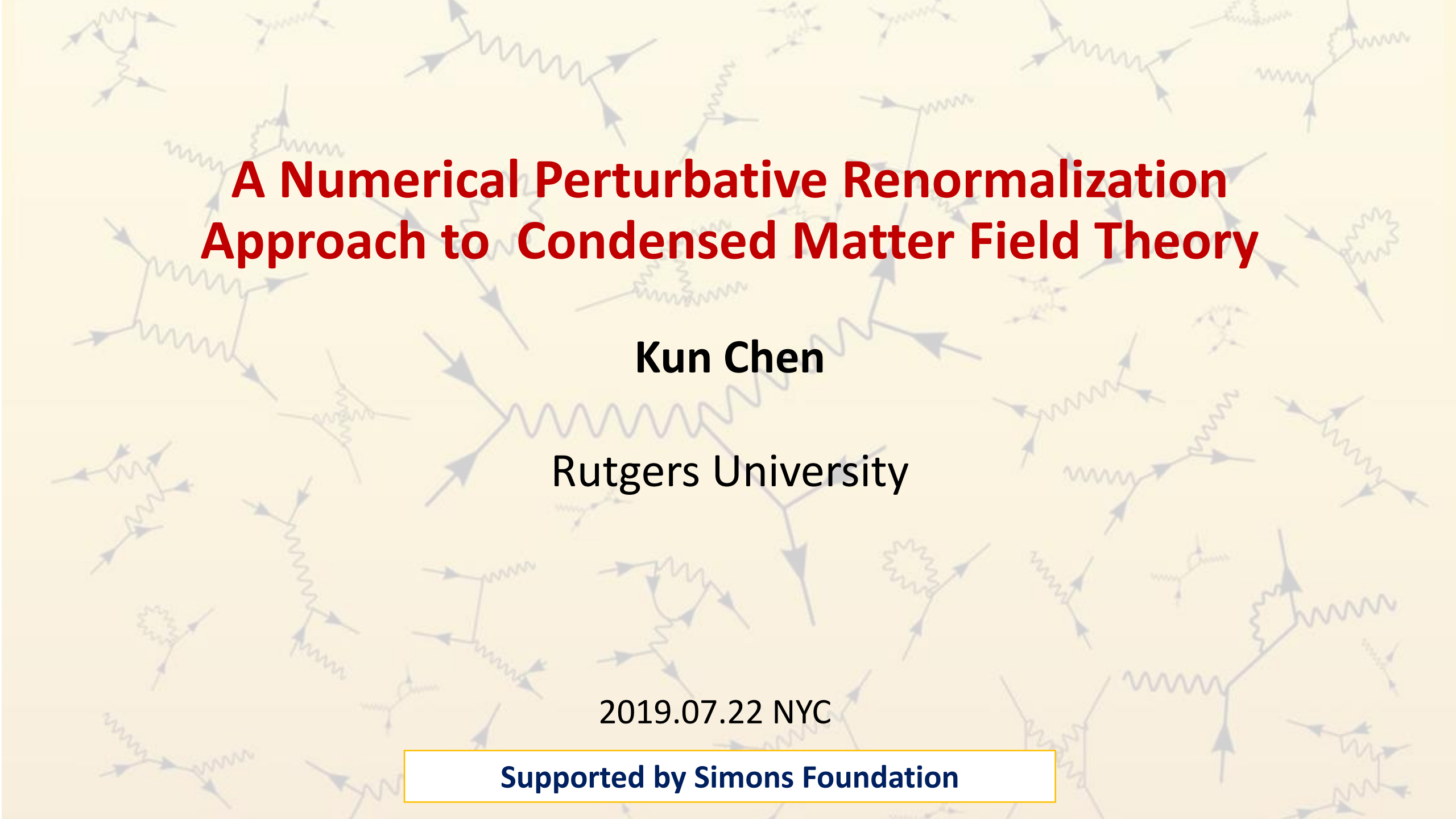


The background of the slide is a light beige color with a repeating pattern of faint, grey Feynman diagrams. These diagrams represent various particle interactions, including vertices, propagators, and loops, typical of quantum field theory.

A Numerical Perturbative Renormalization Approach to Condensed Matter Field Theory

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Kristjan Haule



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Large Parameter in Feynman diagrams

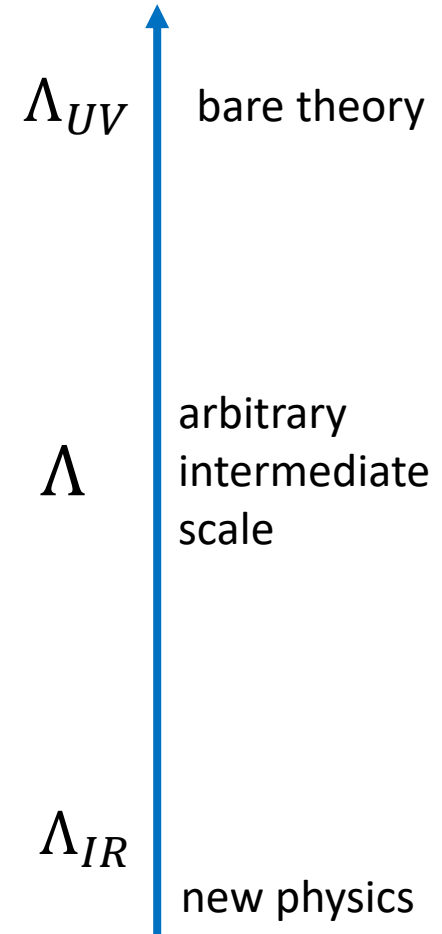
- Large parameters from UV:**

e.g. a ϕ^4 model with a small mass m at $d = 4$: $S_{bare} = \int \frac{d\mathbf{k}}{(2\pi)^d} k^2 \phi_{-\mathbf{k}} \phi_{\mathbf{k}} + u \int \frac{d\mathbf{k} d\mathbf{k}' d\mathbf{q}}{(2\pi)^d} \phi_{\mathbf{k}} \phi_{\mathbf{k}' - \mathbf{q}} \phi_{\mathbf{k}' + \mathbf{q}} \phi_{\mathbf{k}}$

$$\text{Diagram } \Gamma_4 = \text{Diagram } u - \frac{1}{2} \times \text{Diagram } (\text{loop with } k \text{ and } k+\Lambda) \times 3 + \dots$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^2 (k + \Lambda)^2} = \# \ln \frac{\Lambda_{UV}}{\Lambda}$$

