

The fine print of the magneto-hydrostatics problem

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and Matthew Hole², Robert Dewar², Dean Muir², Keneth Duru², Kooper de Lacy¹, Jack Thompson¹,
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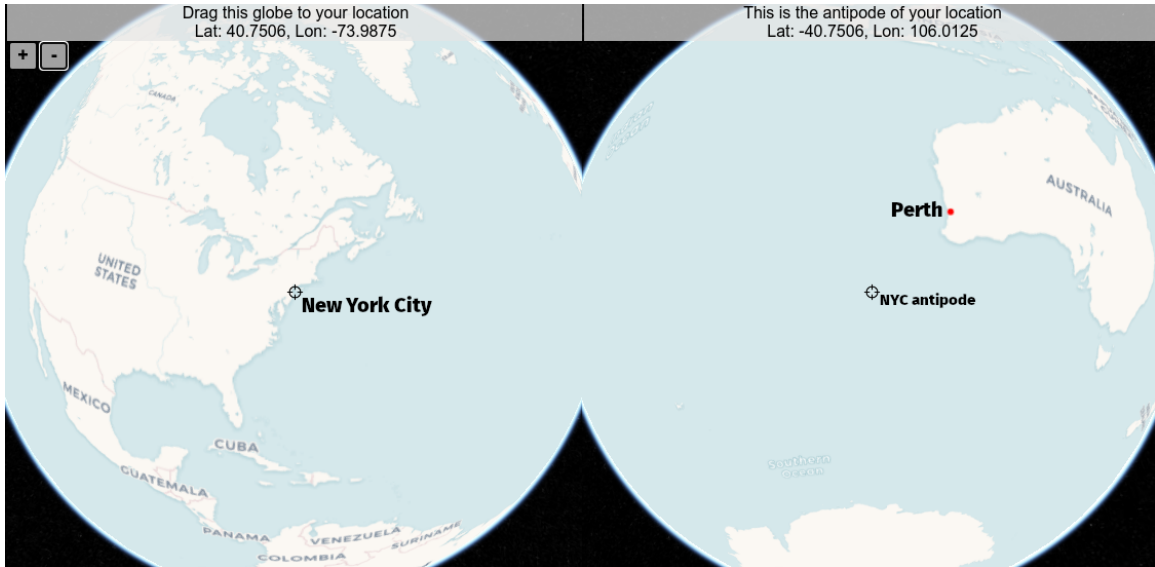
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I come from a land downunder...

<https://engaging-data.com/antipodes-map/>



Avec des “si”, on mettrait Paris en bouteille

If wishes were horses, beggars would ride

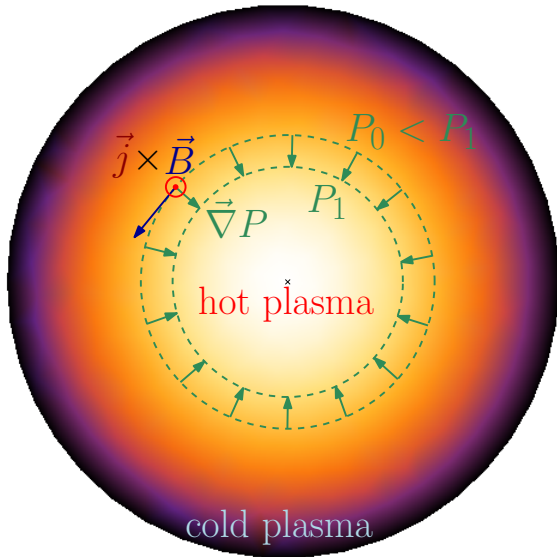


Statue of King Leonidas I in modern-day Sparta

Philip [II of Macedon] wrote to them [Spartans],
***"If** I invade Laconia, I will drive you all out of
house and home,"*
they only wrote back,
"If."

