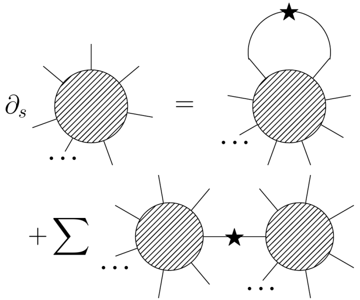
# Space-Time Dependence of Correlation Functions in Turbulence and Related Models




**Léonie Canet**

Simons Collaboration on Wave Turbulence Annual Meeting 2024

# Presentation outline

1. Fully developed homogeneous isotropic turbulence
   - Why (Functional) Renormalisation Group for turbulence ?
   - Time dependence of correlation functions
   - Comparison with direct numerical simulations
   - Hints of the derivation

2. The stochastic Burgers–Kardar-Parisi-Zhang equation
   - Unpredicted scaling in the inviscid limit
   - Functional RG for Burgers–KPZ equation
   - Inviscid Burgers fixed point

# Acknowledgments

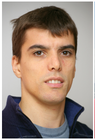
## Collaborators



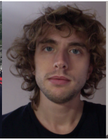



| B.Delamotte | N.Wschebor | G.Balarac | V.Rossetto | M.Brachet |
|---|---|---|---|---|
| Paris | Montevideo | Grenoble | Grenoble | Paris |

## PhD students and post-docs





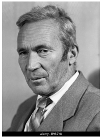



C.Pagani M.Tarpin A.Gorbunova C.Fontaine F.Vercesi L.Gosteva

## References


M. Tarpin, LC, N. Wschebor, Phys. Fluids **30** (2018)
A. Gorbunova, G. Balarac, LC, G. Eyink, V. Rossetto, Phys. Fluids **33** (2021)
C. Pagani, LC, Phys. Fluids **33** (2021)
A. Gorbunova, C. Pagani, G. Balarac, LC, V. Rossetto, Phys. Rev. F **6** (2021)
LC, J. Fluid Mech. *Perspectives* **950** (2022)
C. Fontaine, F. Vercesi, M. Brachet, LC, PRL **131** (2023)
L. Gosteva, M. Tarpin, N. Wschebor, LC, PRE **110** (2024)

# Phenomenology of 3D turbulence
# Kolmogorov statistical theory ...and multi-fractality



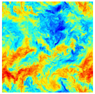
Kolmogorov theory 1941

A.N. Kolmogorov

Dokl.Akad.Nauk.SSSR **30, 31, 32** (1941)

dimensional analysis:

- eddy size $r \sim k^{-1}$
- energy flux $\epsilon$



numerical simulations and experiments

$$S_p(r) \sim r^{\zeta_p} \;\; \text{but} \quad \boxed{\zeta_p \neq p/3}$$

▶ multi-fractality, intermittency

U. Frisch, *Turbulence*, Cambridge University Press

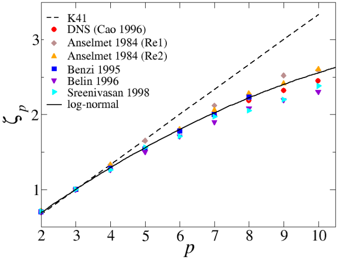
▶ structure functions

$$S_p(r) \equiv \left\langle \left( (\mathbf{v}(\mathbf{r}_0 + \mathbf{r}) - \mathbf{v}(\mathbf{r}_0)) \cdot \hat{\mathbf{r}} \right)^p \right\rangle$$

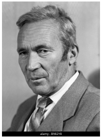
$$S_p(r) = C_p \, \epsilon^{p/3} \, r^{p/3} \qquad \text{(K41)}$$

$$S_3(r) = -\frac{4}{5} \, \epsilon \, r \qquad \text{(exact)}$$

▶ anomalous exponents



# Phenomenology of 3D turbulence
# Kolmogorov statistical theory . . . and random sweeping



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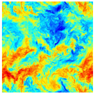
dimensional analysis:

- eddy size $r \sim k^{-1}$
- energy flux $\epsilon$

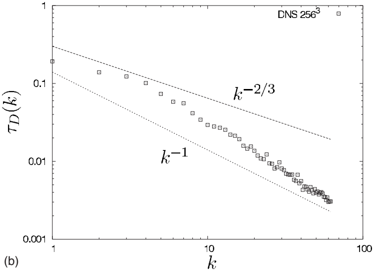
▶ decorrelation time scale in Eulerian framework

$$\tau_D \sim \epsilon^{-1/3} k^{-2/3} \quad \text{(K41)}$$



numerical simulations and experiments

$$\boxed{\tau_D \sim k^{-1}}$$




▶ random sweeping effect

Onsager, Z. Physik (1948), Tennekes, JFM (1975)

Favier, Godeferd, Cambon, Phys. Fluids **22** (2010)

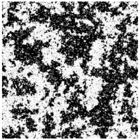
# Statistical theory of turbulence
# Why the Renormalisation Group (RG) ?

- many similarities between critical phenomena
  and homogeneous isotropic turbulence

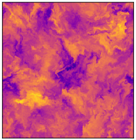
Nelkin, Phys. Rev. A **9** (1974), Rose, Sulem, J. de Phys. **39** (1978)

Eyink, Goldenfeld, Phys. Rev. E **50** (1994), Gawędzki, Nucl. Phys. B **58** (1997)

  - scale invariance, self-similarity
  - universality
  - anomalous critical exponents



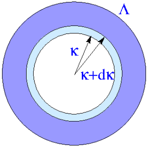
2D Ising magnet



turbulence

- arise from fluctuations at all scales

**Wilson's RG invented for critical phenomena**




  - progressive integration of fluctuation modes
  - build effective theory

Wilson, Kogut, Phys. Rep. C **12** (1974)

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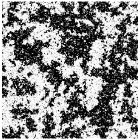

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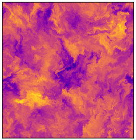


turbulence

Wilson's RG invented for critical phenomena

scale invariance $\Longleftrightarrow$ fixed point of the RG

# Renormalisation Group: Path integral representation of stochastic Navier-Stokes equation

- stochastic Navier-Stokes equation

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha = f_\alpha \qquad \text{with} \qquad \partial_\alpha v_\alpha = 0$$

$\mathbf{f}$: Gaussian random forcing of zero mean and covariance

$$\left\langle f_\alpha(t,\mathbf{x}) f_\beta(t',\mathbf{x}') \right\rangle = 2\delta_{\alpha\beta}\delta(t-t') N_L(|\mathbf{x}-\mathbf{x}'|)$$

- Path integral for stochastic Navier-Stokes equation

Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

LC, J. Fluid Mech. **950** (2022)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{v}\,\mathcal{D}\bar{\mathbf{v}}\,\mathcal{D}\pi\,\mathcal{D}\bar{\pi}\; e^{-\mathcal{S}_{\rm NS} + \text{ source terms}}$$

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \bar{v}_\alpha \underbrace{\left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha\right]}_{\text{deterministic}} + \bar{\pi}\underbrace{\left[\partial_\alpha v_\alpha\right]}_{\text{constraint}} - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \underbrace{N_L(|\mathbf{x}-\mathbf{x}'|)}_{\text{noise}}\,\bar{v}_\alpha$$

