

Spectral-Element Magnetohydrodynamics for Stellarators*

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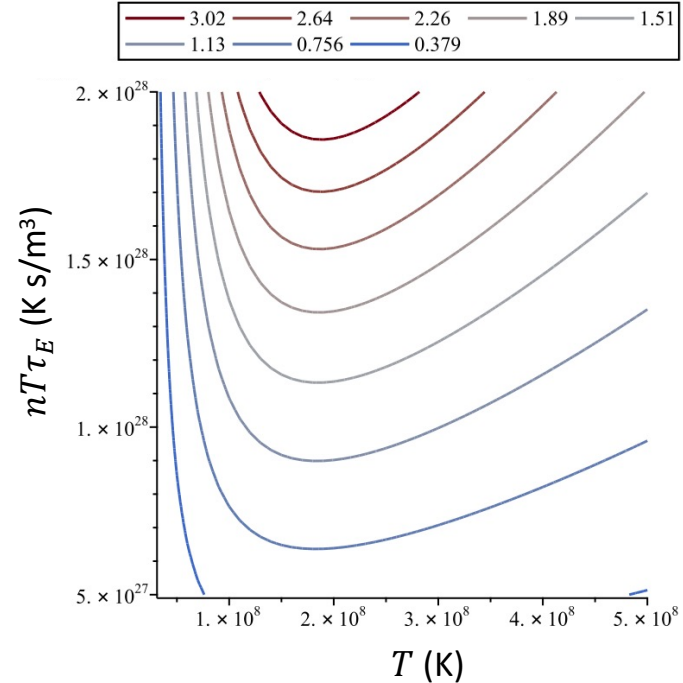
Introduction: Plasma- β is important for fusion.

- Fusion power balance amounts to a relation between the output/input power ratio Q , temperature T , and the $nT\tau_E$ “triple product.”
- Plasma- β appears when rewriting pressure:

$$nk_B T \tau_E = \beta_i \left(\frac{B^2}{2\mu_0} \right) \tau_E$$

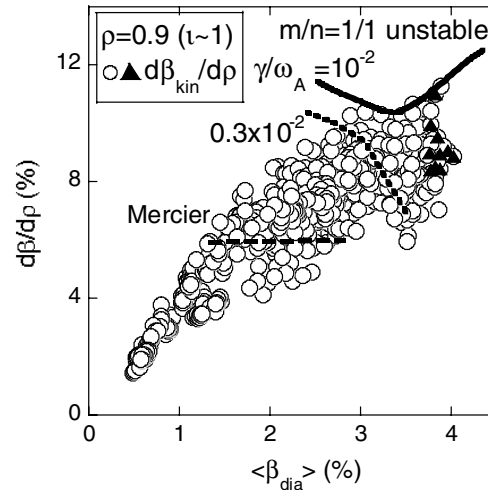
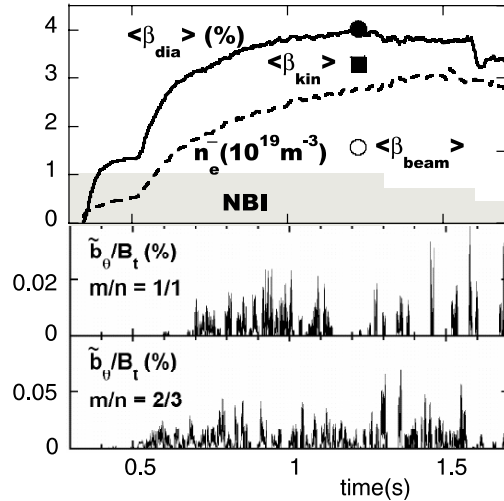
Diagram illustrating the components of the equation:

- β_i is labeled as **confinement efficiency**.
- $\left(\frac{B^2}{2\mu_0} \right)$ is labeled as **engineering & manufacturing costs**.
- τ_E is labeled as **transport physics**.



Contours of scientific- Q for D-T fusion; $Q = 1$ is scientific breakeven.

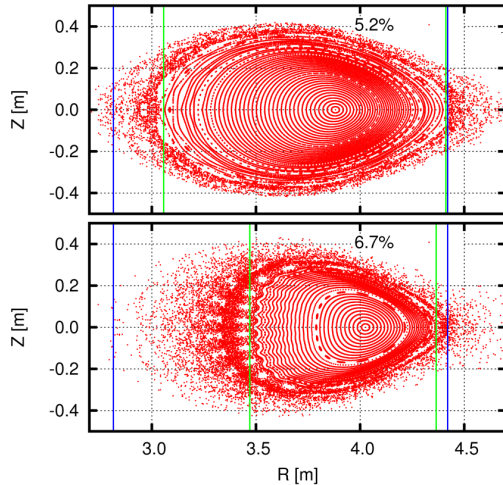
Stellarator β -values rivaling tokamaks have been achieved.



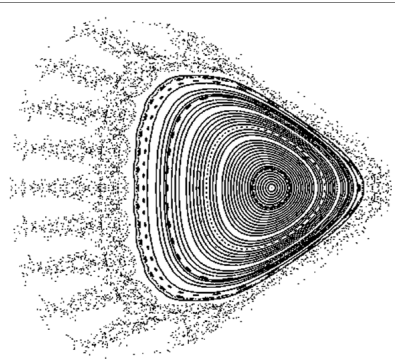
- LHD achieved $\beta = 4\%$ at large aspect ratio above linear ideal-MHD stability limits [Watanabe, *et al.*, NF **45**, 1247 (2005)].

- Comparing high- β operation in W7-AS (low shear) and LHD (high shear), Weller, *et al.* note that the Shafranov shift and heating power are more relevant than linear stability [NF **49**, 065016 (2009)].
- W7-X operation with island divertor achieves central $\beta \cong 4\%$ [Sunn Pederson, *et al.*, NF **62**, 042022 (2022)].

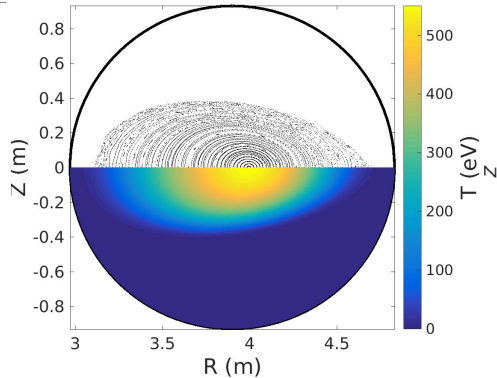
High- β effects are studied with relaxation and time-dependent computations.



