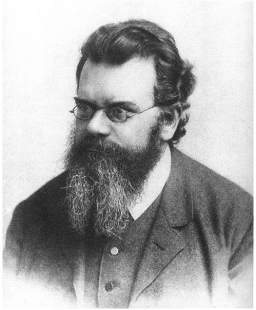
> be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua. Conversely one might try to

From atomistic to continua

Such limiting process consists of two steps:

- the **kinetic limit** that goes from particle systems to the Boltzmann (kinetic) equation; and
- the **hydrodynamic limit** that goes from the Boltzmann equation to the fluid equations.

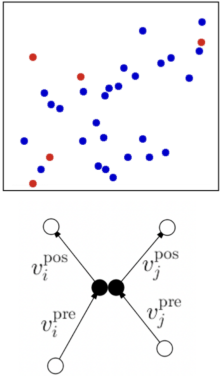
In this talk we will focus on the first kinetic limit, with the aim of deriving the **Boltzmann equation**.



L. Boltzmann

The kinetic limit

- Consider N particles (atoms, molecules etc.) in a fixed 3 dimensional domain, with $N \to \infty$ in the kinetic limit.
- Assume these particles are hard spheres of diameter $\varepsilon \to 0$, and interact only via elastic collisions (see next page).
- We assume the **Boltzmann-Grad scaling** relation $N\varepsilon^2 = 1$ (i.e. mean free path ~ 1).



Particles and collisions

Hard sphere system

The hard sphere dynamics can be described as follows (with $z_j = (x_j, v_j)$, x_j being position and v_j being velocity)

$$\begin{cases} \partial_t x_i = v_i, \quad \partial_t v_i = 0, & \text{if } |x_i - x_j| > \varepsilon; \\ v_i^{\text{pos}} = v_i^{\text{pre}} - ((v_i^{\text{pre}} - v_j^{\text{pre}}) \cdot \omega)\omega, & \\ v_j^{\text{pos}} = v_j^{\text{pre}} + ((v_i^{\text{pre}} - v_j^{\text{pre}}) \cdot \omega)\omega, & \text{if } |x_i - x_j| = \varepsilon, \\ \quad \omega = \varepsilon^{-1}(x_i - x_j). & \end{cases}$$

With a few technical manipulations, this can be defined as a time reversible, and volume preserving dynamical system.

Statistical viewpoint

We shall take the **statistical point of view**, which is represented by the ***s*-particle density functions** $f_s(t, z_1, \cdots, z_s)$.

- These density functions evolve in time deterministically, under the hard sphere system (via Liouville's equation).
- Initially ($t = 0$), they are **factorized** into the one-particle density function $f_{\text{in}} = f_{\text{in}}(x, v)$.
- Question: how do we describe the time evolution of these density functions, in the limit $N \to \infty$?

Propagation of chaos

- Note that, the initial factorization (at $t = 0$) indicates the independence between the states of different particles.

- It is physically intuitive, although mathematically highly nontrivial, that such independence should be preserved in the limit $N \to \infty$, even with time evolutions.

- This is referred to as the **stosszahlansatz**, or **propagation of chaos**, in the physics literature.

The Boltzmann equation

Assuming propagation of chaos, then the time $t > 0$ statistics is determined by the one-particle density $f = f(t, x, v)$.

This one-particle density is expected to satisfy the **Boltzmann equation**

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{Q}(f), \quad f(0) = f_{\mathrm{in}},$$

$$\mathcal{Q}(f)(t,x,v) = \int_{(\mathbb{R}^3)^3} \delta(v - v_1 + v_2 - v_3) \times \delta(|v|^2 - |v_1|^2 + |v_2|^2 - |v_3|^2) \underset{\sim\sim\sim\sim\sim\sim}{(f_1 f_3 - f f_2)} \, \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}v_3$$

where $f_j := f(t, x, v_j)$.

Timeline (before 1975)

- Boltzmann 1870s: obtained the Boltzmann equation, under the propagation of chaos assumption.
- Grad 1949: first precise mathematical formulation of the kinetic limit.
- Grad, Cercignani 1950s: introduced the BBGKY hierarchy to the hard sphere system.
- Lanford, King 1975: first rigorous derivation of Boltzmann equation, **for short time** $|t| \ll 1$.

Timeline (after 1975)

- Illner-Pulvirenti 1986: obtained long-time extensions in the **near vacuum case** (i.e. $N\varepsilon^{d-1} \ll 1$)
- Gallagher-Saint Raymond-Texier 2014: fixed gap in the 1975 proof of Lanford, obtained explicit convergence rates.
- Bodineau-Gallagher-Saint Raymond 2016+: derivation of various linear models based on a **linearized** Boltzmann equation.
- Bodineau-Gallagher-Saint Raymond-Simonella 2016+: derived fluctuations around Boltzmann solutions, which satisfy **linearized** stochastic Boltzmann equations.

The long time problem

Note that all the previous results are restricted to either **short time**, or other **perturbative** settings.