# Why Functional Renormalisation Group ?

▶ Original Wilsonian RG intimately linked with the "$\varepsilon$-expansion"

Wilson, Kogut, *The RG and the $\varepsilon$-expansion* , Phys. Rep. C **12** (1974)

$\cdots$ but for stochastic Navier-Stokes equation

$$\begin{cases} \partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = -\frac{1}{\rho}\partial_\alpha \pi + \nu \nabla^2 v_\alpha + f_\alpha \\ \left\langle f_\alpha(t,\mathbf{x}) f_\beta(t',\mathbf{x}') \right\rangle = 2\delta_{\alpha\beta}\delta(t-t') N_L(|\mathbf{x}-\mathbf{x}'|) \end{cases}$$

▷ absence of a small expansion parameter $\varepsilon$

▷ introduction of an $\varepsilon$ via forcing covariance $N_L(\mathbf{k}) \propto k^{d-\varepsilon}$

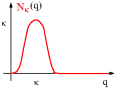
de Dominicis, Martin, PRA **19** (1979), Fournier, Frisch, PRA **28** (1983), Yakhot, Orszag, PRL **57** (1986)

- 3D kinetic energy spectrum $E(k) \propto k^{1-2\varepsilon/3} \Longrightarrow$ K41 scaling for $\varepsilon \to 4$ !
- "freezing" mechanism should occur for $\varepsilon > 4$ for universality

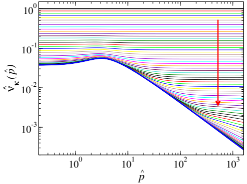
modern implementation of Wilson's RG:
the functional (non-perturbative) RG

# Functional Renormalisation Group for Navier-Stokes: RG fixed point for large-scale forcing

- RG fixed point for 3D homogeneous isotropic stationary turbulence
  - for a large-scale forcing
  - simple approximation




renormalised viscosity $\nu \longrightarrow \nu_\kappa(p)$



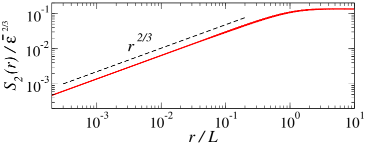

Tomassini, Phys. Lett. B **411** (1997)

Mejía-Monasterio, Muratore-Ginnaneschi, PRE **86** (2012)

LC, Delamotte, Wschebor, PRE **93** (2016)

LC, J. Fluid Mech. *Perspectives* **950** (2022)

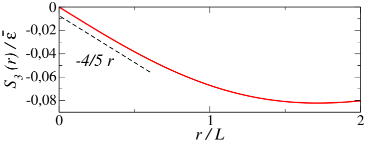
statistical properties: universal and with K41 scaling




$S_2(r) \sim C_2(\bar{\epsilon}r)^{2/3}$ FRG: $C_2 \simeq 2.06$

exp.: $C_2 \simeq 2.0 \pm 0.4$

K.R. Sreenivasan, Phys. Fluids **7** (1995)




$S_3(r) \sim C_3(\bar{\epsilon}r)$ FRG: $C_3 = -0.80$

exact: $C_3 = -4/5$

LC, J. Fluid Mech. *Perspectives* **950** (2022)

# Functional Renormalisation Group for Navier-Stokes: Time dependence of generic $n$-point correlation functions

- space-time $n$-point connected correlation functions

$$C^{(n)}_{\alpha_1 \ldots \alpha_n}(\{t_i, \mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1, \mathbf{x}_1) \cdots v_{\alpha_n}(t_n, \mathbf{x}_n) \Big\rangle_c$$

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- FRG: exact asymptotic behaviour in the limit of all $|\mathbf{k}_i|$ large

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau^2}\left|\sum_\ell \mathbf{k}_\ell t_\ell\right|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \ll \tau \\ \exp\left(-\alpha_\infty \frac{L^2}{\tau}\,|t|\,\sum_{k\ell}\mathbf{k}_k\cdot\mathbf{k}_\ell + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \gg \tau \end{cases}$$

  - for $C^{(2)}$, at small times $\tau_D \propto k^{-1} \neq k^{-2/3} \Longrightarrow$ random sweeping
  - rigorous and generalised for any $n$-point correlations
  - prediction of a new regime at large time

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   - Unpredicted scaling in the inviscid limit
   - Functional RG for Burgers–KPZ equation
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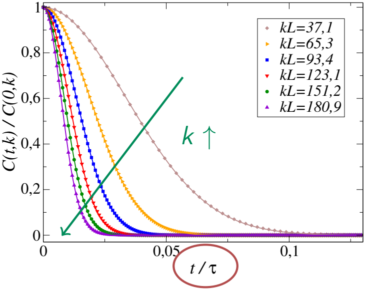
# Two-point correlation function at large wave numbers
# Small time delays: random sweeping effect

- result from functional renormalisation group (FRG):

$$C(t,\mathbf{k}) = C(0,\mathbf{k})\ \underbrace{\exp\left(-\alpha_0\,(L/\tau)^2\,(kt)^2\right)}_{\text{Gaussian in } kt}$$

- results from direct numerical simulations (DNS):



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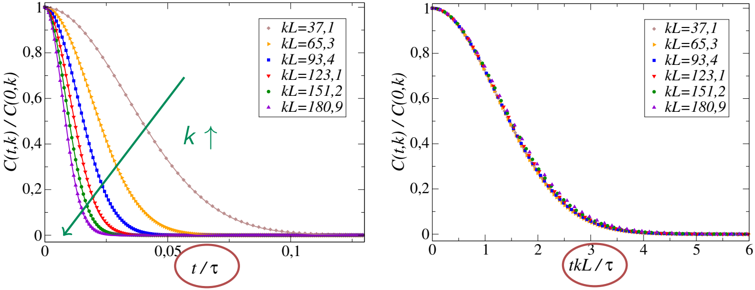
$\Longrightarrow$ $C(t,\mathbf{k})/C(0,\mathbf{k})$ should collapse onto a single Gaussian against $kt$

- results from direct numerical simulations (DNS):



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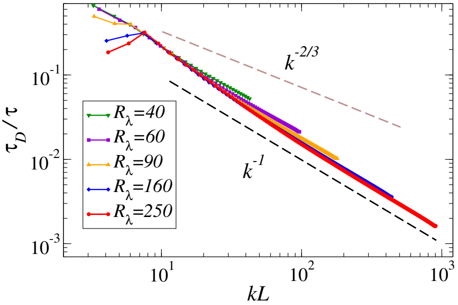
- decorrelation time $\tau_D$
  - from Gaussian fit $\exp(-(t/\tau_D)^2)$
  - from FRG:

    $$\tau_D = (\sqrt{\alpha_0} U_{\mathrm{rms}} k)^{-1}$$

  - similar to

    Favier, Cambon *et al*, Phys. Fluids **22** (2010)




  - implies frequency spectrum $E(\omega) \sim \omega^{-5/3}$

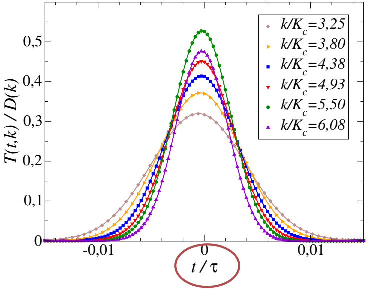
    Chevillard, Roux, Leveque, Mordant, Pinton, Arneodo, PRL **95** (2005).