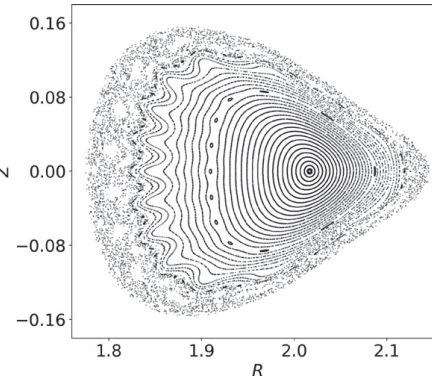
HINT2 computation of LHD topology and profile relaxation for LHD [Y. Suzuki, *et al.*, IAEA 2008 TH/P9-19].



PIES computation of W7-AS force balance at $\beta = 2.5\%$ [A. Reiman, *et al.*, NF **47**, 572 (2007)].



NIMROD MHD evolution of a torsatron subject to heating [T. Bechtel, PhD Dissertation, Univ. WI-Madison, 2021].

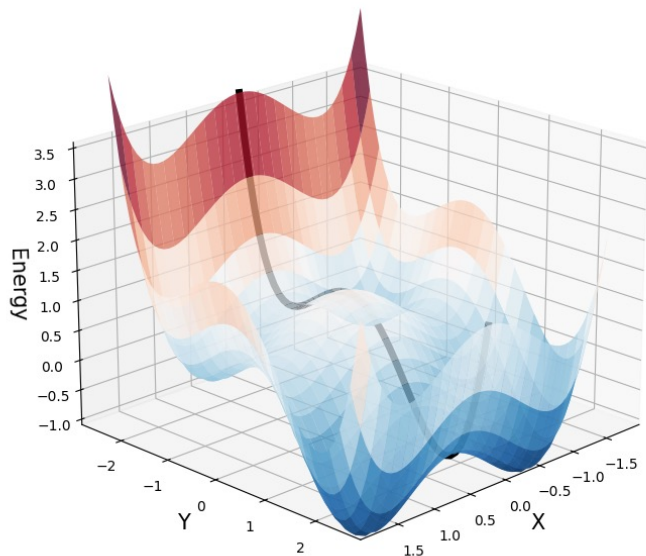


M3D-C1-S MHD evolution from W7-AS VMEC result [Y. Zhou, NF **61**, 086015 (2021)].

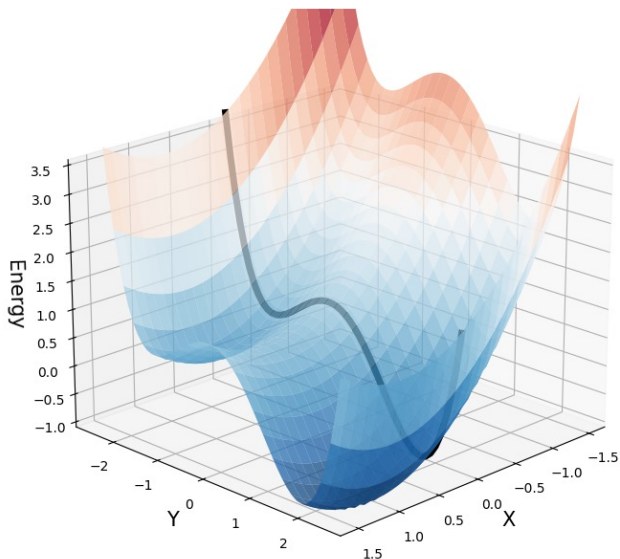
Time-dependent nonlinear computations have advantages.

- Physical time provides a meaningful connection among states.
 - Prescriptions and constraints for relaxation are not required.
- Instabilities and their consequences are modeled when they arise.
- Magnetic topology evolution is described.
- Physical effects beyond base MHD models may be added.
- Externally generated states may be used:
 - We may perturb about the state if it is an accurate force balance.
 - Alternatively, the state may be used as an initial condition.

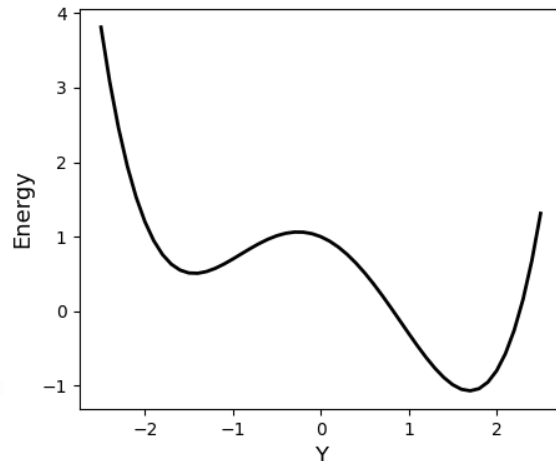
Perturbing about externally generated states is precarious.



Constraint of $X = 0$ is consistent with force balance, $-\partial E/\partial X = 0$, without the constraint.



Constrained configuration is not consistent with force balance after removing the $X = 0$ constraint.



A constrained force balance computation would not distinguish the two, possibly apart from residual error.

For nonlinear time-dependent computations, non-ideal magnetohydrodynamics is the base model.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = -\nabla \cdot \mathbf{\Gamma}_a$$

particle continuity with artificial diffusive flux

$$mn_i \left(\frac{\partial}{\partial t} \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \nabla \cdot \left[\frac{1}{\mu_0} \mathbf{B} \mathbf{B} + \mathbf{\Pi} - \left(P + \frac{B^2}{2\mu_0} \right) \mathbf{I} \right] + \mathbf{S}_v$$

momentum density; $n_i = n/Z$;
 $P = (1 + 1/Z)nT$ for single- T

$$\frac{n_i}{(\gamma - 1)} \left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T \right) = -n_i T \nabla \cdot \mathbf{V} - \nabla \cdot \mathbf{q}_i + S_{Q_i}$$

temperature evolution

$$\underline{\sigma} \cdot \left(\frac{\partial}{\partial t} \mathbf{A} + \nabla \chi \right) = \underline{\sigma} \cdot \mathbf{V} \times \mathbf{B} - \nabla \times \nabla \times \mathbf{A} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

resistive-MHD Ohm's law
 & vector potential

$$\text{or } \frac{\partial}{\partial t} \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) + \kappa_B \nabla \nabla \cdot \mathbf{B}$$

combined Faraday / Ohm's laws
 with error correction

Closures approximate plasma transport; source terms may include numerical corrections.

