

Translation functors and restriction of representations with coherent cohomology of $U(p,q)$ to $U(p-1,q)$

Simons Symposium on Periods and L-values of Motives.

June 10-14 2024 at Schloss Elmau

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I will report on **Joint work in Progress** with M. Harris and T. Kobayashi on

Translation functors and restriction of coherent cohomology of Shimura varieties.

which was mentioned by Michael Harris in his lecture.

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Translation functors and restriction of coherent cohomology of Shimura varieties.

In this talk I will concentrate on

Translation/restriction of discrete series representations of $U(p,q)$ to $U(p-1,q)$ and their applications to periods and $(\mathcal{P}, K)$ -cohomology

My main example will be the pair $G=U(2,1)$, $G'=U(1,1)$

Outline

- I. Motivation and some review
- II. Coherent cohomology of discrete series representation
- III. Restricting of discrete series representations of $U(p,q)$ to $U(p-1,q)$
- IV. Setting the stage: Elementary and coherent pairs of representations
- V. Translating symmetry breaking operators
- VI. What comes next : Translating $(\mathcal{P}, K)$ -cohomology

Motivation:

Suppose $G' \subset G$ are connected semi simple Lie groups, Π , π irreducible representations of G and G' . We consider the restriction of representation Π to G' . In particular we are interested in the symmetry breaking operators

$$\text{Hom}_{G'}(\Pi^\infty, \pi^\infty)$$

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Period integrals:

Suppose Π , π irreducible representations of G and G' realized in $L^2(G)$ and $L^2(G')$ respectively. For K-finite functions f_Π and f_π in Π , respectively $\pi^\vee$ we consider the period integral

$$\int_{G'} f_\Pi(g') f_\pi(g') dg'$$

Observe that this integral depends only on the restriction of f_Π to G' . If the integral converges and is not zero, it defines a continuous linear functional in

$$\mathrm{Hom}_{G'}(\Pi|_{G'}^\infty \otimes (\pi^\infty)^\vee) \neq 0.$$

Hence

$$\mathrm{Hom}_{G'}(\Pi|_{G'}^\infty, \pi^\infty) \neq 0.$$

and π^∞ is a quotient of $\Pi|_{G'}^\infty$.

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Answer: **Yes, under some assumptions.** Suppose that Π and π are discrete series representations and their infinitesimal characters are sufficiently far from the wall and that the minimal K-type of π is a direct summand of the minimal K-type of Π . Then π is a quotient of Π . Result of Li-Harris and independently of J.Vargas.

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Question 2: Suppose that π is a quotient of Π . Is the minimal K-type of π a direct summand of the minimal K-type of Π ?

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Question 2: Suppose that π is a quotient of Π . Is the minimal K-type of π a direct summand of the minimal K-type of Π ?

Answer No except in special cases.

One important aspect of our joint project: Understand better the symmetry breaking operators $\text{Hom}_{G'}(\Pi_{|G'}^\infty, \pi^\infty)$ for discrete series reps and the subset defined by period integrals (in particular those which are nontrivial on the minimal K-types).

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We are restricting our considerations to pairs of discrete series reps of $G=U(p,q)$, $G'=U(p-1,q)$ which have nontrivial nontrivial $(\mathcal{P}.K)$ - cohomology in the same degree. I will report on some of the ideas/ problems /progress.

Discrete series representations of $U(p,q)$ and their coherent cohomology:

We assume from now on that $G=U(p,q)$

Discrete series reps of fixed infinitesimal character are parametrized by $W_K \backslash W_G$ where W_G is the Weyl group of G and W_K is the compact Weyl group

A He decoration δ is a sequence which contains p $+$ and q $-$. The Weyl group W_G acts transitively by permutations on the decorations and we obtain all other decorations by applying W_G to $(-, -, \dots, -, +, \dots, +)$. The set of decorations is in one to one correspondence with the Weyl chambers for which ρ_K is dominant., We write $C(\delta)$ for the Weyl chamber corresponding to δ .