UV “divergence”: Our ignorance
beyond the UV scale

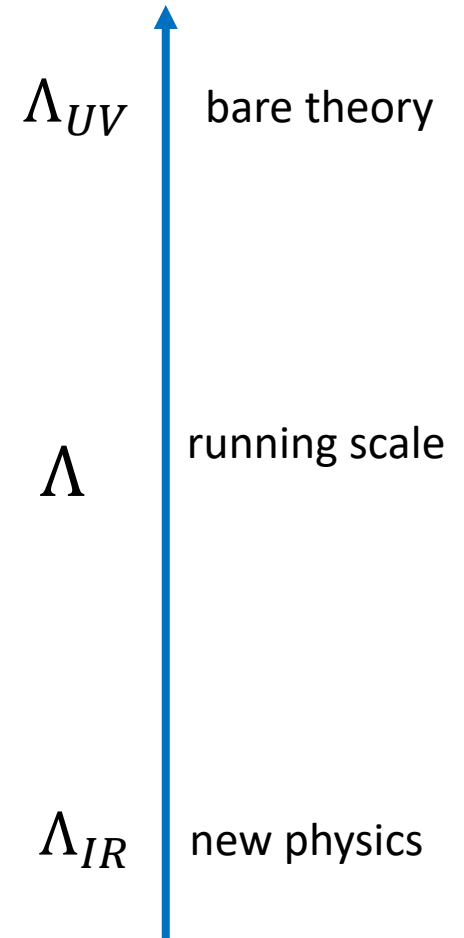


- Vertex Problem (curse of dimensionality):**

- The UV “divergence” has to be resummed (e.g., with certain skeleton diagrammatic technique)
- However, $\Gamma_4(k_1, k_2, k_3)$ can be difficult to store and calculate!

Relevance of Vertex Functions

- **Scaling analysis:**
 - Assume UV and IR scales to be well-separated, the system must be scale invariant!
 - Dimensionless $\Gamma_n = \Psi_{universal} \left(\frac{k_i}{\Lambda}; \frac{u_j}{\Lambda^{\delta_j}} \right) + \text{regular corrections}$
 - Relevant couplings: $\delta_j > 0$; Marginal: $\delta_j = 0$; Irrelevant: $\delta_j < 0$.
- **The dominating vertex functions can be parameterized with the relevant/marginal couplings only (up to certain non-universal corrections)!**
- **Three challenges:**
 - a) How to find all relevant/marginal couplings?
 - b) With known relevant/marginal couplings, how to perturbatively recover physical observables?
 - c) How to non-perturbatively calculate relevant/marginal couplings?



Step 1.

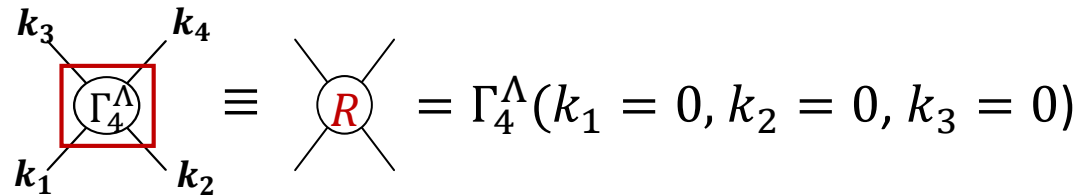
Find Relevant/Marginal Vertex Functions
in Different Field Theories

Example 1: Critical Field Theory

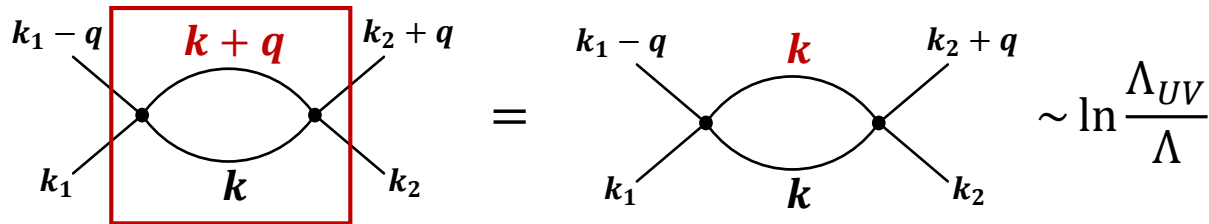
- Critical scalar field theory at $d = 4$:

$$S_\Lambda = \int \frac{d^d k}{(2\pi)^d} (k^2 + \Lambda^2) \phi_{-k} \phi_k + u \int \frac{d^d k d^d k' d^d q}{(2\pi)^d} \phi_k \phi_{k'-q} \phi_{k'+q} \phi_k$$

- Relevant/marginal vertices: $\Gamma_{2,rel}^\Lambda(k) = \# + \#k^2$ and $\Gamma_{4,rel}^\Lambda = \#$



$$\boxed{\Gamma_4^\Lambda} \equiv \text{circle with } R \text{ and 4 lines} = \Gamma_4^\Lambda(k_1 = 0, k_2 = 0, k_3 = 0)$$