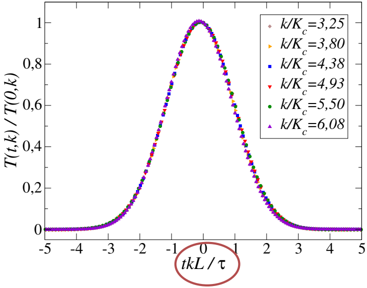
# Three-point correlation function at large wave numbers
# Small time delays

- advection-velocity correlation function from FRG

$$T(t,\mathbf{k}) = -ik_n P^{\perp}_{\ell m} \sum_{\mathbf{k}'} C^{(3)}_{mn\ell}(t,\mathbf{k}',t,\mathbf{k}-\mathbf{k}')\Big|_{\text{large }\mathbf{k}'} \propto \exp\left(-\alpha_0 U^2_{\mathrm{rms}}|\mathbf{k}|^2 t^2\right)$$

- results from direct numerical simulations



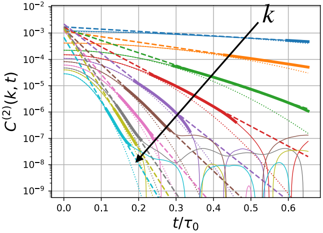

$\Longrightarrow$ same coefficient $\alpha_0$ as for two-point function

Gorbunova, Balarac, LC, Eyink, Rossetto, Phys. Fluids **33** (2021)

# What about large time delays ?

$$C(t,\mathbf{k}) = C(0,\mathbf{k}) \underbrace{\exp\left(-\alpha_\infty\,(L^2/\tau)\,k^2|t|\right)}_{\text{exponential in } k^2 t}$$

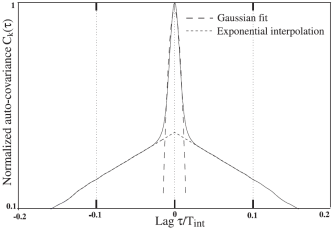
- hard to observe in DNS of Navier-Stokes flow




Gorbunova, Balarac, LC, Eyink, Rossetto,

Phys. Fluids **33** (2021)

- experiment of turbulence air jet




but correlations of modulus

Poulain, Mazellier, Chevillard, Gagne, Baudet,

Eur. Phys. J. B **53** (2006)

# Passively advected scalars in turbulent flows

▶ diffusion-advection equation

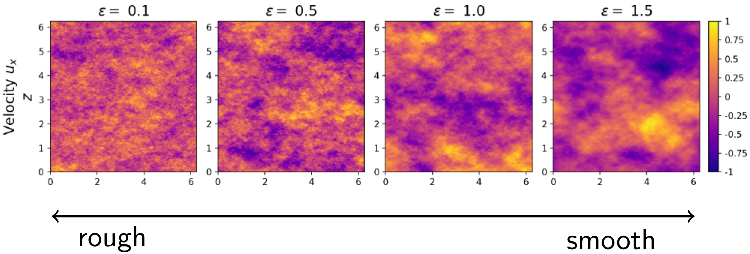
$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta - \kappa \nabla^2 \theta = f$$

**v** synthetic random field Kraichnan, Phys. Fluids **11** (1968)

$$\left\langle \hat{v}_i(t, \mathbf{k}) \hat{v}_j(t', -\mathbf{k}) \right\rangle = P_{ij}^{\perp}(\mathbf{k}) \frac{D_0}{k^{d+\varepsilon}} T_\tau(t - t')$$

©Walter Baxter

$\tau$: finite correlation time, Kraichnan limit $T_\tau(t - t') \xrightarrow{\tau \to 0} \delta(t - t')$



# Correlations in time-correlated Kraichnan model

- result from functional renormalisation group

$$C_\theta(t,\mathbf{k}) \propto \begin{cases} \exp\Big(-\gamma_0 (L/\tau)^2\,(tk)^2\Big) & |t| \ll \tau \\ \exp\Big(-\gamma_\infty (L^2/\tau)\,|t|\,k^2\Big) & |t| \gg \tau \end{cases}$$

Pagani and LC

Phys. Fluids **33** (2021)

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Pagani and LC

Phys. Fluids **33** (2021)

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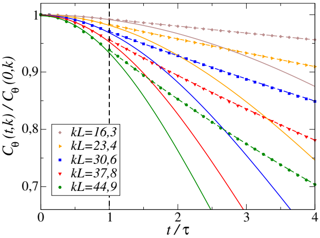
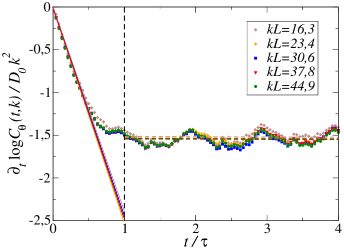



  - Gaussian at $|t| \leq \tau$
  - exponential at $|t| \geq \tau$

$$\frac{1}{k^2}\frac{d}{dt}\log\left[C_\theta(t,\mathbf{k})\right] \propto \begin{cases} -2\gamma_0 (L/\tau)^2\, t \\ -\gamma_\infty (L^2/\tau) \end{cases}$$

Gorbunova, Pagani, Balarac, LC, Rossetto, Phys. Rev. F **6** (2021)

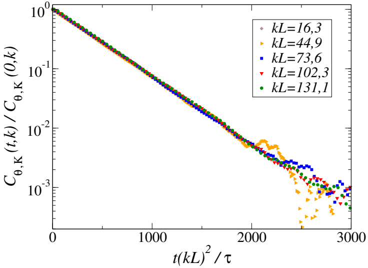
# Correlations in delta-correlated Kraichnan model $\tau \to 0$

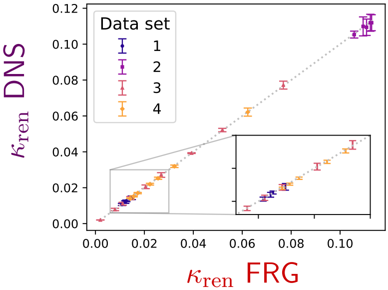
- result from functional renormalisation group (FRG)

$$C_\theta(t,\mathbf{k}) \propto \exp\left(-\kappa_{\rm ren} k^2 |t|\right) , \quad \kappa_{\rm ren} = \kappa + \frac{1}{3}\int_{\mathbf{k}} \frac{D_0}{(k^2+m^2)^{(3+\varepsilon)/2}} d^3\mathbf{k}$$

Pagani, LC, Phys. Fluids **33** (2021), similar to Kraichnan, PRL **72** (1994), Mitra, Pandit, PRL **95** (2005)

- results from direct numerical simulations (DNS)



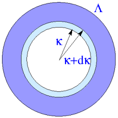




Gorbunova, Pagani, Balarac, LC, Rossetto, Phys. Rev. F **6** (2021)