– Plutarch, On Talkativeness, 100BC

MHS is balance of Lorentz force and pressure gradient



Magneto-hydrostatics

force-balance

$$\vec{j} \times \vec{B} = \nabla p$$

Maxwell equations

$$\nabla \times \vec{B} = \vec{j}$$

$$\nabla \cdot \vec{B} = 0$$

boundary conditions

$$p|_{\partial M} = \text{const}$$

$$(\vec{B} \cdot \vec{n})|_{\partial M} = 0$$

if $j \times B = \nabla p$

then

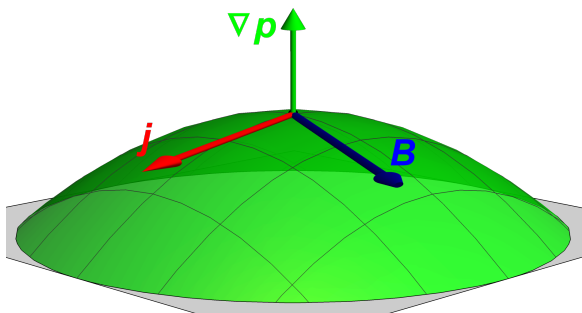
$$B \cdot \nabla p = 0 \text{ and } j \cdot \nabla p = 0$$



p is an **invariant** of B and j



j and B are tangent to the level sets of p



Regularity of pressure implies nested flux surfaces

Let $M \subset \mathbb{R}^3$ compact and connected manifold with boundary

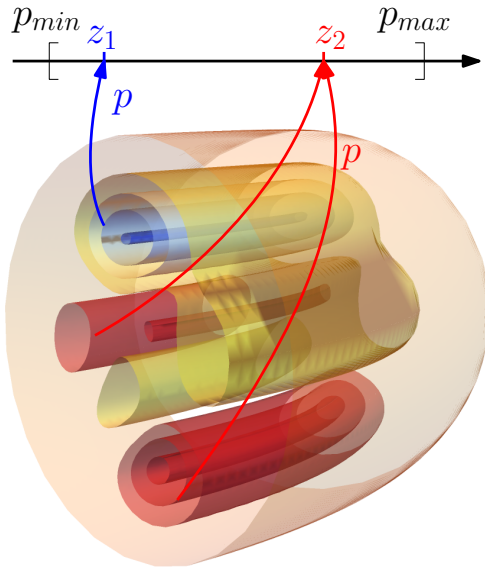
By Sard's lemma [Morse, 1939; Sard, 1965],

if $p \in C^3(M)$, then regular values are **dense** in $[p_{min}, p_{max}]$.

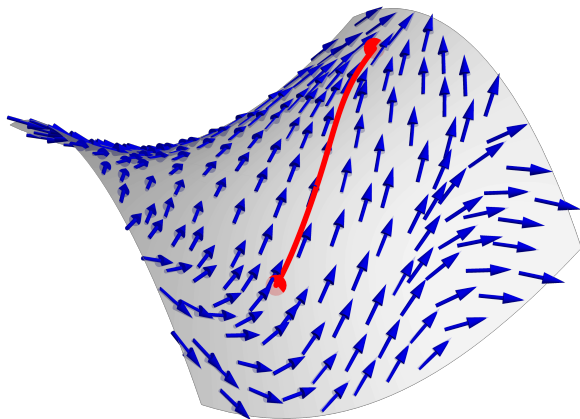
$\exists$ “plenty” of **regular** surfaces $p(x) = z$ with $|\nabla p| > 0$.

On regular surfaces, magnetic field and current density are non-vanishing pointwise linearly independent

$$j \times B = \nabla p \neq 0 \quad \Rightarrow \quad B \neq 0, j \neq 0$$



Vector fields are ODEs on manifolds



Given vector field X on a manifold M ,
an **integral curve** through x_0 solves IVP

$$\gamma'(t) = X(\gamma(t))$$

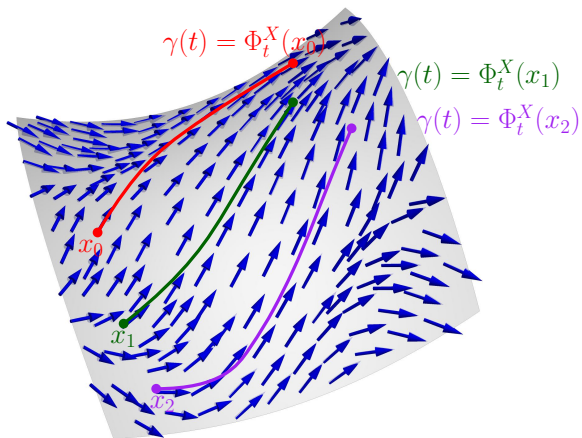
$$\gamma(0) = x_0$$

Tech note: if M compact and X smooth and tangential^a, then

$$\gamma : \mathbb{R} \rightarrow M$$

$${}^a X \cdot n|_{\partial M} = 0$$

Vector fields are generators of “flow”



Collecting integral curves with fixed “time” t as a function of the origin $x \in M$ yields the **flow**

$$\begin{aligned}\Phi_t^X : M &\rightarrow M \\ x &\mapsto \Phi_t^X(x) = \gamma(t)\end{aligned}$$

where γ is an integral curve passing through x at $t = 0$.