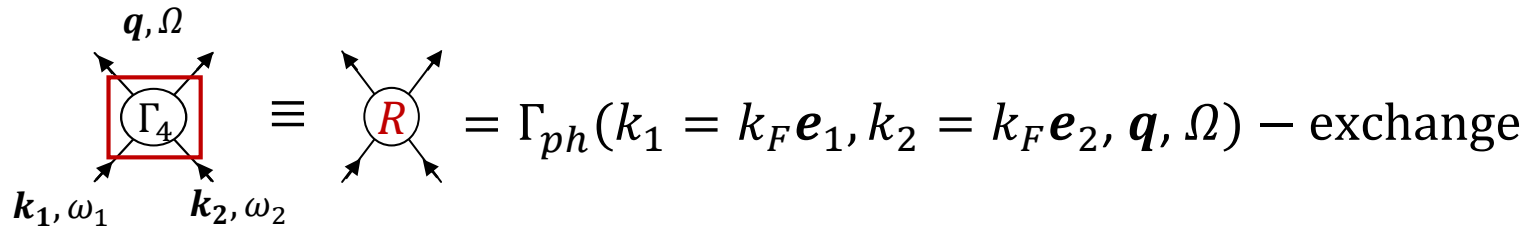
$$\boxed{\text{two-loop diagram}} = \text{two-loop diagram} \sim \ln \frac{\Lambda_{UV}}{\Lambda}$$

Example 2: Fermi Liquid Theory

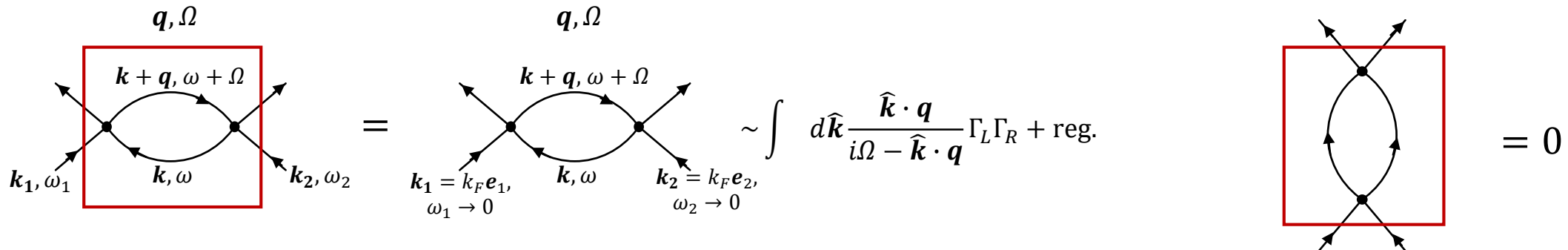
- A spinless Fermi liquid at $d = 2$:

$$L = \sum_k \psi_{k,\tau}^+ \left(\frac{\partial}{\partial \tau} + k^2 - \mu \right) \psi_{k,\tau} + \frac{1}{2V} \sum_{q,k,k'} \frac{8\pi}{q^2 + \lambda} \psi_{k,\tau}^+ \psi_{k-q,\tau}^+ \psi_{k'+q,\tau} \psi_{k',\tau}$$

- Relevant/marginal vertices:** $\Gamma_{2,rel}(k, \omega_n) = \# + \#\omega_n + \#(k - k_F)$
- $\Gamma_{4,rel}$ = particle-hole channel effective interaction near the Fermi surface



$$\Gamma_4 \equiv R = \Gamma_{ph}(k_1 = k_F \mathbf{e}_1, k_2 = k_F \mathbf{e}_2, \mathbf{q}, \Omega) - \text{exchange}$$



$$\sim \int d\hat{\mathbf{k}} \frac{\hat{\mathbf{k}} \cdot \mathbf{q}}{i\Omega - \hat{\mathbf{k}} \cdot \mathbf{q}} \Gamma_L \Gamma_R + \text{reg.}$$

$$= 0$$

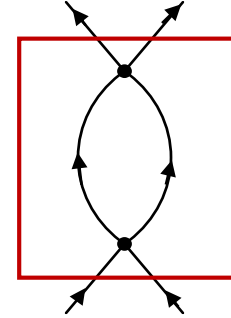
Other Examples

- **Superconductivity:**

pp channel effective interaction

- **2D/3D dilute Bose gas:**

pp channel (B. Svistunov, E. Babaev, and N. Prokof'ev, *Superfluid states of matter*)



- **Conformal field theory (e.g., many common critical theories, like the Ising criticality):**

- 2-point and 3-point correlations are fixed up to a few universal constants and critical exponents.
- All high order correlations can be constructed using operator product expansion.
- Conformal bootstrap.

K. Wilson, On products of quantum field operators at short distances. Cornell Report (1964)

A. Polyakov, Conformal symmetry of critical fluctuations JETP let. 12, 381 (1970).

- **Many other examples in condensed matter field theory books...**

e.g., S. Sachdev, *Quantum Phase Transitions*, (2011)

Step 2.

Q: With known relevant/marginal couplings, how to perturbatively calculate physical observables?