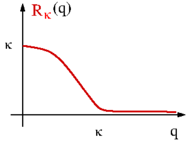
# Presentation outline

# Functional Renormalisation Group

- based on Wilson's RG ideas




  - progressive integration of fluctuation modes
  - Effective average action $\Gamma_\kappa$ instead of effective action $\mathcal{S}_\kappa$




- separation of fluctuation modes

$$\mathcal{Z}_\kappa = \int \mathcal{D}\varphi \, e^{-\mathcal{S}[\varphi] - \Delta\mathcal{S}_\kappa[\varphi] + \int_{t,\mathbf{x}} J\varphi} \quad \text{with } \Delta\mathcal{S}_\kappa = \frac{1}{2}\int_{\mathbf{q}} \varphi R_\kappa(\mathbf{q})\varphi$$

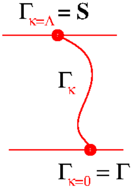
- effective average action: Legendre transform of $\mathcal{W}_\kappa = \ln \mathcal{Z}_\kappa$

$$\Gamma_\kappa[\psi] + \Delta\mathcal{S}_\kappa[\psi] = -\mathcal{W}_\kappa[J] + \int_{t,\mathbf{x}} J\psi \qquad \text{with } \psi = \langle\varphi\rangle = \frac{\partial \mathcal{W}_\kappa}{\partial J}$$

- exact RG equation for $\Gamma_\kappa$

Wetterich, Phys. Lett. B **301** (1993), Dupuis, *et al*, Phys. Rep. 910 (2021)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa R_\kappa(\mathbf{q}) \left[\Gamma_\kappa{}^{(2)} + R_\kappa\right]^{-1}(-\mathbf{q})$$



# Space-time correlations from Functional Renormalisation Group

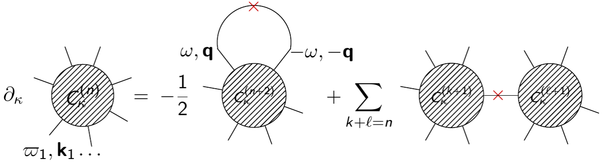
- space-time $n$-point connected correlation functions

$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i, \mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1, \mathbf{x}_1) \cdots v_{\alpha_n}(t_n, \mathbf{x}_n) \Big\rangle_c$$

- exact (but infinite hierarchy of) FRG flow equations for $C^{(n)}$
  - derived from flow equation for generating functional $\mathcal{W}_\kappa = \ln \mathcal{Z}_\kappa$

$$\partial_\kappa \mathcal{W}_\kappa = -\frac{1}{2}\mathrm{Tr}\int_{t_x,t_y,\mathbf{x},\mathbf{y}} \partial_\kappa [R_\kappa]_{\alpha\beta}(\mathbf{x}-\mathbf{y}) \left\{ \frac{\delta^2 \mathcal{W}_\kappa}{\delta j_\alpha(t_x,\mathbf{x})\delta j_\beta(t_y,\mathbf{y})} + \frac{\delta \mathcal{W}_\kappa}{\delta j_\alpha(t_x,\mathbf{x})}\frac{\delta \mathcal{W}_\kappa}{\delta j_\beta(t_y,\mathbf{y})} \right\}$$

Polchinski, Nucl. Phys. B **231** (1984), Wetterich, Phys. Lett. B **301** (1993)



# Key ingredient for closure: Extended symmetries and Ward identities

- extended symmetry: infinitesimal transformation such that the variation of $\mathcal{S}$ is linear in the fields

$\Longrightarrow$ yields exact functional Ward identities:

$$\boxed{\delta\Gamma[\psi,\bar\psi] = \delta\mathcal{S}[\varphi,\bar\varphi]\Big|_{\psi=\langle\varphi\rangle,\bar\psi=\langle\bar\varphi\rangle}}$$

- infinite set of exact identities for vertices

$$\Gamma^{(m,n)}_{\alpha_1\cdots\alpha_{m+n}}(t_1,\mathbf{x}_1,\cdots,t_{m+n},\mathbf{x}_{m+n}) = \frac{\delta^{m+n}\Gamma}{\underbrace{\delta\psi_{\alpha_1}(t_1,\mathbf{x}_1)\cdots}_{m\ \psi}\underbrace{\delta\bar\psi_{\alpha_{m+1}}(t_{m+1},\mathbf{x}_{m+1})\cdots}_{n\ \bar\psi}}$$

$$\boxed{\left\{\Gamma^{(m,n)}_{\alpha_1\cdots\alpha_{m+n}}\right\} \Longleftrightarrow \left\{C^{(m,n)}_{\alpha_1\ldots\alpha_{m+n}}\right\}}$$

# Extended symmetries and Ward identities of the Navier-Stokes action

- time-gauged Galilean invariance: $\mathcal{G} = \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\epsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \partial_t \vec{\epsilon}(t) \end{cases}$

infinite set of exact Ward identities for all vertices with $\mathbf{q} = 0$ on a $\mathbf{u}$

$$\Gamma^{(\mathbf{m+1},n)}_{\alpha\alpha_1\cdots\alpha_{n+m}}(\omega, \mathbf{q} = 0; \{\nu_i, \mathbf{p}_i\}) = \mathcal{D}_\alpha(\omega)\Gamma^{(\mathbf{m},n)}_{\alpha_1\cdots\alpha_{n+m}}(\{\nu_i, \mathbf{p}_i\})$$

[$\mathcal{D}_\alpha(\omega)$ shift operator]

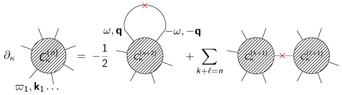
- time-gauged shift symmetry: $\mathcal{R} = \begin{cases} \delta\bar{v}_\alpha(t,\mathbf{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta\bar{\pi}(t,\mathbf{x}) & = v_\beta(t,\mathbf{x})\bar{\epsilon}_\beta(t) \end{cases}$
  - not identified yet! LC, B. Delamotte, N. Wschebor, Phys. Rev. E **91** (2015)

infinite set of exact Ward identities for all vertices with $\mathbf{q} = 0$ on a $\bar{\mathbf{u}}$

$$\Gamma^{(m,n)}_{\alpha_1\cdots\alpha_{m+n}}(\nu_1, \mathbf{p}_1, \cdots, \nu_{m+1}, \mathbf{q} = 0, \cdots) = 0$$

# Exact closure in the large wave-number limit

- flow for $C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$




exact (but infinite hierarchy of) flow

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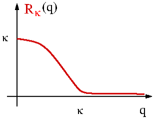
exact (but infinite hierarchy of) flow

asymptotic flow at large wavenumber

(1) large wave-number expansion: all $|\mathbf{k}_i|$ and $\left|\sum_i \mathbf{k}_i\right| \gg \kappa$




$\triangleright$ $\partial_\kappa R_\kappa(\mathbf{q})$ : $|\mathbf{q}| \lesssim \kappa \Longrightarrow |\vec{q}| \ll |\vec{k}_i|$

$\Longrightarrow$ set $\vec{q} = 0$ in all vertices

asymptotically exact for $|\mathbf{k}_i| \gg \kappa \sim L^{-1}$ and in a scaling regime