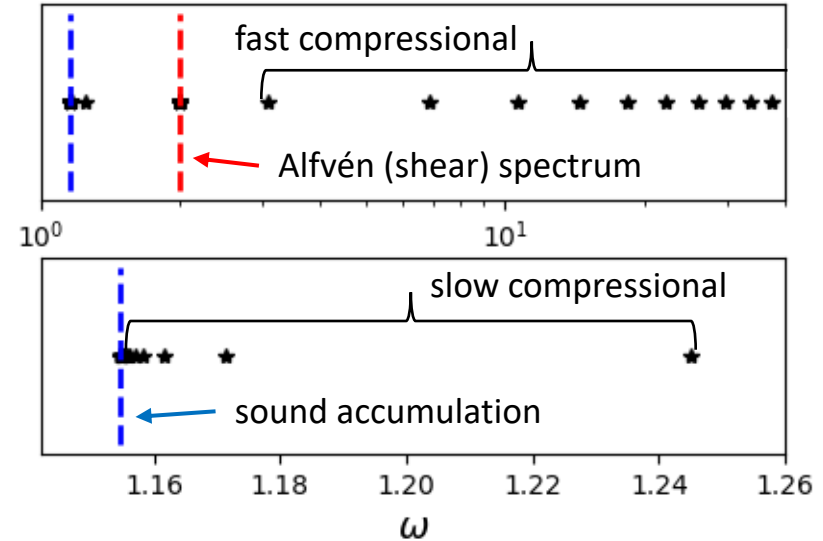
- Thermal conduction and viscous stress are anisotropic.
 - $\mathbf{q} = -n[(\chi_{\parallel} - \chi_{iso})\hat{\mathbf{b}}\hat{\mathbf{b}} + \chi_{iso}\mathbf{I}] \cdot \nabla T. \quad \hat{\mathbf{b}} = \mathbf{B}|\mathbf{B}|^{-1}$
 - $\mathbf{\Pi} = \nu_{\parallel}mn_i(\mathbf{I} - 3\hat{\mathbf{b}}\hat{\mathbf{b}})\hat{\mathbf{b}} \cdot \mathbf{W} \cdot \hat{\mathbf{b}} - \nu_{iso}\mathbf{W} \quad \mathbf{W} = \nabla\mathbf{V} + \nabla\mathbf{V}^T - \frac{2}{3}\mathbf{I}\nabla \cdot \mathbf{V}$
- Equations include numerical error-correcting terms.
 - Diffusive particle flux: $\mathbf{\Gamma}_a = -D_n\nabla n + D_h\nabla\nabla^2 n$
 - Momentum correction: $\mathbf{S}_v = m\mathbf{V} \nabla \cdot \mathbf{\Gamma}_a$
 - Ohmic and viscous heating + energy correction:

$$S_{Qi} = f_i[\eta J^2 - \mathbf{\Pi} : \nabla\mathbf{V}] - \left[\frac{f_i m V^2}{2} - \frac{T}{Z(\gamma-1)} \right] \nabla \cdot \mathbf{\Gamma}_a$$

MFE conditions of interest allow one simplification.

✓ Shock propagation is not a concern.

- Temporal scales are separated.
- Geometry is nontrivial.
- There is a divergence constraint on $\mathbf{B}$.
- Transport is anisotropic.
- Localized features arise.
- Successful simulation depends on distinguishing different modes.

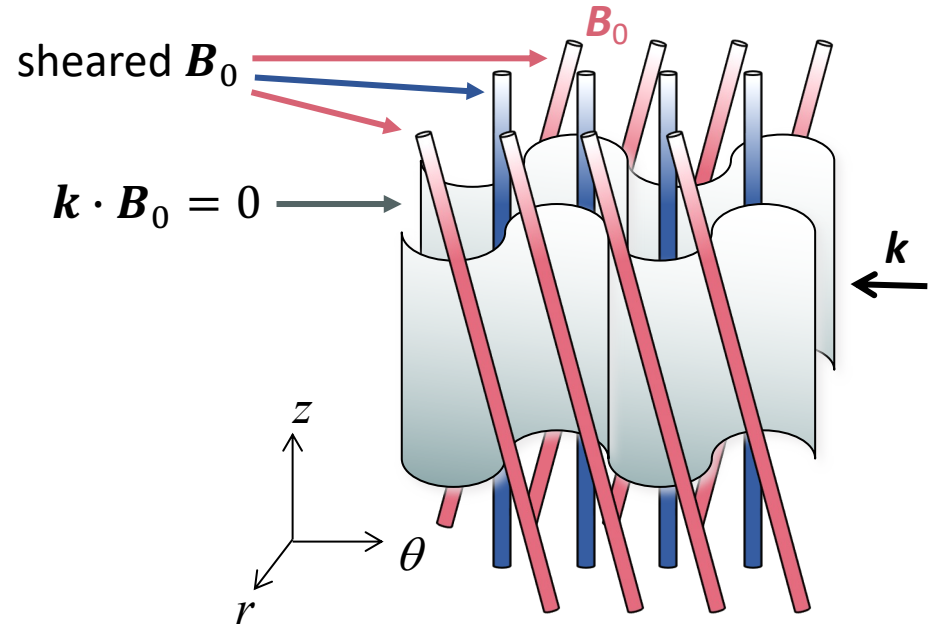
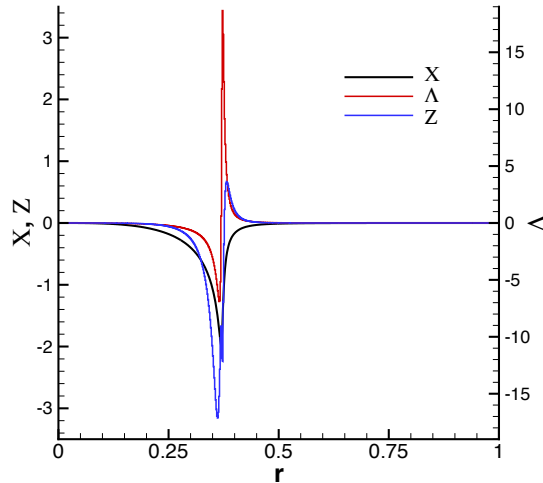


Ideal-MHD eigenvalues in a straight cylinder,
 $B_0 = B_0 \hat{z}$, $\gamma = 1$, $\beta = 1$, $m = 1$, $k_z = 2$.

Unstable modes tend to tap free energy near spatial resonances.

- Modes are driven by $\nabla P_0 \parallel \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$ and by $\nabla(J_{\parallel 0}/B_0)$.
- Localization implies that numerical results depend on limit of resolution.

Eigenmode for ideally unstable interchange.



The compressible slab g -mode indicates the subtleties.

- Ideal-MHD eigenmodes satisfy

$$\frac{\partial}{\partial y} \left[\left(\frac{AS}{k^2 S - \rho^2 \omega^4} \right) \frac{\partial}{\partial y} \xi_y - \rho \omega^2 J_{z0} B_{x0} A \xi_y \right] - \left(A + \frac{\rho g}{L} \right) \xi_y = 0$$
$$A = \rho \omega^2 - F^2 ; \quad F = \mathbf{k} \cdot \mathbf{B}_0 ; \quad S = \rho \omega^2 (B_0^2 + \gamma P_0) - F^2 \gamma P_0$$

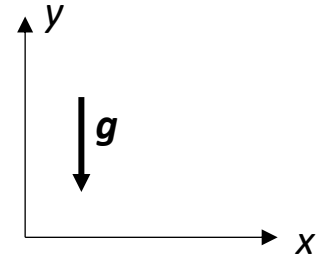
subject to $\xi_y \rightarrow 0, y \rightarrow \pm\infty$.