A: Perturbative Renormalization

Renormalized Expansion

- Represent any physical observables with the relevant/marginal couplings:

Bare expansion

$$\Gamma_4 = \xi^1 u - \frac{1}{2} \xi^2 \times 3 + \dots$$

$$\boxed{\Gamma_4} \equiv R$$

Renormalized expansion

$$\Gamma_4 = R - \frac{1}{2} \left[\text{bubble}(R, R) \right] \times 3 + \dots$$

$$\left[\text{bubble}(R, R) - \boxed{\text{bubble}(R, R)} \right]$$

BPHZ Renormalization Scheme

- Zimmermann's forest formula:**

Identify all 1PI 4-vertex subgraphs, then subtract the “divergent” pieces one by one (smaller vertices first).

$$\begin{aligned}
 & (1 - \hat{P}_{\text{loop}})(1 - \hat{P}_{\text{tadpole}}) \text{ (diagram with vertices } R_1, R_2, R_3) \\
 &= \text{(diagram with vertices } R_1, R_2, R_3) - \boxed{\text{(diagram with vertices } R_1, R_2)} + \boxed{\text{(diagram with vertices } R_1, R_2, R_3)} - \boxed{\text{(diagram with vertices } R_1, R_2)}
 \end{aligned}$$

N. N. Bogoliubov and O. S. Parasiuk, Acta. Math. 97, 227 (1957).
 K. Hepp, Comm. Math. Phys. 2, 301 (1966).
 W. Zimmermann, Math. Phys. 11,1 (1968) .
 W. Zimmermann, Math. Phys. 15, 208 (1969).

- Greatly simplifies when interactions are point-wise(all boxes are simple constants), e.g. QED, ϕ^4 theory, ...
- Too expansive for generic condensed matter field theories with complicated interactions, e.g. Fermi liquid theory, superconductivity, ...

Dyson-Schwinger Equations

- **Dyson-Schwinger Equations:** A set of EXACT equations connecting different 1PI vertex functions

The diagram illustrates the Dyson-Schwinger Equations (DSEs) for 1PI vertex functions. It shows two equations:

1. The equation for the 4-point vertex function Γ_4 :

$$\Gamma_4 = u - \frac{1}{2} \text{ (bubble diagram) } \times 3 + \frac{1}{2} \text{ (rainbow diagram) } \times 3 - \frac{1}{6} \text{ (self-energy diagram) } \Gamma_6$$

2. The equation for the 6-point vertex function Γ_6 :

$$\Gamma_6 = \text{ (diagram with two } \Gamma_4 \text{ sub-diagrams) } \times 15 + \dots$$

Blue curved arrows indicate the loop order: n loops for the first equation and $n+1$ loops for the second equation.

- **The key ideas:**

1. DSEs can recursively generate all Feynman diagrams.
2. If one can renormalize DSEs, then the renormalized DSEs can be a generator for all renormalized diagrams.

DSE Renormalization Scheme

- Derivation of the recursive rules:

$$(1) \quad \text{cross}(u) = \text{circle}(\Gamma_4) + \frac{1}{2} \text{diagram1} \times 3 - \frac{1}{2} \text{diagram2} \times 3 + \dots$$

$$(2) \quad \text{cross}(u) = \text{circle}(R) + \frac{1}{2} \boxed{\text{diagram1}} \times 3 - \frac{1}{2} \boxed{\text{diagram2}} \times 3 + \dots$$

$$(3)=(1)-(2) \quad \text{circle}(\Gamma_4) = \text{circle}(R) - \frac{1}{2} \text{diagram1} \times 3 + \frac{1}{2} \text{diagram2} \times 3 - \dots$$

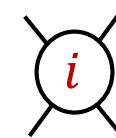
$$\text{cross}(k_1, k_2, k_3, k_4) \equiv \text{circle}(\Gamma_4^{ref})$$

$$\boxed{\text{cross}(u)} = \text{cross}(u)$$

- The key insight: two dual equations

$$\text{cross}(u) = \text{circle}(R) \xi^1 - \boxed{2} \xi^2 - \boxed{3} \xi^3 + \dots$$

$$\text{circle}(\Gamma_4) = \text{circle}(R) \xi^1 + \boxed{2} \xi^2 + \boxed{3} \xi^3 + \dots$$



contain i number of renormalized couplings

DSE Renormalization Scheme

$$\text{crossed } u = \text{circle } R + \frac{1}{2} \left[\text{diagram with } \Gamma_4 \text{ and loop} \right] \times 3 - \frac{1}{2} \left[\text{diagram with } \Gamma_4 \text{ and loop} \right] \times 3 + \dots$$

$$\text{crossed } \Gamma_4 = \text{circle } R - \frac{1}{2} \left[\text{diagram with } \Gamma_4 \text{ and loop} \right] \times 3 + \frac{1}{2} \left[\text{diagram with } \Gamma_4 \text{ and loop} \right] \times 3 - \dots$$

$$\text{crossed } u = \text{circle } R \xi^1 - \boxed{2} \xi^2 - \boxed{3} \xi^3 + \dots$$

$$\text{crossed } \Gamma_4 = \text{circle } R \xi^1 + \boxed{2} \xi^2 + \boxed{3} \xi^3 + \dots$$

$$\boxed{1} = \text{circle } R$$

$$\boxed{2} = -\frac{1}{2} \left[\text{diagram with } R \text{ and loop} \right] \times 3$$

$$\boxed{3} = -\frac{1}{2} \left[\text{diagram with } R \text{ and } \boxed{2} \right] \times 3 + \frac{1}{2} \left[\boxed{2} \text{ and } R \right] \times 3 + \left[\text{diagram with } R \text{ and } R \right] \times 3$$

Bottom-Up Recursive DiagMC

- Renormalized DSEs leads to a bottom-up recursive DiagMC.

$$\begin{aligned}
 \text{Diagram } N &= - \sum_{i+j=N} \frac{1}{2} \left[\text{Diagram } i \text{ connected to Diagram } j \right] \times 3 + \sum_{i+j+k=N} \frac{1}{2} \left[\text{Diagram } i \text{ connected to Diagram } j \text{ connected to Diagram } k \right] \times 3 - \sum_{i+j=N} \frac{1}{6} \left[\text{Diagram } i \text{ connected to Diagram } j \right]
 \end{aligned}$$

- Build higher order diagrams from lower order vertices.