Blaizot, Wschebor, Mendez-Galain, Phys. Lett B **832** (2006), Tarpin, LC, Wschebor, Phys. Fluids **30** (2018)

# Exact closure in the large wave-number limit

- ▶ flow for $C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$


exact (but infinite hierarchy of) flow

asymptotic flow at large wavenumber


closed flow at large wavenumber

with $\mathcal{K}^{(2)}(\{\omega_i,\mathbf{k}_i\}) = \int_\omega J(\omega)\mathcal{D}_\alpha\mathcal{D}_\alpha$ and $J(\omega) = \int_{\mathbf{q}} \kappa\tilde{\partial}_\kappa C^{(2)}_\kappa(\omega,\mathbf{q})$

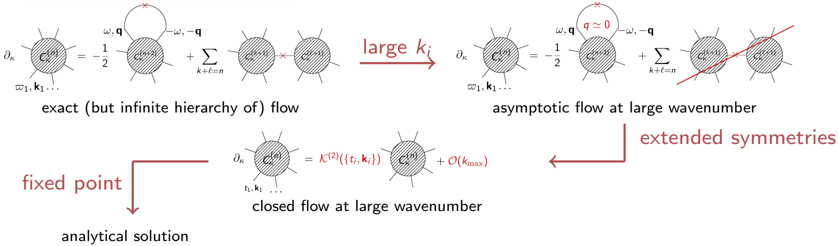
(2) Ward identities related to extended symmetries

- time-gauged Galilee
- time-gauged response shift

infinite set of exact Ward identities for all vertices with a $\mathbf{q} = 0$

# Exact closure in the large wave-number limit

- flow for $C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$




$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) = C^{(n)}_{\alpha_1\dots\alpha_n}(\{0,\mathbf{x}_i\}) \times \textbf{dominant term}$$

## (3) solution at the fixed point

$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau^2} \left|\sum_\ell \mathbf{k}_\ell t_\ell\right|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \ll \tau \\ \exp\left(-\alpha_\infty \frac{L^2}{\tau} |t| \sum_{k\ell} \mathbf{k}_k \cdot \mathbf{k}_\ell + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \gg \tau \end{cases}$$

M. Tarpin, LC, N. Wschebor, Phys. Fluids **30** (2018), LC, J. Fluid Mech. *Perspectives* **950** (2022)

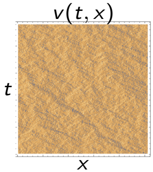
# Outline

# The stochastic Burgers and Kardar-Parisi-Zhang equations

- **Burgers equation for randomly stirred fluids** Burgers (1948)

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v} + \mathbf{f}$$

$\mathbf{f}$: stochastic Gaussian forcing at large scale




Brachet (2021)

$\Longrightarrow$ mapping to:

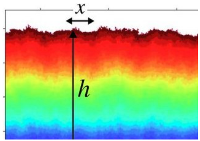
- **KPZ equation for stochastically growing interfaces**

Kardar, Parisi, Zhang, PRL **56** (1986)

$$\partial_t h - \frac{\lambda}{2}(\nabla h)^2 = \nu\nabla^2 h + \eta$$

$\eta$: stochastic Gaussian noise with correlations

$$\langle \eta(t,\mathbf{x})\eta(t',\mathbf{x}')\rangle = 2D\delta(t-t')\delta^d(\mathbf{x}-\mathbf{x}')$$




Takeuchi *et al* Sci. Rep. **1** (2011)

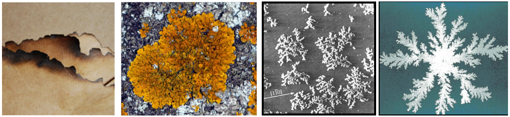
for $\mathbf{v} = -\lambda\nabla h$ with $\nabla\times\mathbf{v} = 0$ and $\mathbf{f} = -\lambda\nabla\eta$

# Stochastic interface growth and self-organised criticality

- kinetic roughening is a non-equilibrium critical phenomena
  - generic scale-invariance
  - universality

Halpin-Healy, Zhang, Phys. Rep. **254** (1995)

Krug, Adv. Phys. **46** (1997)



- correlation function takes Family-Vicsek scaling form

$$C(t,\mathbf{x}) = \left\langle \left(h(t,\mathbf{x}) - h(0,0)\right)^2 \right\rangle \sim \begin{cases} |\mathbf{x}|^{2\chi} & t=0 \\ t^{2\beta} & \mathbf{x}=0 \end{cases}$$

$\longrightarrow$ collapse onto a universal scaling function

$$C(t,\mathbf{x}) \sim |\mathbf{x}|^{2\chi} F\left(t/|\mathbf{x}|^{z}\right), \quad z = \chi/\beta$$

linear growth (EW): $\lambda = 0$

Edwards, Wilkinson, Proc.R.Soc.Lond. A (1982)

$$\partial_t h = \nu \nabla^2 h + \eta$$

- time-reversal symmetry

$\longrightarrow$ in all dimensions

$$\chi = \frac{2-d}{2}, \quad z = 2$$

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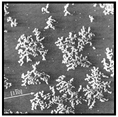
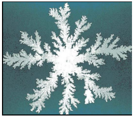
- kinetic roughening is a non-equilibrium critical phenomena
  - generic scale-invariance
  - universality

Halpin-Healy, Zhang, Phys. Rep. **254** (1995)

Krug, Adv. Phys. **46** (1997)



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$\longrightarrow$ collapse onto a universal scaling function

$$C(t,\mathbf{x}) \sim |\mathbf{x}|^{2\chi} F\left(t/|\mathbf{x}|^{z}\right), \quad z = \chi/\beta$$

nonlinear growth (KPZ)

- Galilean invariance in all $d$

$$\chi + z = 2$$

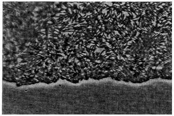
- time-reversal symmetry

$\longrightarrow$ in one dimension only

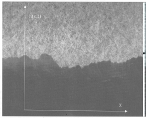
$$\chi = \frac{1}{2}\,, z = \frac{3}{2}$$

# Extremely large universality class

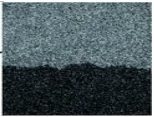
## ▶ KPZ in stochastic growth processes



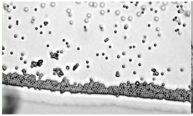
bacteries
PRL (1997)



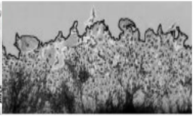
combustion
JPSJ (1997)



liquid crystals
PRL (2010)



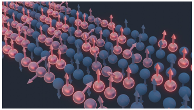
deposition
Nature (2011)

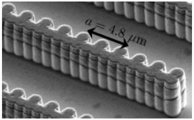


cancer cells
PRE (2012)

## ▶ KPZ in quantum systems

- integrable Heisenberg spin chains
  - ▷ solid-state magnets Nature Phys. **17** (2021)
  - ▷ ultra-cold gases Science **316** (2022)
- driven-dissipative Bose-Einstein condensates
  - ▷ exciton-polaritons Nature **608** (2022)
- entanglement growth PRX **7** (2017)
- Anderson localisation PRL **132** (2024)