- Terms from compression at fine scale
- Terms from field-line bending at fine scale

- Reduced systems satisfy

$$\frac{\partial}{\partial y} \left[(\omega^2 - y^2) \frac{\partial}{\partial y} \phi \right] - D \phi = 0$$

$D > 1/4$ allows instability, $\omega^2 < 0$.



$$J_{z0} B_{x0} = \rho_0 \left(1 + \frac{y}{L} \right) g$$

slab equilibrium configuration



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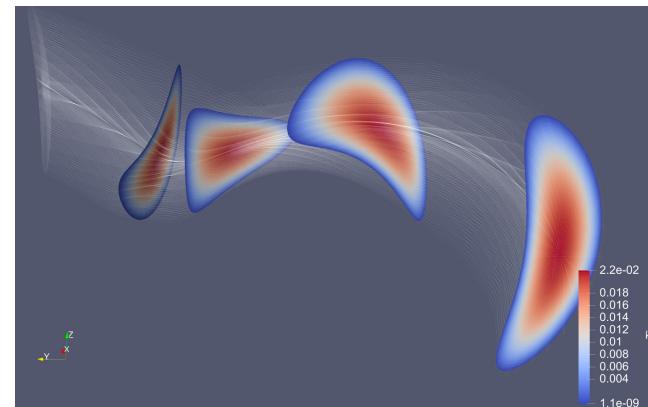
Methods: NIMSTELL is a stellarator variant of NIMROD*.

- Both use a plane of 2D elements with finite Fourier series for a periodic coordinate.
- FE nodes are based on Legendre polynomials, allowing either h or p refinement.
- NIMSTELL's geometry is expanded as a Fourier series in a generalized toroidal angle.

$$R(\xi, \eta, \zeta) = R_0(\xi, \eta) + \sum_{n=1}^N [R_n(\xi, \eta) e^{in\zeta} + cc.]$$

$$Z(\xi, \eta, \zeta) = Z_0(\xi, \eta) + \sum_{n=1}^N [Z_n(\xi, \eta) e^{in\zeta} + cc.]$$

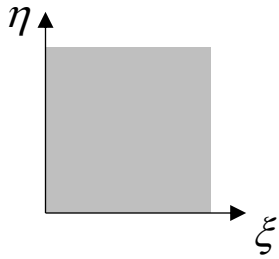
$$\phi(\xi, \eta, \zeta) = \phi_0(\xi, \eta) + \sum_{n=1}^N [\phi_n(\xi, \eta) e^{in\zeta} + cc.] + \zeta$$



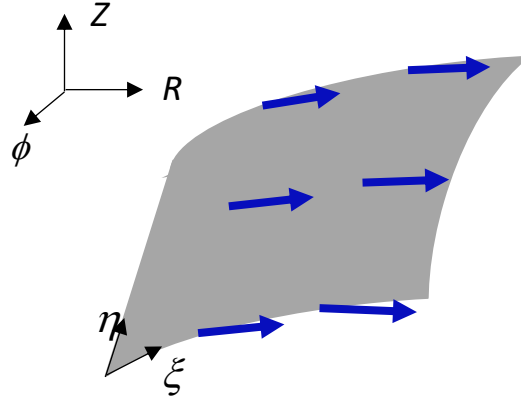
Mesh and T contours from anisotropic conduction in WISTELL-A configuration.

Vector potential is expanded in the $H(\text{curl})$ space.

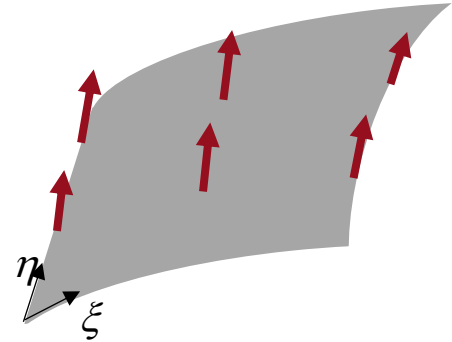
- Nédélec edge elements* form a basis for the $H(\text{curl})$ function space.



Standard 2D element.



Linear-quadratic nodal
 $\sigma_k(\xi)\lambda_l(\eta)\nabla\xi$ basis vectors.



Quadratic-linear nodal
 $\lambda_k(\xi)\sigma_l(\eta)\nabla\eta$ basis vectors.

- The $\sigma_k(x)$, $x \in \{\xi, \eta\}$ are discontinuous.
- The $\lambda_k(x)$, $x \in \{\xi, \eta\}$ are continuous expansions and 1 degree higher.



Magnetic field satisfies expected constraints.

$$\mathbf{A} = \sum_{k,l} [A_{\xi k,l}(\zeta) \sigma_k(\xi) \lambda_l(\eta) \nabla \xi + A_{\eta k,l}(\zeta) \lambda_k(\xi) \sigma_l(\eta) \nabla \eta + A_{\zeta k,l}(\zeta) \lambda_k(\xi) \lambda_l(\eta) \nabla \zeta]$$

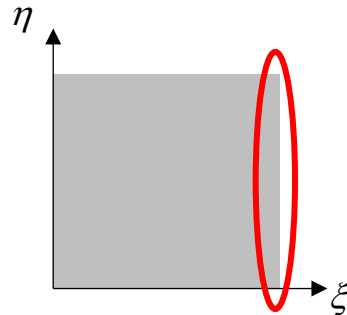
- Divergence constraint $\nabla \cdot \mathbf{B} = 0$ within elements follows from $\mathbf{B} = \nabla \times \mathbf{A}$.
- Continuity of $\mathbf{B} \cdot \hat{\mathbf{n}}$ across element boundaries is inherent from the formulation.
 - For example, consider $\hat{\mathbf{n}} = \frac{\nabla \xi}{|\nabla \xi|} \propto \frac{\partial \mathbf{R}}{\partial \eta} \times \frac{\partial \mathbf{R}}{\partial \zeta}$ element surface normal:
 - $\frac{\partial \mathbf{R}}{\partial \eta}$ and $\frac{\partial \mathbf{R}}{\partial \zeta}$ are continuous by construction.