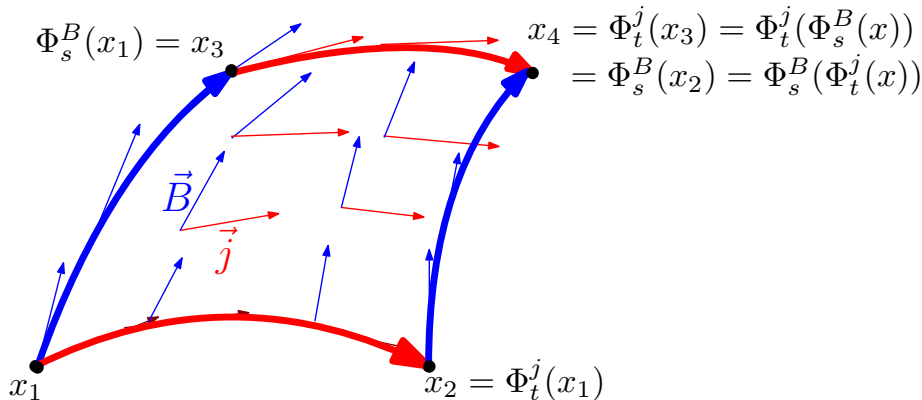
Force-balance implies B and j flows commute

Geometric interpretation of force-balance

$$\nabla \times (j \times B) = \nabla \times \nabla p = 0 \iff (j \cdot \nabla)B - (B \cdot \nabla)j =: [j, B] = 0$$



$$\Phi_t^j \circ \Phi_s^B = \Phi_s^B \circ \Phi_t^j, \quad \text{flow order does not matter!}$$



Proposition

With $\nabla \cdot \mathbf{X} = 0$, $\nabla \cdot \mathbf{Y} = 0$,

$$\begin{array}{lcl} \mathbf{X} \times \mathbf{Y} = \nabla p & \iff & [\mathbf{X}, \mathbf{Y}] = 0 \\ |\nabla p| > 0 & & \mathbf{X}, \mathbf{Y} \text{ lin.ind.} \end{array}$$

- MHS force-balance $\mathbf{j} \times \mathbf{B} = \nabla p$ is the statement that $\mathbf{j}$ is a symmetry of $\mathbf{B}$ (found it!)

Arnold structure theorem and Hamada coordinates

[Hamada, 1962; Arnold, 1965, 1966, 1974]

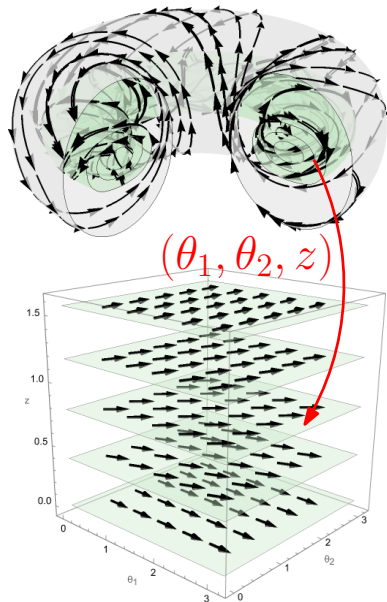
Let $M \subset \mathbb{R}^3$ compact and connected manifold with boundary

if $j \times B = \nabla p$, $p|_{\partial M} = \text{const}$

- 1 compact connected components of regular surfaces are **tori**
- 2 $\exists(\theta_1, \theta_2, z) : U \subset M \rightarrow \mathbb{R}^2/\mathbb{Z}^2 \times I$, i.e.
Hamada coordinates s.t.

$$j = a_1(z) \frac{\partial}{\partial \theta_1} + a_2(z) \frac{\partial}{\partial \theta_2}$$
$$B = b_1(z) \frac{\partial}{\partial \theta_1} + b_2(z) \frac{\partial}{\partial \theta_2}$$

the current and magnetic field-lines are straight and the vectors have been **rectified**.



Rectifiability of a vector field is special

Rectifiability is a well understood dynamical problem [Sternberg, 1969].

Theorem

if X is a **nowhere zero** vector field on a 2-torus S . Then, the following are equivalent:

- ① X preserves an area-form on S
- ② X is winding ($\sim$ has rotational transform)
- ③ X is semi-rectifiable.