We get a bijection between discrete series of G and the set $\text{Decorations} \times \{(x_1, \dots, x_{p+q}) \text{ with } x_i \in \mathbb{Z} + 1/2 \cdot (p+q+1), \text{ satisfying } x_1 > x_2 > \dots > x_{p+q}\}$

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Example: Discrete series representations of $U(2,1)$

Representation	decoration δ
holomorphic d.s	$(-, +, +)$
anti holomorphic d.s.	$(+, +, -)$
generic d.s.	$(+, -, +)$

We have a direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus p^+ \oplus p^- \quad (1)$$

The complex structures on G/K are defined so that the natural map $\mathfrak{g} \rightarrow T_{X,e}$ identifies p^- (resp. p^+) with the *antiholomorphic tangent space* $T_{X,e}^-$ (resp. the holomorphic tangent space $T_{X,e}^+$).

We let $\mathcal{P} = \mathfrak{k} \oplus p^-$. This is a maximal θ -stable parabolic subalgebra of $\mathfrak{g}$ with unipotent radical p^- and Levi K .

Any $(\mathfrak{g}, K)$ -module Π defines an $(\mathcal{P}, K)$ -module by restriction. Let W be a finite-dimensional $(\mathcal{P}, K)$ -module. An irreducible $(\mathcal{P}, K)$ -module is a $(\mathfrak{k}, K)$ -module, and thus just an irreducible representation of the compact connected group K .

The coherent cohomology of Π with coefficients in W_Π is the relative Lie algebra cohomology

$$H^\bullet(\mathcal{P}, K; \Pi \otimes W_\Pi).$$

The coherent cohomology of discrete series Π with coefficients in W_Π has been completely calculated by W.Schmid. For every discrete series there exists a unique representation W_Π and a degree q in which the coherent cohomology is one dimensional. Otherwise it is zero, the degree q depends only on δ . So we use the notation $q(\delta)$.

Example: Discrete series representations of $U(2,1)$

Representation	decoration δ	$q(\delta)$
holomorphic d.s	$(-, +, +)$	0
anti holomorphic d.s.	$(+, +, -)$	2
generic d.s.	$(+, -, +)$	1

Discrete series representations of $U(1,1)$

Representation	decoration δ	$q(\delta)$
holomorphic d.s	$(-, +,)$	0
anti holomorphic d.s.	$(, +, -)$	1

Restriction of discrete series representations of $U(p,q)$ to $U(p-1,q)$

A short summary of the results by He. Let Π be a discrete series representation of $U(p,q)$ with decoration δ and parameter λ and π a discrete series with decoration δ' and parameter λ' .

The interlacing pattern of λ and λ' induces an interlacing pattern $[\delta, \delta']$ of the decorations δ, δ' .

We say the pair (Π, π) satisfies the GGP interlacing relation if all pairs of adjacent symbols in $[\delta, \delta']$ belong to the following list:

$((+), +), (+, (+)), (-, (-)), ((-), -), (+, -), (-, +), ((+), (-)), ((-), (+))$

He's theorem

The space $Hom_{G'}(\Pi, \pi)$ is of dimension 1 if and only if the decorations of (Π, π) satisfies GGP interlacing relation.

Example : Interlacing patterns for discrete series representations of $U(2,1)$, $U(1,1)$

Case 1 ; $q=q'=0$

Holomorphic d.s.of $U(2,1)$	$(-, +, +), (2, 0, -2)$
holomorphic d.s of $U(1,1)$	$((-), (+)), (a, b)$

Allowed interlacing pattern $[\delta, \delta']$

$$((-), -, +, (+), +), \quad a > 2 > 0 > b > -2$$

Case 2 : $q=1, q'=0$

generic d.s.of $U(2,1)$	$(+, -, , +), (2, 0, -2)$
holomorphic d.s of $U(1,1)$	$((-), (+)), (a, b)$

Allowed interlacing patterns $[\delta, \delta']$

$$A.) ((-), (+), +, -, +) \quad a > b > 2 > 0 > -2$$

and

$$B.) (+, (+), (-), -, +) \quad 2 > a > b > 0 > -2$$

Case 3 $q=q'=1$

generic d.s.of $U(2,1)$	$(+, -, , +), (2, 0, -2)$
antiholomorphic d.s of $U(1,1)$	$((+), (-)), (c, d)$

Allowed interlacing patterns $[\delta, \delta']$

$$C.) (+, (+), (-), -, +) \quad 2 > c > d > 0 > -2$$

and

$$D.) (+, -, +, (+), (-)) \quad 2 > 0 > -2 > c > d$$

Observations in the Examples:

In **Case 2** : $q=1, q'=0$

generic d.s.of $U(2,1)$	$(+, -, , +), (2, 0, -2)$
holomorphic d.s of $U(1,1)$	$((-), (+)), (a, b)$

The representations of G and G' do not have cohomology the same degree. Hence we will **not consider** this case from now on.