$$\hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \frac{\nabla \xi}{|\nabla \xi|} \cdot \sum_{k,l} [A_{\zeta k,l}(\zeta) \lambda_k(\xi) \lambda'_l(\eta) - \lambda_k(\xi) \sigma_l(\eta) A'_{\eta k,l}(\zeta)] \frac{1}{J} \frac{\partial \mathbf{R}}{\partial \xi}$$

continuous by Fourier expansion

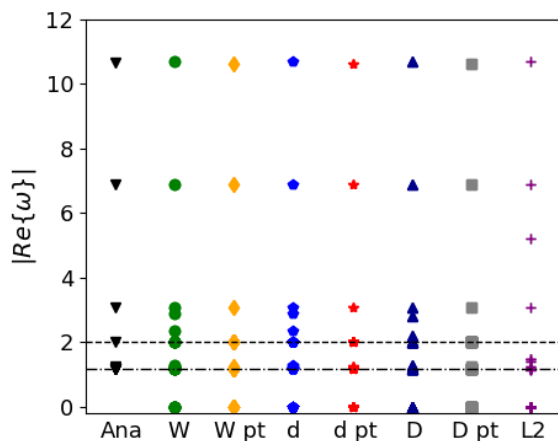
$\lambda_k(\xi) = 0$ or 1

$$\text{and } \frac{\nabla \xi}{|\nabla \xi|} \cdot \frac{1}{J} \frac{\partial \mathbf{R}}{\partial \xi} = \left| \frac{\partial \mathbf{R}}{\partial \eta} \times \frac{\partial \mathbf{R}}{\partial \zeta} \right|^{-1} \lambda'_l(\eta) \text{ and } \sigma_l(\eta) \text{ are continuous for } 0 < \eta < 1$$

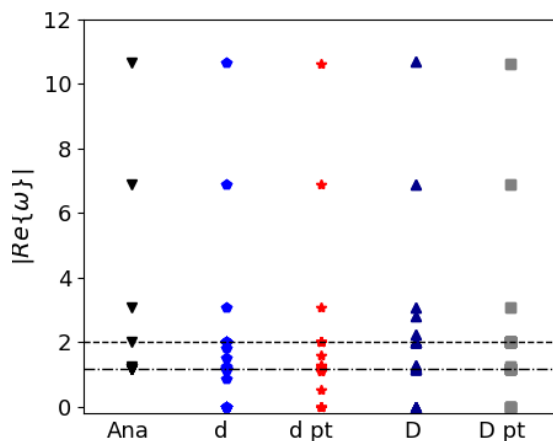


1D cylindrical ideal-MHD eigenvalue results compare formulations.

- $H(\text{curl})$ has continuous rA_θ and A_z over r , discontinuous A_r of 1 degree lower.
- Results here are for the basic uniform- $\mathbf{B}_0$ profile.



Numerical eigenvalues with A_r of the same degree as V and p .



Numerical eigenvalues with rA_θ and A_z of the same degree as V and p .

Labeling

W: Weyl, $\chi = 0$

d: damping, $\nabla^2 \chi = d_A \nabla \cdot \mathbf{A}$

D: diffusive, $\chi = -D_A \nabla \cdot \mathbf{A}$

L2: diffusive with χ in $L2$, $\mathbf{A}$ in $H1$

pt: total, not plasma, pressure

← Alfvén spectrum

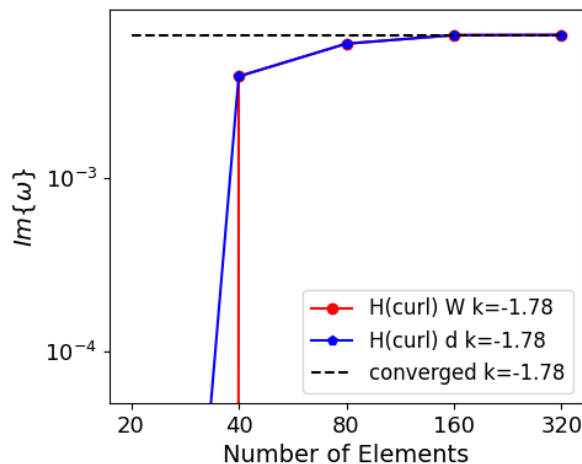
→ sound accumulation

- All formulations with $\mathbf{A}$ have $\omega = 0$ modes; Weyl has the most.
- Having χ in $L2$, hence $\mathbf{A}$ in $H1$, misses the Alfvén spectrum.

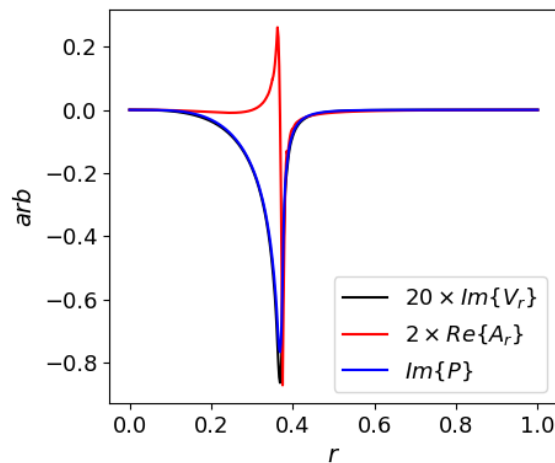
Eigenvalue results on interchange consider fidelity to fine-scale response.

- Equilibrium profile* has $B_{z0} = 1$, $q(r) = (1 + c_2^2 r^2)/Rc_1$,
 $D_s(r) = -\frac{1}{rB_{z0}} \left(\frac{q}{q'}\right)^2 P'_0 = c_1^2 [(1 + c_2^2 r^2) c_2^4 r^2]^{-1}$
- Conditions with $c_1 = 4/7$, $c_2 = 10/7$ are stable for $r_s > 0.466$.
- Computations for $m = 4$, $k = -1.784$ have $D_s = 0.443$, cubic V and p , and the damping formulation.
- Mesh is uniform in r .

*Gruber and Rappaz, *FE Methods in Linear Ideal MHD*, Springer, 1985.



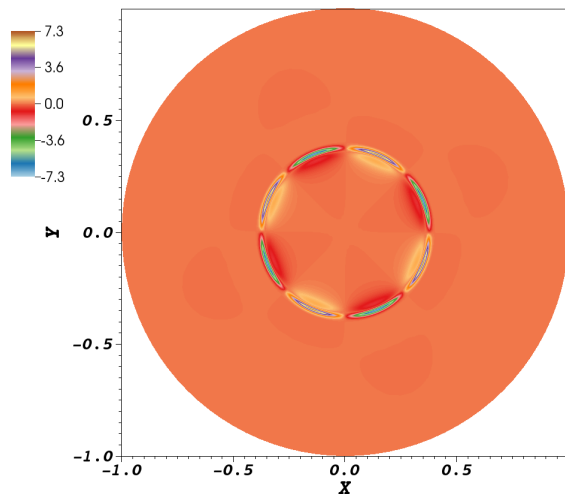
Convergence “from the stable side” is important.