Moreover, **if** X has Diophantine winding number, then X is rectifiable.

Semi-rectifiability is the weaker requirement that $\exists \rho > 0$ s.t.

$$X/\rho = a_1 \frac{\partial}{\partial \theta_1} + a_2 \frac{\partial}{\partial \theta_2}$$

namely that field-lines are straight but the vector field X is not “constant”.

For magnetic fields, slightly less can be assumed...

Theorem ([Perrella et al., 2022])

Let S be an embedded 2-torus in $M \subset \mathbb{R}^3$ and $\mathbf{n}$ be a unit normal.

if B satisfies

$$(\nabla \cdot B)|_S = 0, \quad (\nabla \times B)|_S \cdot \mathbf{n} = 0, \quad B|_S \cdot \mathbf{n} = 0$$

then, the following are equivalent

- ① $\exists u \in C^\infty(S)$ solution to the cohomological equation

$$B|_S \cdot \nabla u = [\mathbf{n}, B] \cdot \mathbf{n}$$

- ② either $B|_S \equiv 0$ or

$B|_S$ is nowhere zero and semi-rectifiable with $\rho = |B|^2|_S$.

An application assuming flux-surfaces

Let $B \neq 0$ be a magnetic field ($\nabla \cdot B = 0$).

if p is an invariant of B with toroidal regular surfaces (flux function)

$$S \subset \{x \in M : p(x) = z, |\nabla p|(x) > 0\} \cong \mathbb{R}^2 / \mathbb{Z}^2$$

if $(\nabla \times B)|_S \cdot \nabla p|_S = 0$

then,

- $u = -\ln |\nabla p|_S$ is a solution to the cohomological equation
- $B|_S$ is nowhere zero and semi-rectifiable
- naturally

$$(\nabla \times B) \times B|_S = \lambda \nabla p|_S$$

for $\lambda \in C^\infty(S)$ nowhere zero.

Reeb cylinder as obstruction to semi-rectifiability

despite nested flux-surfaces (no islands) [Peralta-Salas et al., 2021]

In cylindrical coordinates (R, φ, Z) , consider

[Perrella et al., 2023]

$$B = \nabla\psi \times \nabla\varphi + f\nabla\varphi$$

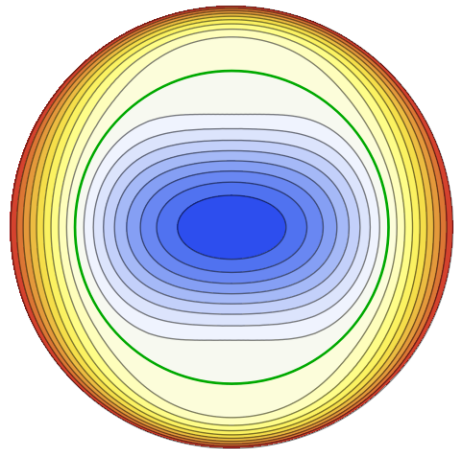
where

$$\psi = (\tilde{R}^2 + Z^2 - 1)(1 + Z^2(\tilde{R}^2 + Z^2 - 2))$$

$$f = Z + (\tilde{R}^2 + Z^2 - 1)$$

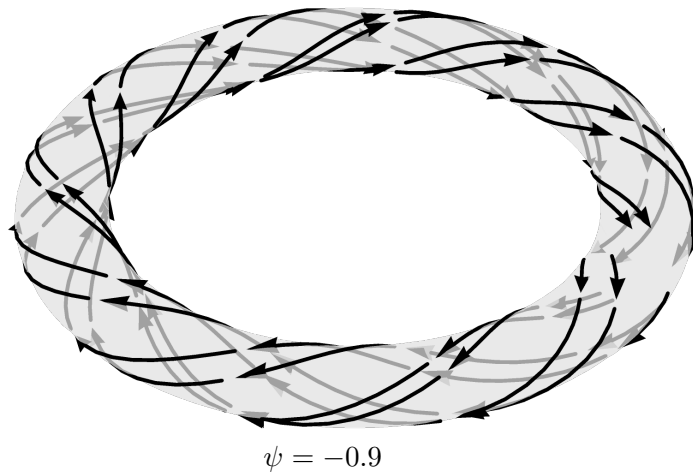
and $\tilde{R} = R - 2$.