In **Case 1** $q=q' = 0$

Holomorphic d.s. Π of $U(2,1)$	$(-, +, +), (2, 0, -2)$
holomorphic d.s π of $U(1,1)$	$((-), (+)), (a, b)$

The minimal K -type of the representation Π restricted to $K' = K \cap G' \subset K$ contains the minimal K type of π for the representations for $(a,b) = (5/2, -1/2)$ and $(a,b)=(5/2,-3/2)$.

In Case 3

generic d.s.of $U(2,1)$	$(+, -, , +), (2, 0, -2)$
antiholomorphic d.s of $U(1,1)$	$((+), (-)), (c, d)$

The minimal K -type of the representation Π restricted to $K' = K \cap G' \subset K$ contains the minimal K type of π in the interlacing pattern C.

$$(+, (+), (-), -, +) \quad 2 > c > d > 0 > -2$$

and $(c, b) = (3/2, 1/2)$ and not in the other interlacing pattern D.

Elementary and coherent pairs of representations

Definition:

We call a pair of representations Π, π **elementary** if

- $\dim \operatorname{Hom}_{G'}(\Pi, \pi) \neq 0$, i.e the interlacing relations are satisfied.
- both representations have nontrivial coherent cohomology in the same degree q with respect to the coefficients W_{Π} , respectively W_{π} and W_{π} has multiplicity one in the restriction of W_{Π} to K' .
- the minimal K -type of π has multiplicity one in the restriction of the minimal K -type of Π to K' .

Example:

In the previous slide we saw that

- the generic representation Π with $(+, -, , +), (2, 0, -2)$ and the antiholomorphic representation with $((+), (-))$ and $(c, d) = (3/2, 1/2)$ are an elementary pair.
- The holomorphic d.s of $U(2, 1)$ with parameter $(-++), (2, 0, -2)$ and the d.s of $U(1, 1)$ with $(-, +), (5/2, -1/2)$ or $(5/2, -3/2)$ are elementary pairs.

Why are we interested in elementary pairs?

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Because of

$(\mathcal{P}, K)$ -cohomology Theorem : Suppose that Π, π is an elementary pair , then we have a nontrivial map

$$H^{q(\delta)}(\mathcal{P}, K, \Pi \otimes W_{\Pi}) \rightarrow H^{q(\delta')}(\mathcal{P}', K', \pi \otimes W_{\pi})$$

We would like to generalize the $(\mathcal{P}, K)$, cohomology theorem

We would like to *find/ describe /understand the pairs of d.s. representations* (Π, π) with cohomology in the same degree $q=q'$, whose decorations satisfy the He interlacing relations and have the following

Coherent Property:

There exists an elementary pair (σ, σ') with the same decorations $\delta(\sigma) = \delta(\Pi)$ and $\delta(\sigma') = \delta(\pi)$.

Furthermore the interlacing patterns of the decorations $[\delta(\sigma), \delta(\sigma')]$ and $[\delta(\Pi), \delta(\pi)]$ are identical.

We refer to $[\delta(\sigma), \delta(\sigma')]$ as the coherent decoration.

Examples of pairs of representations with coherent decorations in Case 1 and Case 3 C.

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We define **coherent pairs of representations** as pairs of discrete series representation with coherent decorations and $(\mathcal{P}, K)$ in the same degrees

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Summary:

Our examples show : There are many coherent pairs and very few elementary pairs with $(\mathcal{P}, K)$ -cohomology in same degree. So are only be a few symmetry breaking operators defined using period integrals of minimal K -types and our chance of generalizing the $(\mathcal{P}, K,)$ -cohomology theorem are not very good

So we need a different idea and a better understanding of the symmetry breaking operator

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We try

Translating a symmetry breaking operator

Translating a symmetry breaking operator

For this we would like to use tensor functors. Essentially we would like to understand the following

Let Π, π be a coherent pair. Let F be the defining representation of $U(p, q)$ and F' an irreducible subrepresentation of $F|_{G'}$. We consider the G' -homomorphism

$$\Pi \otimes F = \bigoplus_j V_j \rightarrow \pi \otimes F'$$

obtained using the symmetry breaking operators in $\text{Hom}_{|G'}(F, F')$ and $\text{Hom}_{G'}(\Pi, \pi')$. If F' is the trivial representation then we have G' -homomorphism

$$\Pi \otimes F \rightarrow \pi.$$

It is important to understand the representations V_j in the kernel of this homomorphism.