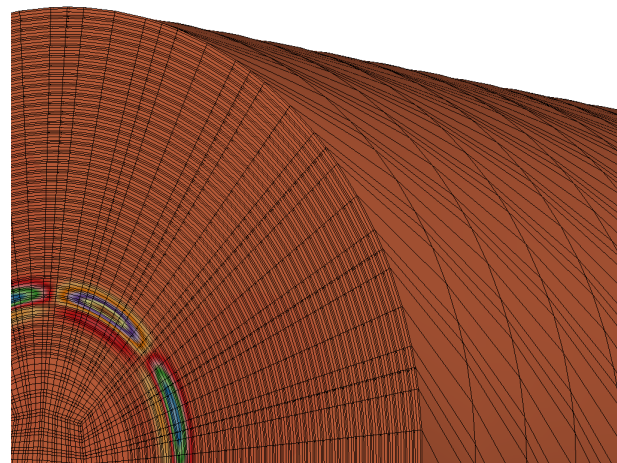
Eigenvector for the 80-element computation displays localization.

NIMSTELL reproduces the resistive version of the mode.

- NIMSTELL FE mesh is 2D, but the formulation is based on the 1D eigenvalue results.



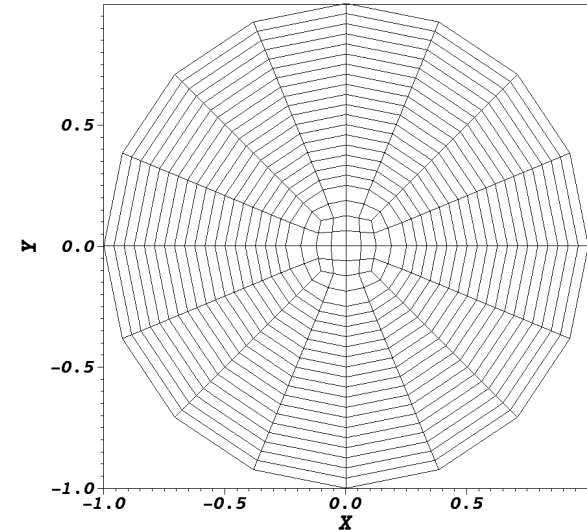
Result on V_z for $\eta = \chi_{iso} = \nu_{iso} = 10^{-7}$ with 48x24 mesh of biquartic elements, $n=0,1$.



Twisted mesh case has $\eta = \chi_{iso} = \nu_{iso} = 10^{-6}$ has $0 \leq n \leq 5$ without aligning the twist with the helical mode.

For a given mesh, we find that a minimum level of resistivity is needed.

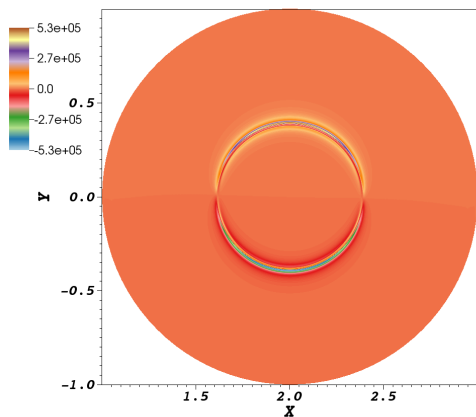
- A series of computations uses biquartic elements without packing near the rational surface.
- Mesh sizes ranged from 24×16 to 48×24 to 96×48 , all having the central region with a non-polar arrangement.
- Unstable noise develops if resistivity is too small for a given resolution.
 - 24×16 needs $\eta \sim 10^{-6}$.
 - 48×24 needs $\eta \sim 10^{-7}$.
- Likely related to $\nabla \times \nabla \times \mathbf{A}$ providing coercivity in the $\mathbf{A}$ -advance—not evident from 1D eigenmode results, apart from $\omega = 0$ modes.



The lowest-resolution mesh is effectively 24×16 . Element curvature is not shown.

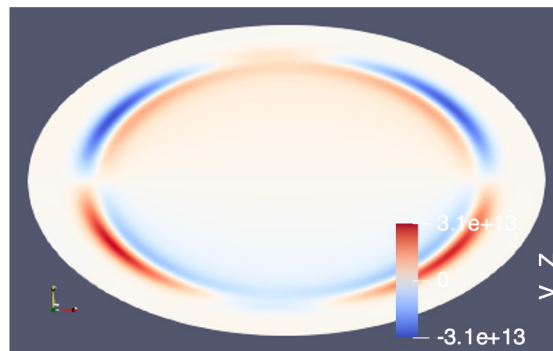
We are also verifying current-driven tearing.

- Cylindrical tearing in a pinch profile is a standard NIMROD verification case.



Contours of V_z for an $S = 10^6$,
 $Pm = 10^{-3}$ computation

- Toroidal tearing in a twisting elliptical geometry required opposite helicity.
- DESC* was used to create the $N_p = 1$, $e = 0.8$ equilibrium.



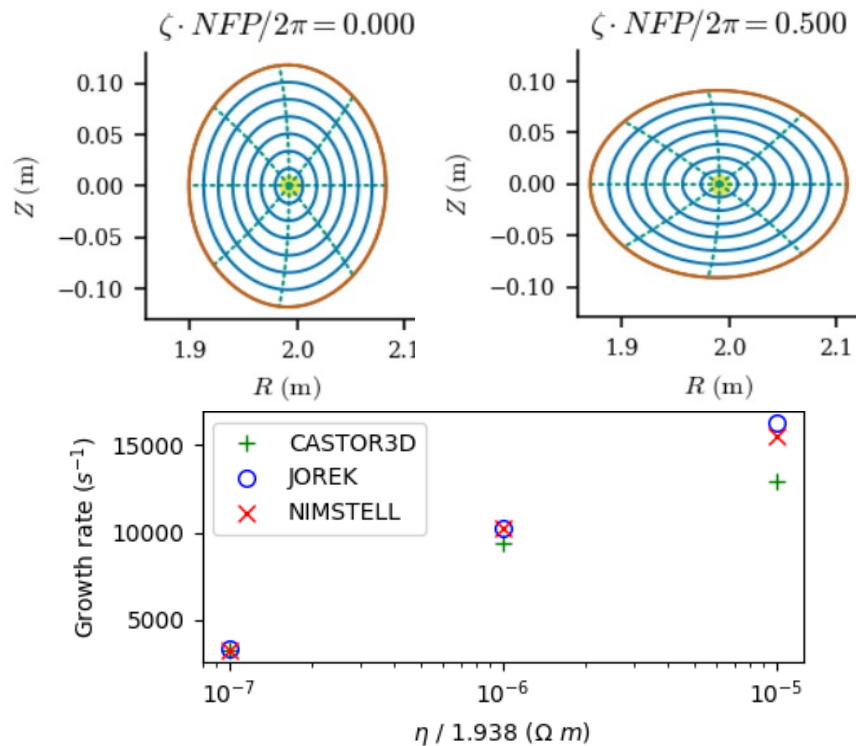
Contours of V_ϕ from a computation with
 $S = 2 \times 10^4$, $Pm = 2 \times 10^{-3}$.