- 1 $\psi = \text{const}$ are nested toroidal surfaces ✓
- 2 $|B| > 0$ ✓

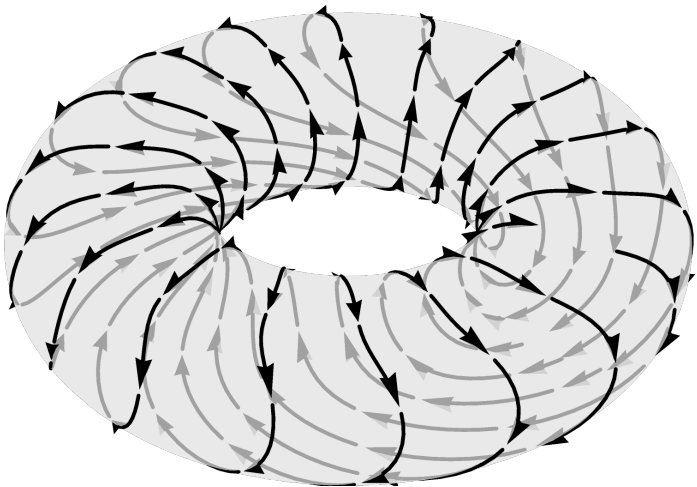


Poloidal cross-section of ψ levels.
Critical surface (green) at $\psi = 0$
where $\tilde{R}^2 + Z^2 = 1$.

Critical surface contains “Reeb component”

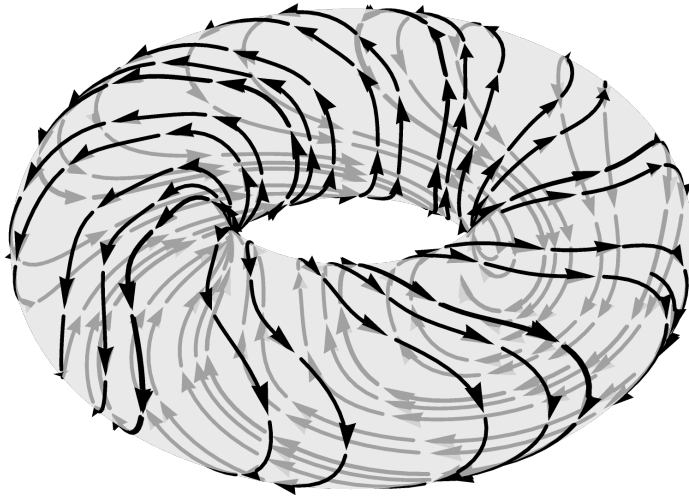


Critical surface contains “Reeb component”



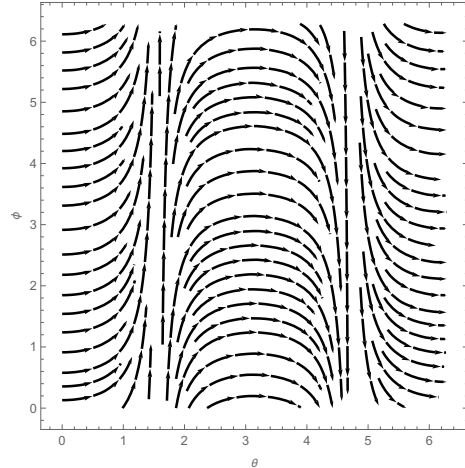
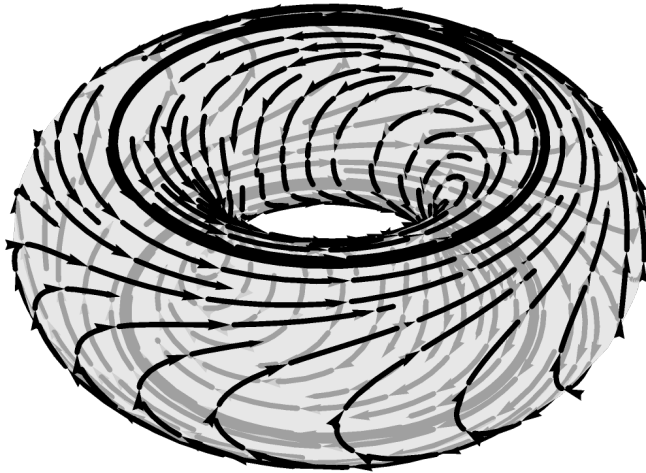
$$\psi = -0.2$$

Critical surface contains “Reeb component”