R. Rossi, Determinant diagrammatic Monte Carlo algorithm in the thermodynamic limit. *PRL*, 119, 045701 (2017).

- Similar efficiency for any representation (space/time, momentum/frequency, or mixed).
- Easy to separate different channels (very important for Fermi liquid/superconductivity).
- If necessary, Γ_6 , Γ_8 , ... can also be renormalized without sacrificing the efficiency.

Step 3.

Q: How to calculate relevant/marginal couplings?

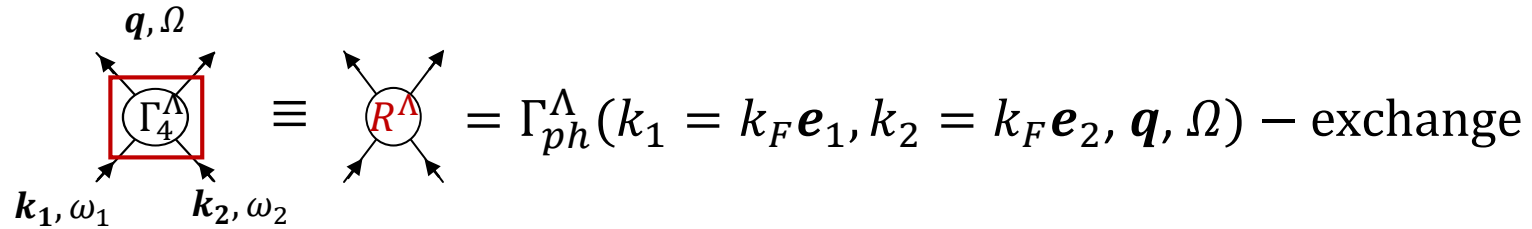
A: Renormalization Group/Skeleton diagrammatic techniques/DSE

Fermi Liquid

- A spinless Fermi liquid at $d = 2$:

$$L = \sum_{\mathbf{k}} \psi_{\mathbf{k},\tau}^+ \left(\frac{\partial}{\partial \tau} + \mathbf{k}^2 - \mu + R_{\mathbf{k}}^\Lambda \right) \psi_{\mathbf{k},\tau} + \frac{1}{2V} \sum_{\mathbf{q} \mathbf{k} \mathbf{k}'} \frac{8\pi}{q^2 + \lambda} \psi_{\mathbf{k},\tau}^+ \psi_{\mathbf{k}-\mathbf{q},\tau}^+ \psi_{\mathbf{k}'+\mathbf{q},\tau} \psi_{\mathbf{k}',\tau}$$

- Effective interaction:

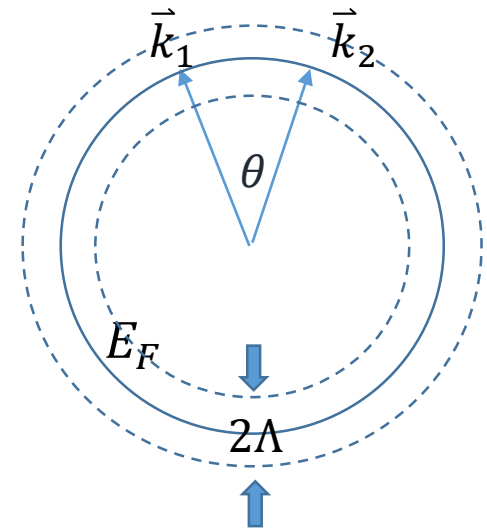


$$\text{Diagram} \equiv \text{Diagram} = \Gamma_{ph}^\Lambda(k_1 = k_F \mathbf{e}_1, k_2 = k_F \mathbf{e}_2, \mathbf{q}, \Omega) - \text{exchange}$$

- The IR regulator suppresses the slow modes near the Fermi surface.

$$G_\Lambda(\mathbf{k}, \tau) = G(\mathbf{k}, \tau) \frac{(k - k_F)^2}{(k - k_F)^2 + \Lambda^2}$$

- Recover the physical system when $\Lambda \rightarrow 0$.