# The KPZ equation: A non-perturbative fixed point

▶ path integral for the KPZ equation

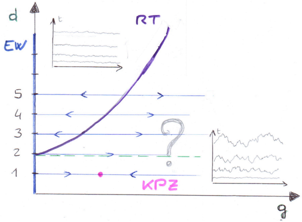
Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

$$\mathcal{S}_{\mathrm{KPZ}}[h,\bar{h}] = \int_{t,\mathbf{x}} \left\{ \bar{h} \left[ \partial_t h - \nabla^2 h - \frac{\sqrt{g}}{2} \left( \nabla h \right)^2 \right] - \bar{h}^2 \right\} \qquad g = \frac{\lambda^2 D}{\nu^3}$$

▶ perturbative Renormalisation Group

expansion at small coupling $g$:

- at 1-loop order FNS, PRL **36** (1976); KPZ, PRL **56** (1986)
- at 2-loop order Frey and Täuber, PRE **50** (1994)
  Sun, Plischke, PRE **49** (1994); Teodorovich, JETP 82 **268** (1996)
- resummed to all-order in $d = 2 + \varepsilon$
  Wiese, PRE **56** (1997), J. Stat. Phys. **93** (1998)




$\longrightarrow$ Roughening transition in $d > 2$ with $z = 2$, $\chi = 0$ for all $\varepsilon$

but fails to find the KPZ strong-coupling fixed-point !

# The KPZ equation: exact results in $d = 1$

- universal height distribution for the KPZ equation

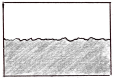
M. Hairer
Fields Medal
2014

- curved geometry – droplet (GUE-TW)

  Sasamoto, Spohn, PRL **104** (2010)

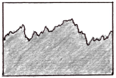
  Amir, Corwin and Quastel, Commun. Pure Appl. Math. **64** (2011)

  Calabrese, Le Doussal, Rosso, EPL **90** (2010)



- flat geometry (GOE-TW)

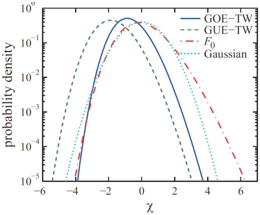
  Calabrese, Le Doussal, PRL **106** (2011),



- Brownian geometry (Baik-Rains $F_0$)

  Imamura, Sasamoto, PRL (2012)

  Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. **18** (2015)




- two-point correlation function: Airy processes

Prahöfer, Spohn, J. Stat. Phys. (2004), Sasamoto, J. Phys. A (2005), Imamura, Sasamoto, PRL (2012)

- large deviations for atypical large fluctuations

Le Doussal, Majumdar, Schehr, EPL **113** (2016)

- yet it still holds surprises !

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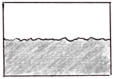

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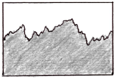

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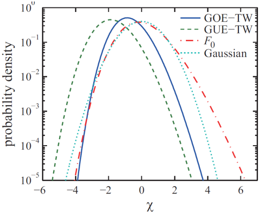


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- **yet it still holds surprises !**

# The KPZ - Burgers equation in the inviscid limit

The Galerkin-truncated Burgers equation: Crossover from inviscid-thermalised to Kardar-Parisi-Zhang scaling

C. Cartes[1], E. Tirapegui[2], R. Pandit[3] and M. Brachet[4]

Phil. Trans. A **380** (2022)

**Family-Vicsek Scaling of Roughness Growth in a Strongly Interacting Bose Gas**

Kazuya Fujimoto,[1,2] Ryusuke Hamazaki,[3,4] and Yuki Kawaguchi[2]

Phys. Rev. Lett. **124** (2020)

(c)

| Model | $\alpha$ | $\beta$ | $z$ |
|---|---|---|---|
| KPZ | $1/2$ | $1/3$ | $3/2$ |
| EW | $1/2$ | $1/4$ | $2$ |
| BHM ($\nu \gg 1$) | $0.517 \pm 0.030$ | $0.255 \pm 0.012$ | $2.07 \pm 0.20$ |
| BHM ($\nu \simeq 1/2$) | $0.500 \pm 0.003$ | $0.489 \pm 0.004$ | $1.00 \pm 0.01$ |

this Letter

**Anomalous ballistic scaling in the tensionless or inviscid Kardar-Parisi-Zhang equation**

Enrique Rodríguez-Fernández,[1,*] Silvia N. Santalla,[2,†] Mario Castro,[3,‡] and Rodolfo Cuerno[1,§]

Phys. Rev. E **106** (2022)

observation of an unpredicted scaling $z = 1$ in the limit $\nu \to 0$

- also found in Burgers-Hopf Majda, Timofeyev, PNAS **96** (2000)
- also predicted in $d \to \infty$, $\mathrm{Re} \to \infty$ Bouchaud, Mézard, Parisi, PRE **52** (1995)

# The KPZ - Burgers equation in the inviscid limit

- simulation of 1D Burgers equation

Cartes, Tirapegui, Pandit, Brachet, Phil. Trans. A **380** (2022)

$$\partial_t v + \lambda v \partial_x v = \nu \partial_x^2 v + \sqrt{D}\partial_x f$$

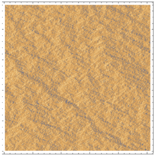
$\longrightarrow$ spectral (Galerkin) truncation preserves all the symmetries

(Galilean invariance, time-reversal symmetry) $\Longrightarrow \chi = 1/2$

- observation of three scaling regimes

EW


KPZ

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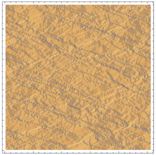
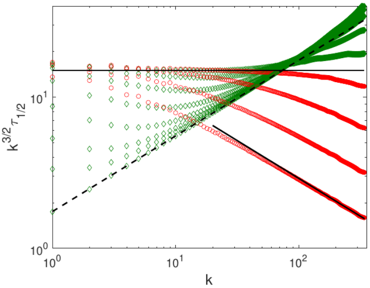



- decorrelation time from the two-point function $C(t,k)$

$$\tau_{1/2}(k) \text{ such that } C(\tau_{1/2}(k),k) = \frac{1}{2}C(0,k)\,, \qquad \tau_{1/2} \sim k^{-z}$$

# New fixed point for the new scaling ?

- decorrelation time from the two-point function $C(t, k)$



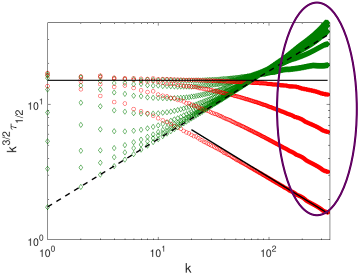

Inviscid Burgers ($\nu = 0$): $z = 1$

Kardar-Parisi-Zhang: $z = 3/2$

Edwards-Wilkinson ($\lambda = 0$): $z = 2$

# New fixed point for the new scaling ?

- decorrelation time from the two-point function $C(t, k)$




▷ large $k$ behaviour
controlled by UV fixed point

▷ one coupling constant
$g = \lambda^2 D/\nu^3$ (or Re)

▷ inviscid limit $\nu \to 0 \iff$ infinite coupling limit $g \to \infty$

non-perturbative side also in $d = 1$ !