Examples show that even if F' trivial and $\Pi \otimes F$ is a direct sum of irreducible inequivalent representations it is difficult to understand the kernel of this homomorphism..

If F' is trivial and an irreducible direct summand Π' of $\Pi \otimes F$ is not in the kernel , we obtain a **translation** $T_{\Pi, \Pi'}$ of the symmetry breaking operator in

$$\text{Hom}_{G'}(\Pi, \pi) \rightarrow \text{Hom}_{G'}(\Pi', \pi)$$

We found a condition which implies that a direct summand Π' of $\Pi \otimes F$ is not in the kernel.. Thus if this condition is satisfied there exists a "translation" $T_{\Pi, \Pi'}$ of a G' symmetry breaking operator in $\text{Hom}_{G'}(\Pi, \pi)$ to a symmetry breaking operator in $\text{Hom}_{G'}(\Pi', \pi)$.

Example:

In Case 1, the holomorphic case, the restriction of $\Pi = (-, +, +)$ $(2, 0, 2)$ to $U(1, 1)$ is a direct sum of holomorphic representations. We get two elementary pairs and a countable number of coherent pairs. We can translate the symmetry breaking operator

$$(-, +, +)(2, 0, 2) \rightarrow (-, +)(9/2, 1/2)$$

to a symmetry breaking operator

$$(-, +, +)(3, 0, -2) \rightarrow (-, +)(9/2, -1/2)$$

We may continue by translating now this symmetry breaking operator to a symmetry breaking operator

$$(-, +, +)(4, 0, -2) \rightarrow (-, +)(9/2, -1/2)$$

Theorem

If (Π, π) is an elementary pair we may obtain nontrivial nontrivial symmetry breaking operators for the other (coherent) pairs of representations with the same decorations and interlacing patterns by iterating the translation of the symmetry breaking operator for the elementary pair.

Example:

Case 3 Recall that $\Pi=(2,0,-2)(+-+)$ and $\pi= (+,-) (3/2,1/2)$ is a elementary pair . We can translate the symmetry breaking operator

$$(+, -, +)(2, 0, -2) \rightarrow (+, -)(3/2, 1/2)$$

to the symmetry breaking operator

$$(+, -, +)(2, 0, -5) \rightarrow (+, -)(3/2, 1/2)$$

by repeatedly applying the above procedure.

Conversely

Theorem

For any coherent pair Π, π with decorations δ, δ' and coherent decoration $[\delta, \delta']$ there exists an elementary pair σ, σ' with the same decorations and coherent decoration $[\delta, \delta']$ whose symmetry breaking operator is a translation of the symmetry breaking operator of Π, π along a iteration of translations.

To finish this project we now work to generalize the $(\mathcal{P}, K)$ -cohomology theorem by working out the .

Translation of $(\mathcal{P}, K)$ -cohomology

Translation of $(\mathcal{P}, K)$ -cohomology

This part is still **work in progress** and we have only results in some special cases

Suppose that Π, π is a coherent pair and Π', π is an elementary pair. The symmetry breaking operator $\Pi \rightarrow \pi$ is obtained from the symmetry breaking operator $\Pi' \rightarrow \pi$ by one translation by the standard representation or its contragredient. So we have the commutative diagram

$$\begin{array}{ccc} \Pi' \times F & \rightarrow & \pi \otimes id \\ \uparrow & & \updownarrow \\ \Pi & \rightarrow & \pi. \end{array}$$

We proved under some mild assumptions and a translation using the standard representation F or its contragredient and a "translation of the coherent cohomology using the commutative diagram.

$$\begin{array}{ccc}
 H^q(\mathcal{P}, K; \Pi \otimes W_{\Pi}) & \rightarrow & H^q(\mathcal{P}', K'; \pi \otimes Id \otimes W_{\pi}) \\
 \uparrow & & \downarrow \\
 H^q(\mathcal{P}, K; \Pi' \otimes W_{\Pi'}) & & H^q(\mathcal{P}', K'; \pi \otimes W_{\pi})
 \end{array}$$

It remains to consider the translation of the coherent cohomology along an iteration of translations. In some special cases we can prove it but unfortunately we cannot yet prove it in generality.

Thank you