*Dudt and Kolemen, Phys. Plasmas **27**, 102513 (2020).



Published JOREK results facilitate toroidal verification.

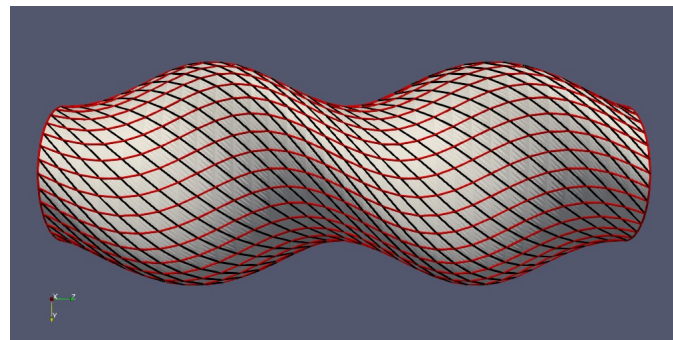
- Tearing verification includes a linear benchmark of the elliptical W7-A JOREK case [N. Nikulsin, *et al.*, Phys. Plasmas **29**, 063901 (2022)].
- The configuration has $N_p = 5$, $e = 0.76$.
- We used DESC to generate the equilibrium for NIMSTELL.
- The (2,1) mode requires $0 \leq n \leq 10$ for minimal toroidal resolution.



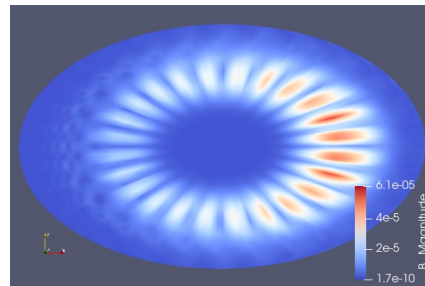
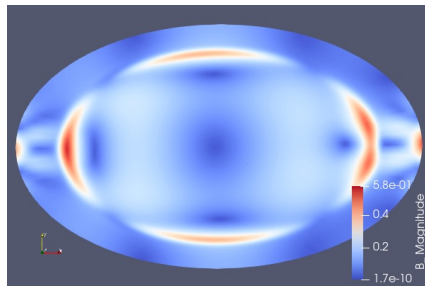
Comparison of tearing growth rates for W7-A case over a scan of resistivity values.

Coercivity is again a concern at small resistivity values.

- ✗ Distorting toroidal meshes lead to unphysical instability with the $\mathbf{A}$ representation — not a resolution issue.
- Twisting meshes avoid the numerical instability.
- Computations have $e = 0.8$, $N_p = 1$, $S = 1 \times 10^5$, $Pm = 0.1$.
- Tearing mode at low- n (left, from twisted mesh) is expected.
- Ballooning-like mode (right, distorting mesh) is not.

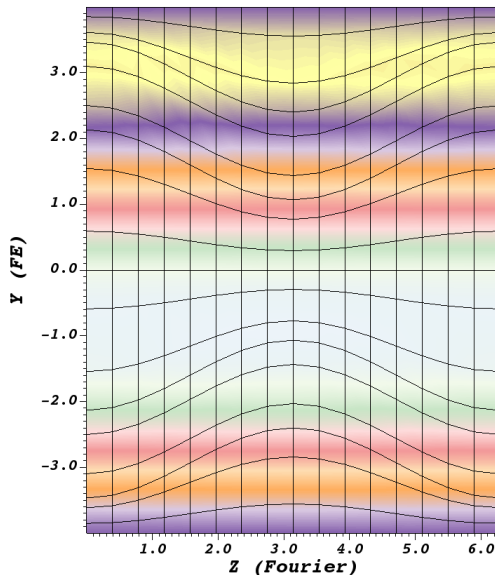


Overlay of unwrapped tori compares twisting (black) and distorting (red) meshes.

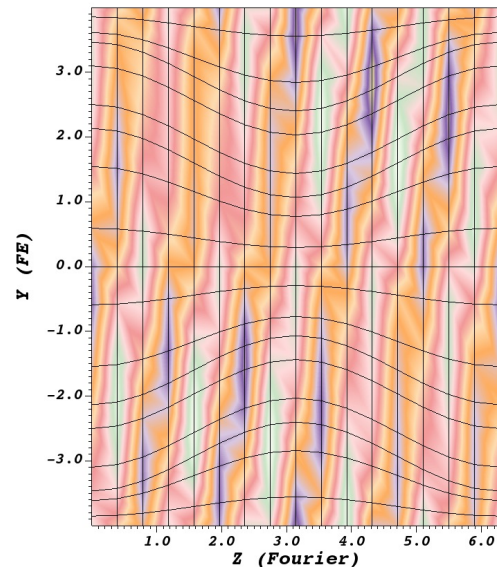


Slab geometry reproduces the challenge for $\mathbf{A}$ in $H(\text{curl})$ at large S .

- Numerical error is reproduced with a mesh distorted over ζ .
- Parameters have $S = 10^6$, $\text{Pm} = 0.5$.
- Tearing mode is symmetric over ζ .
- Mesh distortion is $n = 1$; computations have $0 \leq n \leq 5$ + with dealiasing.
- ❖ Coercivity of $\mathbf{A}$ -equation at small η -values may be the issue.



Slab-geometry computation with $\mathbf{B}$ in $H1$ reproduces symmetric tearing mode. [V_x is shown.]

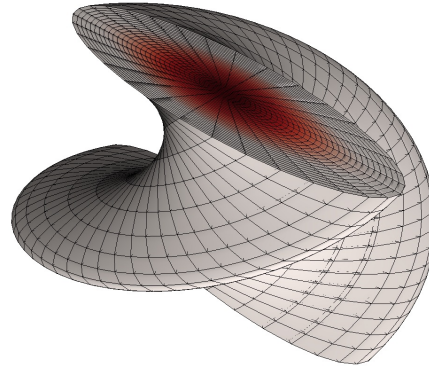


Slab-geometry computation with $\mathbf{A}$ in $H(\text{curl})$ produces numerical mode.

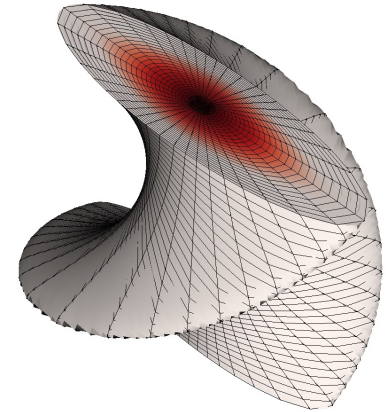


Accurate anisotropic transport is also important.