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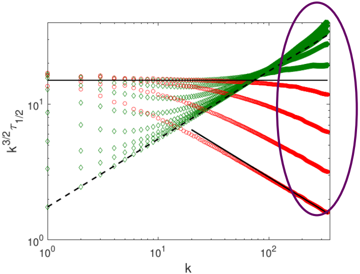


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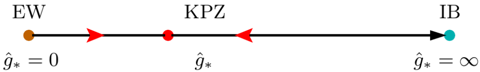
non-perturbative side also in $d = 1$ !

- possible scenario for renormalisation group :

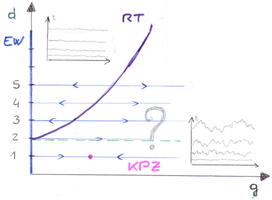


functional RG needed to access
non-perturbative fixed points!

# Functional Renormalisation Group: The KPZ fixed point

- FRG with simplest approximation

$$\Gamma_{\kappa}[\psi,\tilde{\psi}]=\int_{t,\mathbf{x}}\left\{\tilde{\psi}\left(\partial_t\psi-\frac{\sqrt{g_\kappa}}{2}(\nabla\psi)^2-\nabla^2\psi\right)-\tilde{\psi}^2\right\}$$



# Functional Renormalisation Group: The KPZ fixed point

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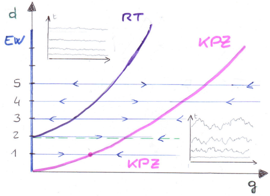
> captures the KPZ fixed point in all $d$!

LC, Chaté, Delamotte, Wschebor, PRL **104**, (2010)



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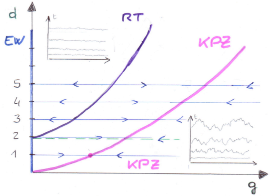
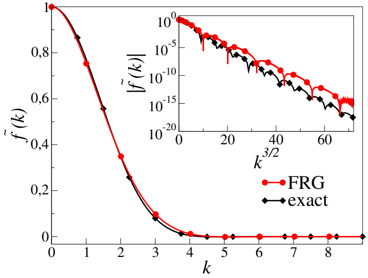
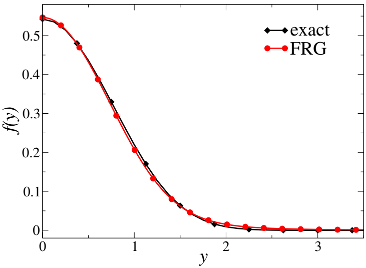



- FRG with functional approximation: scaling function in $d = 1$




LC, Chaté, Delamotte, Wschebor PRE **84**, (2011), Prähofer and Spohn, J. Stat. Phys. **115** (2004)

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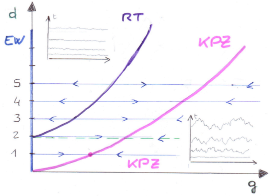
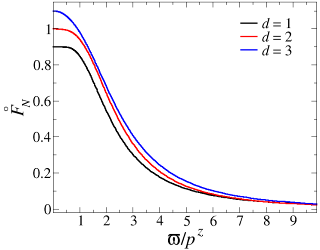
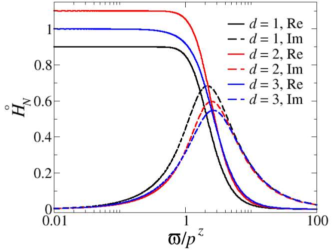



- scaling functions for correlations and response in $d = 2$ and 3




Kloss, LC, Wschebor, PRE **86** (2012)

# Functional Renormalisation Group: The IB fixed point

- FRG with simplest approximation

$$\Gamma_\kappa[\psi,\tilde\psi] = \int_{t,\mathbf{x}} \left\{ \tilde\psi\Big(\partial_t\psi - \frac{\sqrt{g_\kappa}}{2}(\nabla\psi)^2 - \nabla^2\psi\Big) - \tilde\psi^2 \right\}$$

  - effective coupling $\hat g_\kappa = \lambda^2 D_\kappa/\nu_\kappa$
  - define $\hat w_\kappa = \hat g_\kappa/(1+\hat g_\kappa) \in [0,1]$

- FRG flow equation for $\hat w_\kappa$ in $d=1$

$\longrightarrow$ single anomalous dimension $\eta_\kappa = -\partial_s \ln \nu_\kappa$ with $s = \ln(\kappa/\Lambda)$ (RG 'time')

$$\partial_s \hat w_\kappa = \hat w_\kappa \,(2\eta_\kappa - 1)\,(1-\hat w_\kappa) \qquad \text{with} \qquad \eta_\kappa = 0 \ \text{ for } \ \hat w_\kappa = 0$$

- 3 fixed point solutions

- EW: $\hat w_* = 0$

  $\eta_* = 0$ , $z_{\mathrm{EW}} = 2$

  UV stable

- KPZ: $0 < \hat w_* < 1$

  $\eta_* = 1/2$ , $z_{\mathrm{KPZ}} = 3/2$

  IR stable

- IB: $\hat w_* = 1$

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Fontaine, Vercesi, Brachet, LC, PRL 131 (2023)

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Fontaine, Vercesi, Brachet, LC, PRL 131 (2023)

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| | | |
|---|---|---|
| ▶ EW: $\hat{w}_* = 0$ | ▶ KPZ: $0 < \hat{w}_* < 1$ | ▶ IB: $\hat{w}_* = 1$ |
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| UV stable | IR stable | UV stable |

Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

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Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

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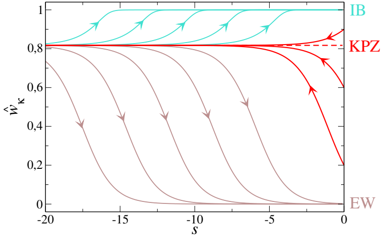



$\leftarrow$: IR flow

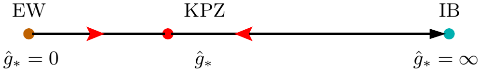
from small scales $\kappa = \Lambda$

to large scales $\kappa \to 0$

$\rightarrow$ and $\rightarrow$: UV flows

backwards from $\kappa \to 0$ to $\kappa = \Lambda$

confirms the existence of the IB fixed point in $d = 1$ !




Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

# Functional Renormalisation Group: Existence of IB fixed-point in all $d$

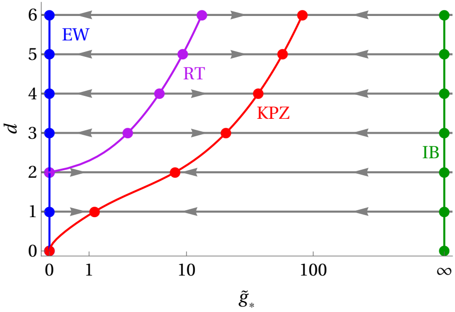
- FRG with simplest approximation

$$\Gamma_\kappa[\psi,\tilde\psi] = \int_{t,\mathbf{x}} \left\{ \tilde\psi \left( \partial_t \psi - \frac{\sqrt{g_\kappa}}{2} (\nabla\psi)^2 - \nabla^2 \psi \right) - \tilde\psi^2 \right\}$$

- FRG flow equation for $\hat{w}_\kappa = \hat{g}_\kappa / (1 + \hat{g}_\kappa)$ in $d > 1$

$\longrightarrow$ two anomalous dimensions $\eta^\nu_\kappa = -\partial_s \ln \nu_\kappa$ and $\eta^D_\kappa = -\partial_s \ln D_\kappa$

$$\partial_s \hat{w}_\kappa = \hat{w}_\kappa \left( d - 2 - \eta^{\mathcal{D}}_\kappa + 3\eta^\nu_\kappa \right) \left( 1 - \hat{w}_\kappa \right)$$




$z = 1$ scaling in all $d$ super-universal!