However, in the context of Hilbert's Sixth Problem, to match the hydrodynamic limit, we need to extend the derivation of the Boltzmann equation to **non-perturbative settings**, as stated in the following question:

Question

Can we derive the Boltzmann equation from hard sphere dynamics, for **arbitrarily long time**, as long as the solution to the Boltzmann equation exists?

We answer this question positively in our main theorem.

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Hilbert's Sixth Problem for waves

At the time of Hilbert's 1900 lecture, the theory of **quantum physics** was still at its infancy.

With the advancement of quantum mechanics in the 20th century, especially the wave-particle duality, physicists have soon developed the parallel statistical theory for waves, i.e. the **wave kinetic theory** or **wave turbulence**.

Naturally, the same question corresponding to the Hilbert's Sixth Problem for particles, has also been asked in quantum and wave settings.

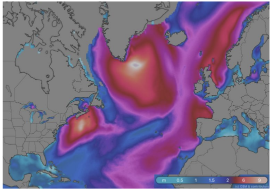
Wave turbulence in the 20th century

- The role of the Boltzmann equation is replaced by that of **wave kinetic equations**.
- The theory started with the works of Nordheim (1928), Peirels (1929) and Uehling-Uhlenbeck (1933).
- Fundamental contributions made by Hasselman (1962, 1963; water waves) and Zakharov (1965; KZ spectrum)

V. E. Zakharov

Wave turbulence in applications

The wave turbulence theory has been applied to plasma theory, oceanography, crystal thermodynamics etc.

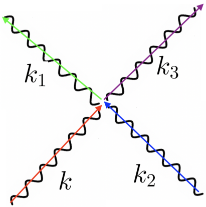


Left: ocean wave forecasting (source: Simons Collaboration on Wave Turbulence). Right: turbulent plasma (source: NASA, ESA, J. Hester, A. Loll (ASU)).

Wave turbulence setup

We now turn to the description of the wave turbulence setup.

- **Particles** are replaced by (homogeneous) **Fourier modes** or (inhomogeneous) **wave packets** in a large (periodic) box of size L;
- Degree of freedom is $\sim L^3$;
- Collisions are replaced by nonlinear interactions, modeled by nonlinear dispersive equations, of strength α.



Four wave interactions: k_j are wave numbers.

The cubic Schrödinger equation

The choice of nonlinear dispersive equation is arbitrary; a canonical case is the **cubic Schrödinger equation**

$$(i\partial_t + \Delta)u = \alpha|u|^2 u$$

in the large periodic box $[0, L]^3$.

- Strength of nonlinearity $\alpha \to 0$ as $L \to \infty$;
- We will be dealing with **homogeneous** cases, which leaves some flexibility in the **scaling relation** between α and L: we assume $\alpha L^\gamma = 1$ for $\gamma \in (0, 1]$.

Statistical ensemble

As in the particle setting, we assume the Fourier modes of the initial data are independent:

$$u(0) = u_{\text{in}} = L^{-3/2} \sum_k \widehat{u_{\text{in}}}(k) e^{2\pi i k \cdot x}, \quad \widehat{u_{\text{in}}}(k) = \sqrt{n_{\text{in}}(k)} \cdot \eta_k.$$

- Here $k \in \mathbb{Z}_L^3 = L^{-1}\mathbb{Z}^3$ and n_{in} is Schwartz, and η_k i.i.d. normalized Gaussian r.v.
- The role of one-particle and s-particle densities in the Boltzmann case are played by two-point and $2s$-point correlations.

Propagation of chaos

- Define the **kinetic time** $T_{\text{kin}} = \alpha^{-2}$; this corresponds to the mean free path in kinetic theory.
- As in the particle setting, we expect propagation of chaos/stosszahlansatz, i.e. independence of Fourier modes is preserved in the limit.
- In other words, for any $\tau > 0$, and any different $k_1, \cdots, k_r$, we expect the r.v.

$$\widehat{u}(\tau \cdot T_{\text{kin}}, k_j), \quad 1 \leq j \leq r,$$

to be asymptotically independent in the limit $L \to \infty$.

The wave kinetic equation

Assuming propagation of chaos, then the general $2r$ point correlations are determined by the two-point correlation

$$n(\tau,k) := \mathbb{E}|\widehat{u}(\tau\cdot T_{\mathrm{kin}},k)|^2.$$

This is expected to satisfy the **wave kinetic equation**

$$\partial_\tau n = \mathcal{C}(n), \quad n(0) = n_{\mathrm{in}},$$

$$\mathcal{C}(n)(\tau,k) = \int_{(\mathbb{R}^3)^3} \delta(k-k_1+k_2-k_3)\delta(|k|^2-|k_1|^2+|k_2|^2-|k_3|^2) \\ \times \underset{\sim\sim\sim}{(n_1n_3(n+n_2)-nn_2(n_1+n_3))}\,\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

where $n_j := n(\tau,k_j)$.