- Thermal transport computations solve $\nabla \cdot [(\kappa_{\parallel} - \kappa_{\perp})\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla T + \kappa_{\perp} \nabla T] + Q = 0$.
- Straight configuration has $\mathbf{B} = \nabla \left[C I_l \left(\frac{N_p r}{R} \right) \cos \left(l\theta - \frac{N_p z}{R} \right) + z \right]$ with $l = 2, R/N_p = 4$.
- The separatrix has x -points at $r = 3.46$, which is inside the domain.
- Computations have $\kappa_{\parallel}/\kappa_{\perp} = 10^6$.

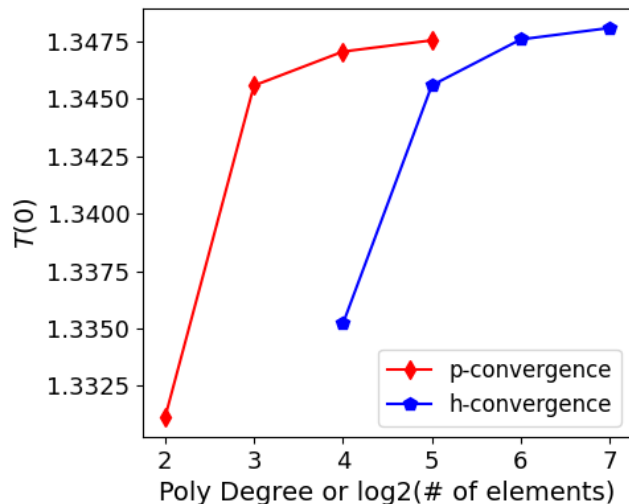


Twisting mesh and contours of temperature. Mesh is $n = 0,1$ only.

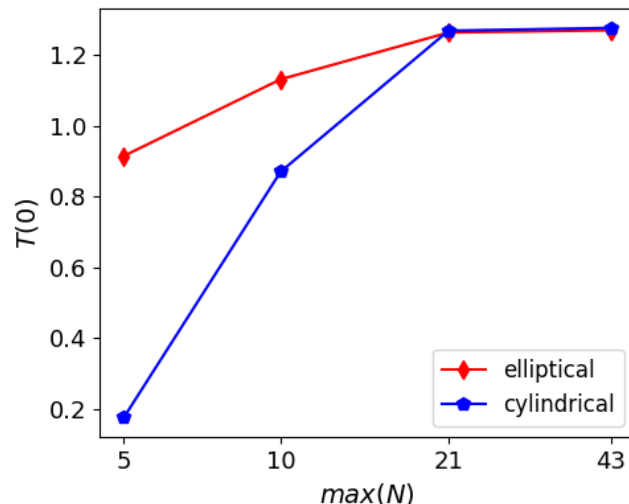


Distorting mesh and contours of temperature. Mesh has a broad n spectrum.

Results support meshing by symmetry, where possible.



Twisting mesh follows symmetry and converges readily. p -refinement uses 960 elements (32 azimuthally); h -refinement uses bicubics.



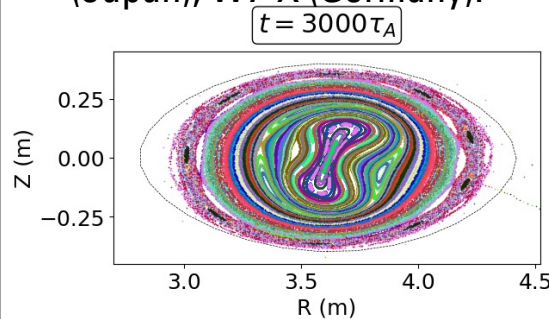
Distorting mesh needs convergence with respect to Fourier expansion, $0 \leq n \leq N$. 1340 biquintic elements are used.

There are commonalities and differences with other efforts.

- The MIPS code [Y. Todo, *et al.*, Pl. Fus. Res. **5**, S2062 (2010)] uses an explicit 4th-order time advance and 4th-order spatial finite differences.
- The stellarator version of M3D-C1 [Y. Zhou, NF **61**, 086015 (2021)] uses Bell 2D (reduced quintic) elements and Hermite cubics for a split-implicit potential representation of full MHD.
- The stellarator version of JOREK [N. Nikulsin, *et al.*, PoP **29**, 063901 (2022)] uses 2D G1 Bezier elements and 1D Fourier for implicit reduced MHD.
- The NIMSTELL time advance is semi-implicit.

Theory-experiment partnership
with HSX upgrade.

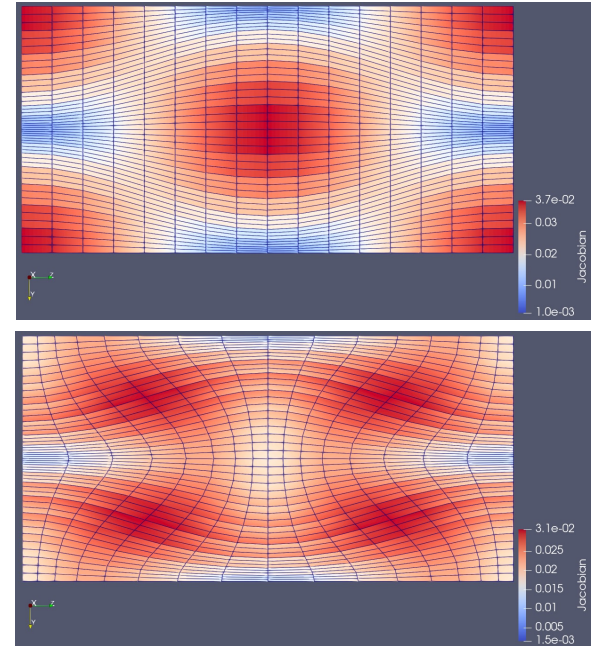
Collaborations with major
international facilities: LHD
(Japan), W7-X (Germany).



Thrusts for the UW-Madison M3D-C1
effort also include design validation.
[Courtesy of A. Wright]

NIMSTELL directions

- Implementation of remaining nonlinear terms and nonlinear applications
- Continue numerical investigation of vector-potential formulation
- Benchmark & apply new $H1$ representation of $\mathbf{B}$
 - Boundary condition equation needs preconditioning
 - Provides an alternative representation
- Enhance Fourier-banded preconditioner for quasi-symmetric configurations
- Explore mesh optimization, as allowed by the generalized toroidal angle



Meshes over one FE-Fourier plane.
Lower panel is optimized to reduce
variation in J along constant- η gridlines.

Conclusions

- While expanding $\mathbf{A}$ in $H(\text{curl})$ respects the physical divergence constraint on $\mathbf{B}$, stability at small η -values is a numerical challenge.
- Having an alternative representation for $\mathbf{B}$ will facilitate applications while the study of vector potential continues.
- Mesh alignment with $\mathbf{B}$ is not required for accurate anisotropic transport, but aligning to symmetries helps.
- Using externally generated states as initial conditions is safest.