Gosteva, Tarpin, Wschebor, LC

PRE **110** (2024)

How do we demonstrate that $z = 1$ in all $d$ at the IB fixed point?

# Path integral for the Burgers-KPZ equation
# To be incompressible or not to be

- action for Burgers-KPZ equation enforcing irrotationality

$$\mathcal{S}_{\text{Burgers}} = \int_{t,\mathbf{x}} \Big\{ \bar{\mathbf{v}} \cdot \left[\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu \nabla^2 \mathbf{v}\right] - \mathcal{D}(\nabla \cdot \bar{\mathbf{v}})^2 + \bar{\mathbf{v}} \cdot (\nabla \times \bar{\mathbf{v}}) + \mathbf{v} \cdot (\nabla \times \mathbf{v}) + \bar{\theta} \nabla \cdot \bar{\mathbf{v}} + \theta \nabla \cdot \mathbf{v} \Big\} .$$

- existence of extended symmetries for Burgers
  - time-gauged Galilean invariance: $\mathcal{G} : \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\varepsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \partial_t \vec{\varepsilon}(t) \end{cases}$

**for Navier-Stokes**:

- time-gauged shift symmetry: $\mathcal{R} = \begin{cases} \delta \bar{v}_\alpha(t,\mathbf{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{\pi}(t,\mathbf{x}) & = v_\beta(t,\mathbf{x}) \bar{\epsilon}_\beta(t) \end{cases}$
  from incompressibility

**for Burgers**:

- time-gauged shift symmetry from irrotationality !

$$\mathcal{R} : \begin{cases} \delta \bar{v}_\alpha(t,\mathbf{x}) & \to \bar{\varepsilon}_\alpha(t) \\ \delta \bar{w}_\alpha(t,\mathbf{x}) & \to \epsilon_{\alpha\beta\gamma} \bar{\varepsilon}_\beta(t) v_\gamma(t,\mathbf{x}) - \bar{\varepsilon}_\alpha(t) \theta(t,\mathbf{x}) \end{cases}$$

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**for Navier-Stokes**:

- time-gauged shift symmetry: from incompressibility $\mathcal{R} = \begin{cases} \delta\bar{v}_\alpha(t,\mathbf{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta\bar{\pi}(t,\mathbf{x}) & = v_\beta(t,\mathbf{x})\bar{\epsilon}_\beta(t) \end{cases}$

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$$+ \bar{\mathbf{v}}\cdot(\nabla\times\bar{\mathbf{v}}) + \mathbf{v}\cdot(\nabla\times\mathbf{v}) + \bar{\theta}\nabla\cdot\bar{\mathbf{v}} + \theta\nabla\cdot\mathbf{v} \Big\}.$$

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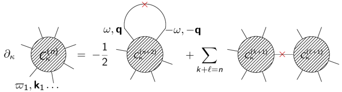
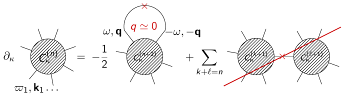
# Exact asymptotic solution at large wavenumbers for inviscid Burgers fixed point

- Exact closure of the flow of $C(t, \mathbf{p})$ in the limit of large wavenumbers



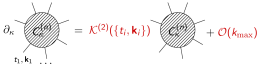
exact (but infinite hierarchy of) flow

large $\mathbf{k}_i$




asymptotic flow at large wavenumber

extended symmetries




closed flow at large wavenumber

fixed point

analytical solution

with $\mathcal{K}^{(2)}(\{\omega_i, \mathbf{k}_i\}) = \int_\omega J(\omega)\mathcal{D}_\alpha\mathcal{D}_\alpha$ and $J(\omega) = \int_{\mathbf{q}} \kappa\tilde{\partial}_\kappa C_\kappa^{(2)}(\omega, \mathbf{q})$

- fixed-point solution at large $\mathbf{p}$ (UV):

$$C(t, \mathbf{p}) \propto \begin{cases} \exp\Big(-\alpha_0\,(|\mathbf{p}|t)^2 + \mathcal{O}(|\mathbf{p}|L)\Big) & t \ll \tau_0 \\ \exp\Big(-\alpha_\infty\,|\mathbf{p}|^2|t| + \mathcal{O}(|\mathbf{p}|L)\Big) & t \gg \tau_0 \end{cases}$$

Fontaine, Vercesi, Brachet, LC, PRL **131** (2023), Gosteva, Tarpin, Wschebor, LC, PRE **110** (2024)

# Exact asymptotic solution at large wavenumbers for inviscid Burgers fixed point

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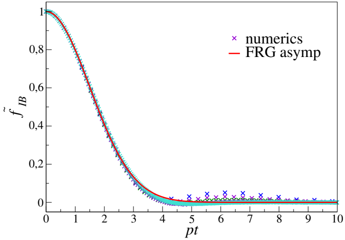
- comparison with numerics in $d=1$: Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

  - proof of $z=1$ scaling at small $t$
  - analytical form of the scaling function



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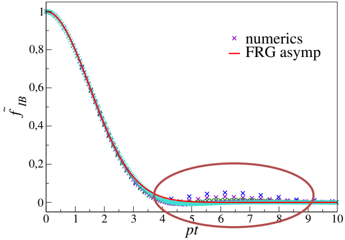
  - hints of crossover at large $t$



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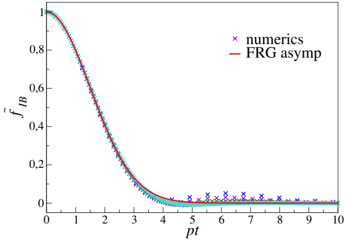


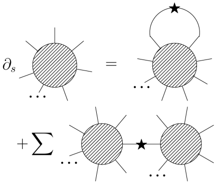

scaling $z=1$ at large $p$ generic
consequence of (gauged) Galilean and shift symmetries

# Summary and perspectives

## time dependence of $n$-point correlations at large $p$

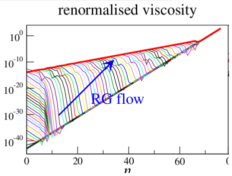
- exact closure of FRG equations based on extended symmetries
  $\Longrightarrow$ generic scaling $z = 1$

▷ Navier-Stokes, Kraichnan, Burgers-KPZ




## towards calculation of intermittency corrections

- KPZ anomalous exponents in high $d$
- anomalous exponents in shell models
- anomalous exponents for Navier-Stokes turbulence ?




Fontaine, Tarpin, Bouchet, LC
SciPost. Phys. **15**(2023)

# Thank you for your attention !