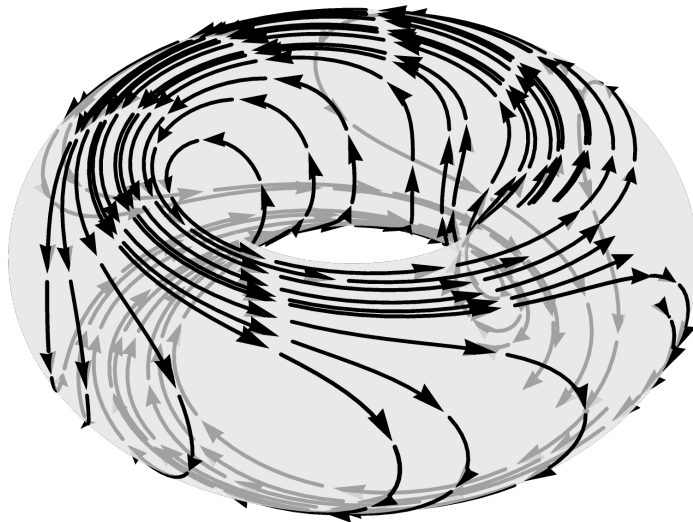
$$\psi = -0.1$$

Critical surface contains “Reeb component”



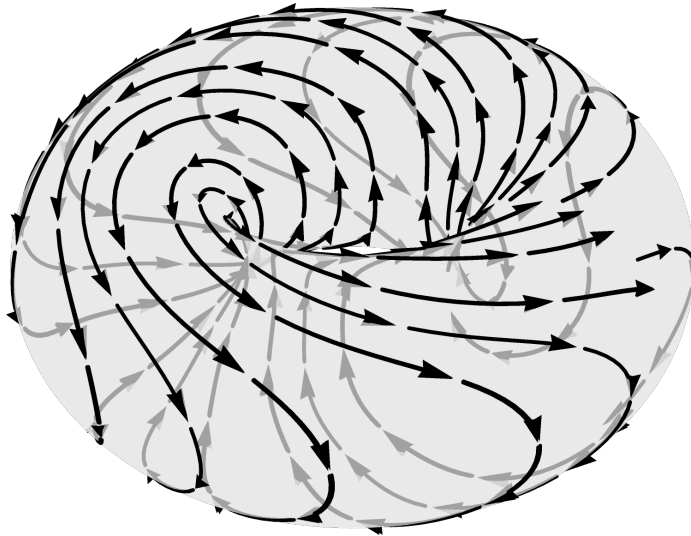
$\psi = 0$. The toroidal surface $\tilde{R}^2 + Z^2 = 1$ does not have straight field-line coordinates!

Critical surface contains “Reeb component”



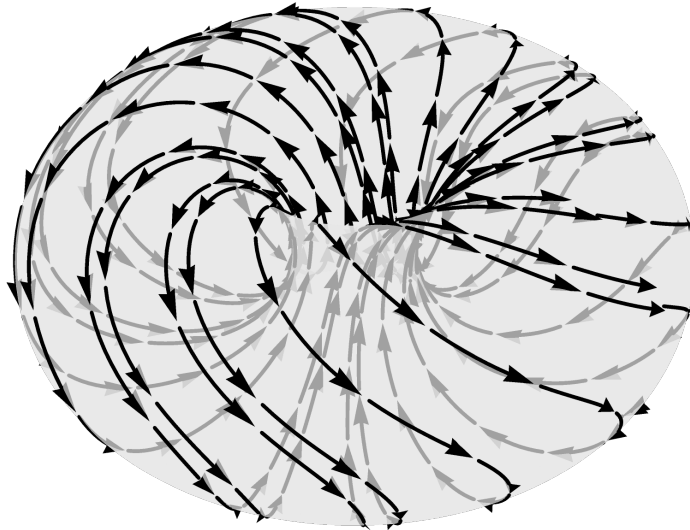
$$\psi = 0.01$$

Critical surface contains “Reeb component”



$$\psi = 0.1$$

Critical surface contains “Reeb component”



$$\psi = 0.9$$

Symmetry gives rise to invariants

Recalling

Proposition

With $\nabla \cdot \mathbf{X} = 0$, $\nabla \cdot \mathbf{Y} = 0$,

$$\begin{aligned} \mathbf{X} \times \mathbf{Y} = \nabla p &\iff [\mathbf{X}, \mathbf{Y}] = 0 \\ |\nabla p| > 0 &\quad \mathbf{X}, \mathbf{Y} \text{ lin.ind.} \end{aligned}$$