Remarks on the WKE

- We are in the homogeneous setting so there is no transport term $k \cdot \nabla_x n$; the inhomogeneous setting is conceptually similar.
- The collision integral here is cubic instead of quadratic as is for Boltzmann, but with the same domain of integration (which is specific for Schrödinger).
- We remark that, the collision integral for the **quantum Boltzmann equation** is just a linear combination of these two quadratic and cubic collision integrals.

Mathematical works

- Spohn 1977, Erdös-Yau 2000, Erdös-Salmhofer-Yau 2008: linear Schrödinger, random potential.
- Lukkarinen-Spohn 2011: nonlinear, but at equilibrium.
- Buckmaster-Germain-Hani-Shatah 2019: nonlinear, off-equilibrium, but with time scale $\ll T_{\mathrm{kin}}^{1/2}$.
- D.-Hani, Collot-Germain 2019: same as above, but with time scale $T_{\mathrm{kin}}^{1-\varepsilon}$.
- D.-Hani 2021–2023: reaches the kinetic time scale $\tau \cdot T_{\mathrm{kin}}$ but **for short time** $|\tau| \ll 1$. (See also Staffilani-Tran 2021 for a result with stochastic forcing.)

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Main result 1: wave kinetic equation

In the first main theorem, we obtain the long-time derivation result for the wave kinetic equation, from the cubic Schrödinger equation:

Theorem 1 (D.-Hani 2023, arXiv:2311.10082)

Suppose the solution $n(\tau, k)$ to the wave kinetic equation exists on **any given time interval** $[0, \tau_*]$. Then we have

$$\mathbb{E}|\widehat{u}(\tau \cdot T_{\mathrm{kin}}, k)|^2 = n(\tau, k) + O(L^{-\nu}), \quad \nu > 0,$$

and all supplementary results (propagation of chaos etc.), **on the full interval** $[0, \tau_*]$.

Main result 2: Boltzmann equation

In the second main theorem, we obtain the same long-time derivation result for the Boltzmann equation, from the hard sphere dynamics (i.e. the first limit process in Hilbert's Sixth Problem):

Theorem 2 (D.-Hani-Ma 2024, arXiv:2408.07818)

Suppose the solution $f(t,x,v)$ to the Boltzmann equation exists on **any given time interval** $[0,t_*]$. Then we have

$$(s\text{-particle density}) = \prod_{j=1}^{s} f(t,x_j,v_j) + O_{L^1_{x,v}}(\varepsilon^\nu), \quad \nu > 0,$$

on the full interval $[0,t_*]$. Here the limit is taken in $L^1_{x,v}$ space with fixed t.

Remarks on the main results

- If the WKE (or Boltzmann equation) has global bounded solutions, Theorems 1 and 2 can be extended to the rescaled time $\tau_* \sim \log\log L$ (or time $t_* \sim \log\log N$), which **diverges** (slowly) in the kinetic limit.
- Theorem 2 will be extended to the periodic case $x \in \mathbb{T}^d$, with minor modifications, in a subsequent work.
- The inhomogeneous (or Hartree interaction) case in Theorem 1, as well as the case of general (short range) potentials in Theorem 2, is also expected to be true with the same ideas, but technically more involved.

Ideas of the proof (I)

The overall strategy of the proof in Theorems 1 and 2:

- Time layering: divide the long time interval $[0, t_*]$ into $\mathfrak{L} \gg 1$ time layers, each of length $\tau := \mathfrak{L}^{-1} t_* \ll 1$.
- Cumulant expansion: by inducting on the layer number ℓ, prove a **cumulant expansion formula**

$$(s\text{-particle density}) = \sum_{H \subset \{1, \cdots, s\}} \prod_{j \notin H} f(t, x_j, v_j) \cdot (\text{cumulants})$$

at time $t = \ell\tau$, where the (cumulants) are given by expansions of **interaction history diagrams** on $[0, t]$.

Ideas of the proof (II)

The idea of controlling the **interaction history diagrams**:

- Analytical component: reduce this estimate to a **combinatorial property** of the diagram, by bounding the associated integrals.
    - ★ This is easier for Boltzmann (Theorem 2), but requires delicate cancellations for WKE (Theorem 1).
- Combinatorial component: devise a **cutting algorithm** to prove the desired combinatorial property.
    - ★ This is easier for WKE (Theorem 1), but requires a complicated combination of algorithms for Boltzmann (Theorem 2).

Future horizon (I)

- Derivation of the **quantum Boltzmann equation**: does is depend on the microscopic model? quadratic or quadratic-cubic?
- Turbulence/cascade and formation of condensates: emergence of **KZ spectrum**? formation of δ singularity?
- Quasilinear equations and **1D turbulence**: water wave, MMT equations, FPUT model etc., where **anomalies** happen in 1D? what if the WKE is locally ill-posed (Ampatzoglou-Leger 2024)?

Future horizon (II)

- Connecting kinetic and hydrodynamic limits: in what range can we justify the predictions of **Hilbert's Sixth Problem**? what about other scaling laws?
- Derivation of the **Landau equation**: how to deal with many collisions in unit time? is there a CLT type phenomenon happening?