Functional RG Approach

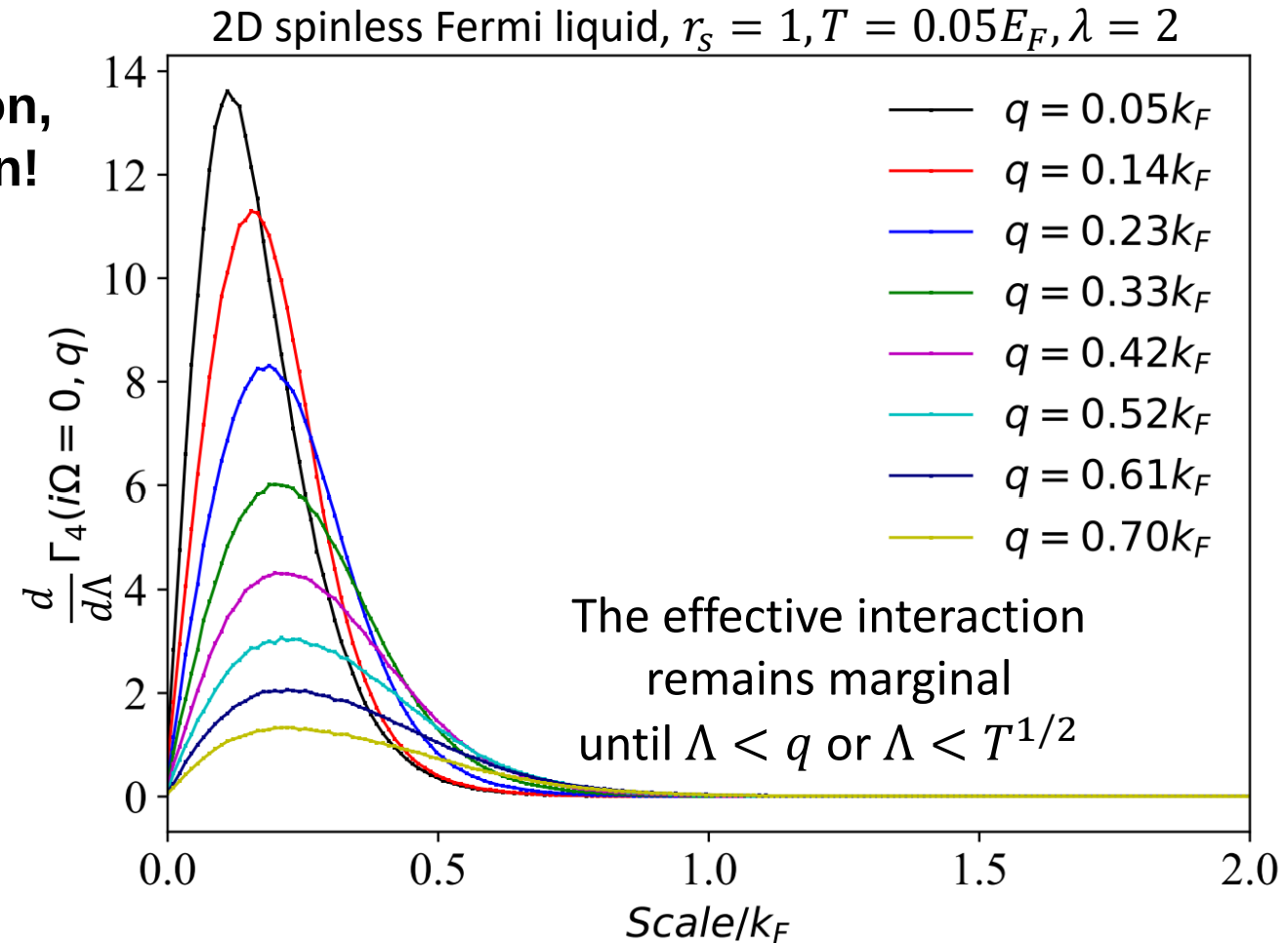
- DiagMC samples the diagrams for β function, and at the same time, solve the RG equation!

$$\frac{d}{d\Lambda} \text{circle}(R_\Lambda) = - \left[\text{diagram with two } \Gamma_4^\Lambda \text{ nodes} \right] \times 3 + \frac{1}{2} \left[\text{diagram with one } \Gamma_6^\Lambda \text{ node} \right]$$

where,

$$\text{circle}(\Gamma_4^\Lambda) = \text{circle}(R_\Lambda) \cdot \xi + \left[\text{circle}(2) \right] \cdot \xi^2 + \left[\text{circle}(3) \right] \cdot \xi^3 + \dots$$

$$\text{circle}(\Gamma_6^\Lambda) = \left[\text{diagram with three } R_\Lambda \text{ nodes} \right] \times 15 \cdot \xi + \dots$$



Consistent with theoretical predictions:

R. Shankar, Rev. Mod. Phys. 66, 129 (1994)

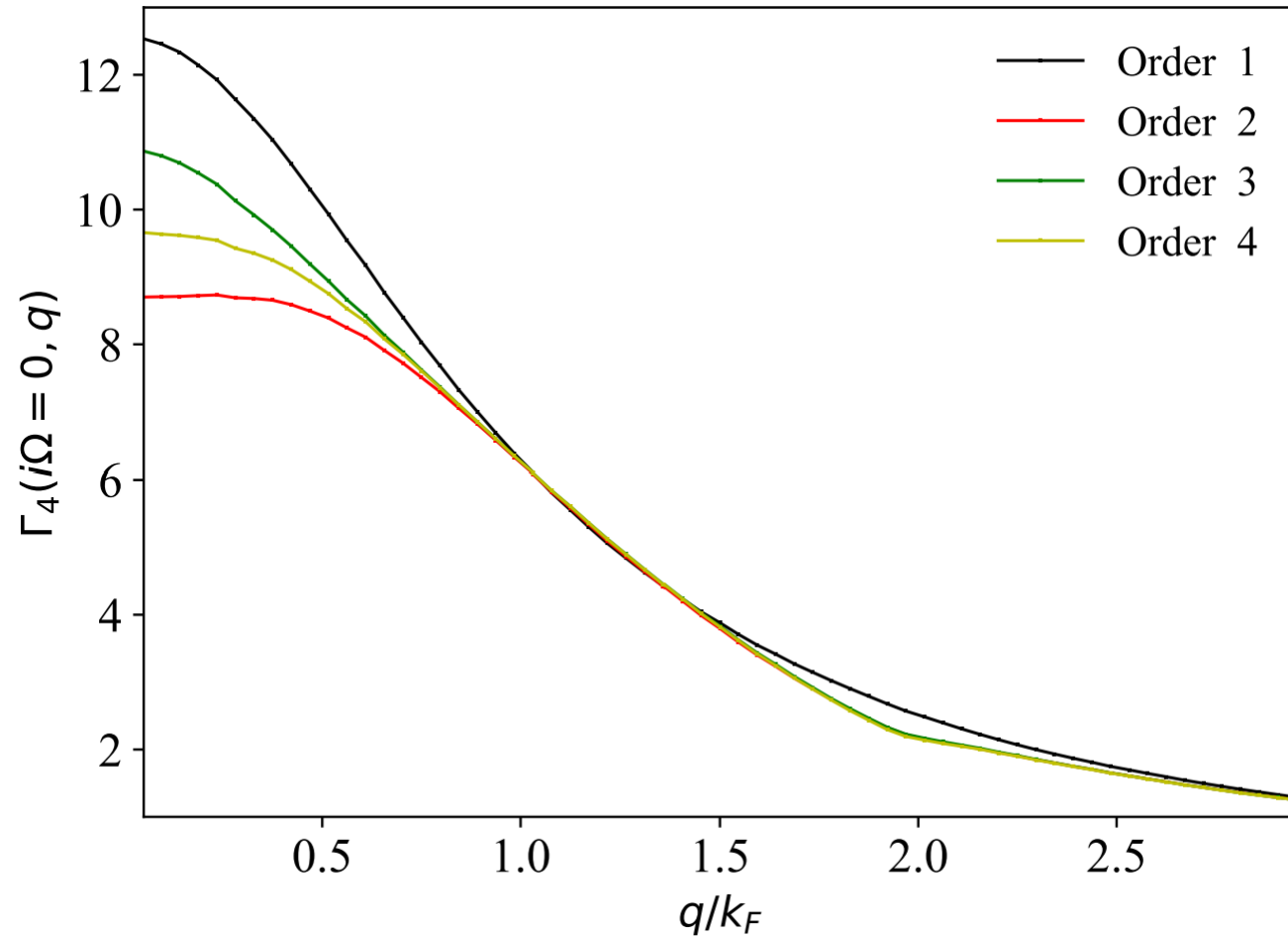
N. Dupuis and G. Y. Chitov, Phys. Rev. B. 54,3040 (1996)