Let B be a magnetic field ($\nabla \cdot B = 0$)

Proposition (Noether theorem for magnetic fields)

conformal symmetry

$\Rightarrow$

invariant

$\exists f > 0$ and field u (lin.ind.) such that

$$\nabla \cdot (fu) = 0$$

$$[u, B/f] = 0$$

$\exists$ non-constant p such that

$$B \cdot \nabla p = 0$$

Proposition (Noether theorem for magnetic fields)

symmetry $\Rightarrow$ *invariant*

no invariant $\Rightarrow$ *no symmetry*

- Clebsch representation imposes two invariant functions α and β

$$B = \nabla\alpha \times \nabla\beta$$

- In contrast, the only invariant functions of ergodic fields B are constants

The converse of the Noether theorem is in general false.

Proposition (falsity of converse Noether)

invariant



symmetry

Adapted fields as sufficient condition

Proposition ([Perrella et al., 2023])

if B is a magnetic field with flux function p

if B has an *adapted vector field* V s.t.

$$B \cdot V > 0,$$

$$(\nabla \times V) \cdot \nabla p = 0,$$

then, setting

$$f = V \cdot B,$$

$$u = \frac{\nabla p \times V}{V \cdot B},$$

yields a *conformal symmetry*

$$[u, B/f] = 0 \iff u \times B = \nabla p$$

- if** magnetic field B
- ① has a flux function p , $B \cdot \nabla p = 0$
 - ② nowhere zero
 - ③ $(\nabla \times B) \cdot \nabla p = 0$



$V = B$ is an adapted vector field

$$u = \frac{\nabla p \times B}{B^2}, \quad \left[u, \frac{B}{B^2} \right] = 0$$

Arnold structure theorem yields

Boozer coordinates

if magnetic field B

① has a flux function p , $B \cdot \nabla p = 0$

② nowhere vanishing toroidal component, $B \cdot \nabla \varphi > 0$

$\Longleftrightarrow$

$V = \nabla \varphi$ is an adapted vector field

$$u = \frac{\nabla p \times \nabla \varphi}{B \cdot \nabla \varphi}, \quad \left[u, \frac{B}{B \cdot \nabla \varphi} \right] = 0$$

Arnold structure theorem yields

PEST coordinates

- u is purely poloidal
- toroidal angle φ remains a coordinate

More examples:

- any field E with $\nabla \times E = 0$ such that $B \cdot E > 0$ is an adapted vector field

Adapted fields relevant to near-axis expansion

Pick an axis curve $\gamma : S^1 \rightarrow M$, $s \mapsto \gamma(s)$ for B .

- 1 we expect $B|_{\gamma} \cdot \nabla s > 0$
- 2 $\exists$ tubular neighbourhood U of γ which gives a projection (retraction)
 $r : U \rightarrow \gamma$
- 3 set $\phi : U \rightarrow S^1$ so that

$$\phi(x) = s(r(x))$$

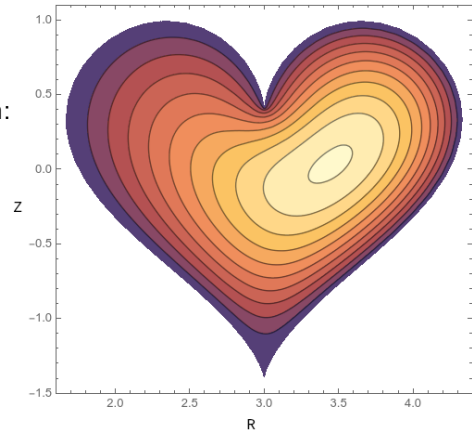
- 4 by continuity, it still holds on a tubular neighbourhood $V \subseteq U$ of γ that

$$B \cdot \nabla \phi > 0$$

if B has a flux function p compatible with axis γ , then $V = \nabla \phi$ is adapted and near-axis straight field-line coordinates are guaranteed.

We did not have time to talk about...