G. Hitov, D. Senechal, Phys. Rev. B. 57, 1444 (1998)

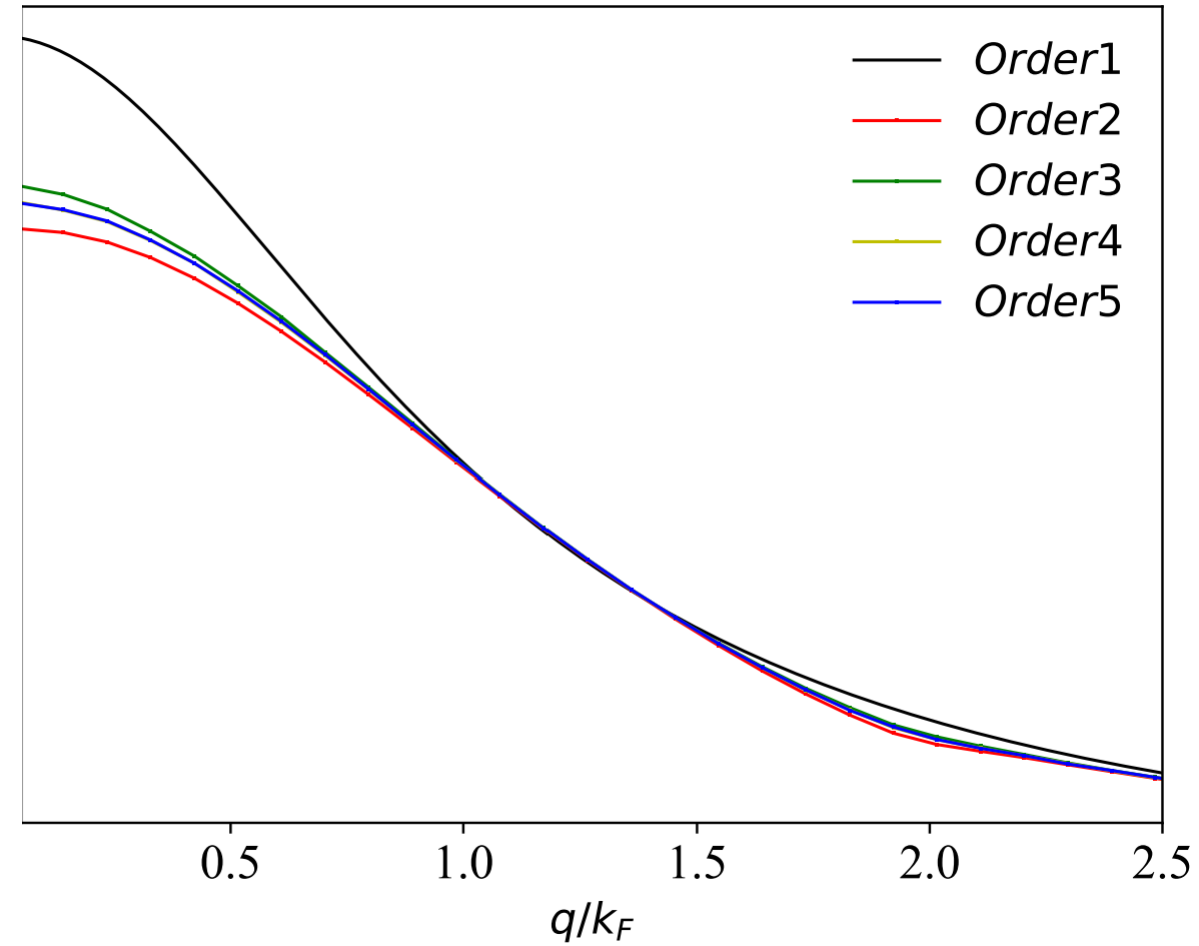
Improved Convergence/Efficiency

$\Lambda \rightarrow 0$: the physical vertex function

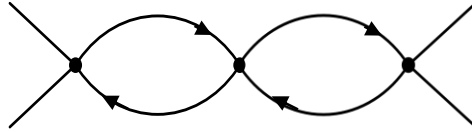
Bare coupling expansion



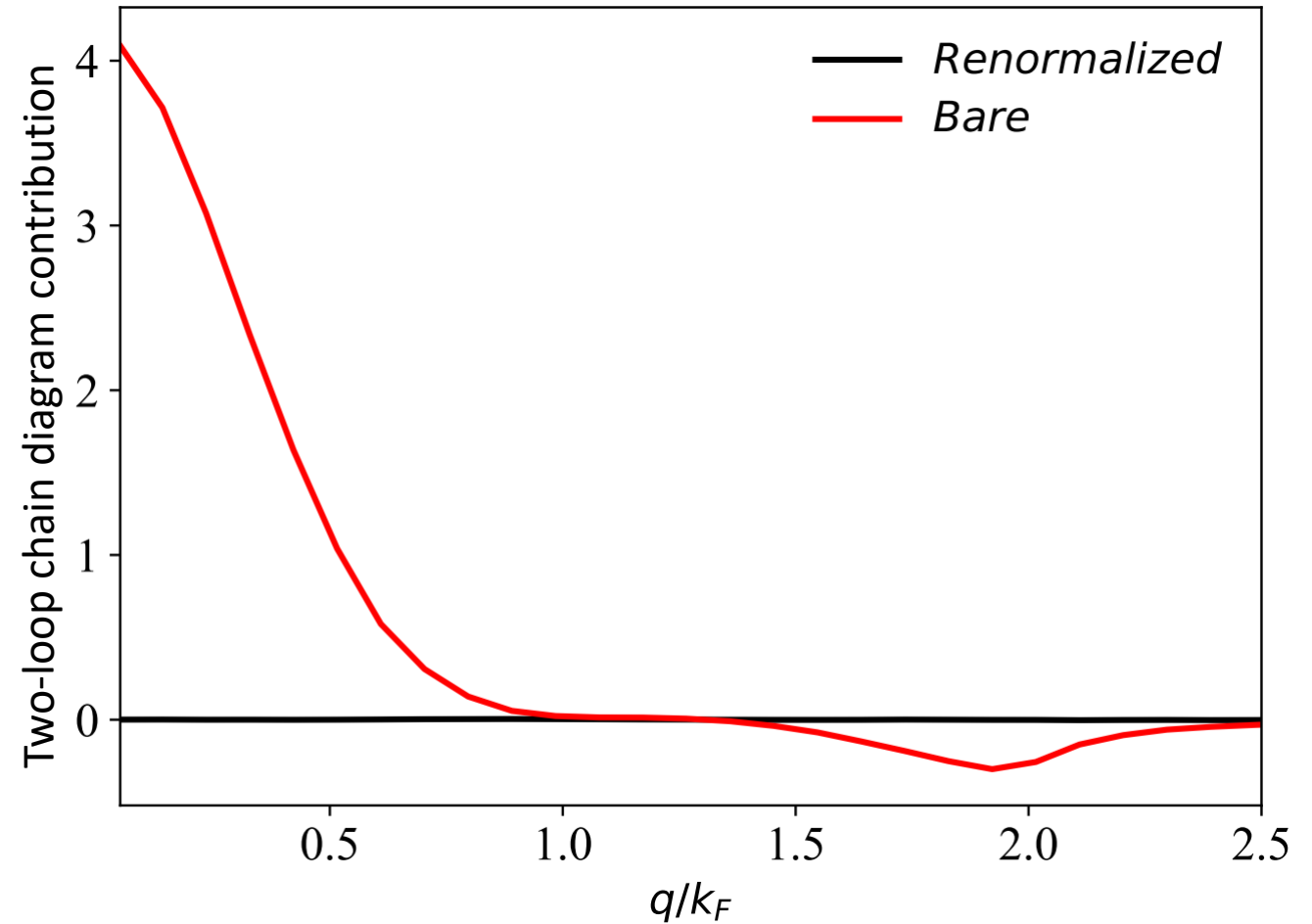
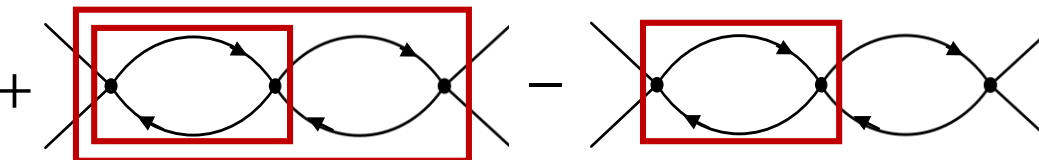
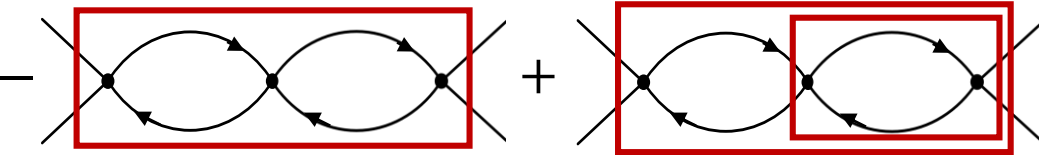
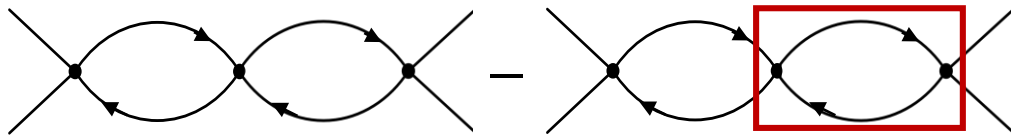
Renormalized expansion+RG



Sign Cancellation Between Diagrams



Bubble diagrams + vertex corrections + ph-ladder-type diagrams



Workflow