- if axisymmetry, MHS reduces to an elliptic problem: well-posedness provided sufficient regularity [Jeyakumar et al., 2021]
- rigidity of smooth solutions under small ideal deformation [Pfefferlé et al., 2020]
- MHS solutions in non-Euclidean geometry without Killing fields [Cardona et al., 2023]
- MHS solutions without axisymmetry when $p|_{\partial M} \neq \text{const}$ [Sato and Yamada, 2023]
- identification of magnetic fields as $1\frac{1}{2}D$ Hamiltonian systems
- observations about the MRxMHD model



solution to Grad-Shafranov equation on domain bounded by

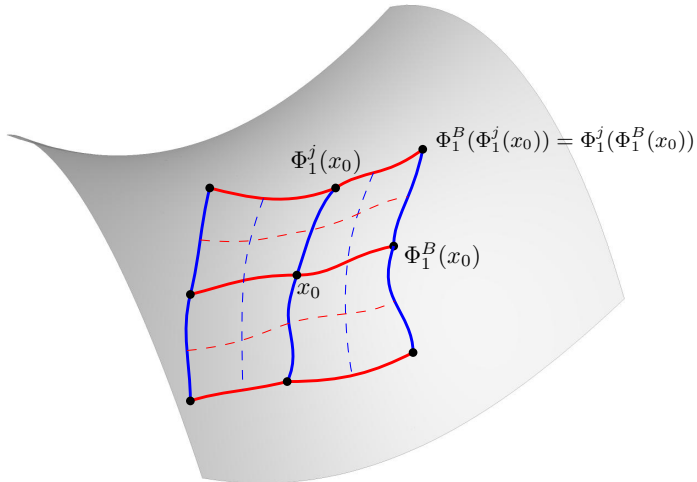
$$R(\theta) = 3 + \sin(t) - \frac{1}{3} \sin(3t),$$

$$Z(\theta) = \frac{13 \cos(t) - 5 \cos(2t) - 2 \cos(3t) - \cos(4t)}{12}$$

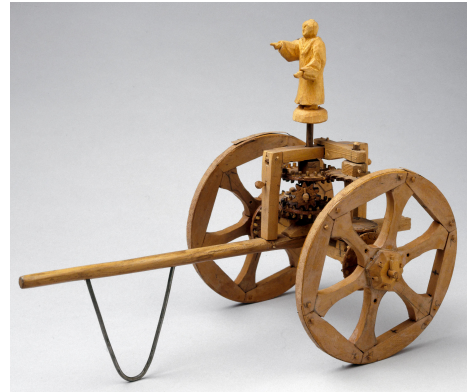
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Regular pressure surfaces can be charted



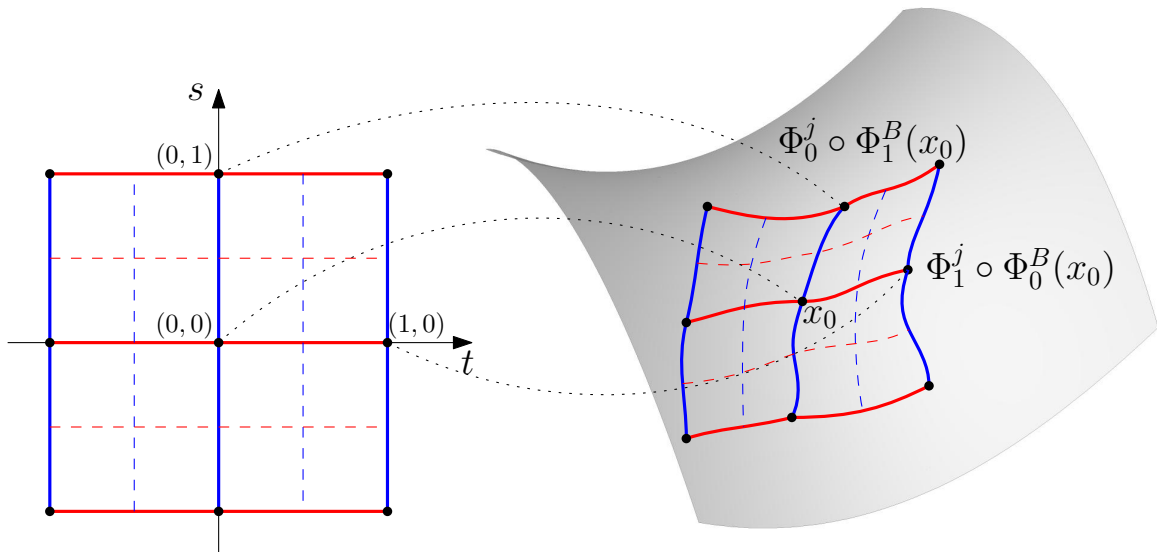
We can explore pressure surfaces by flowing along B and j without getting lost.



south-pointing cart for navigation before compasses (1050BC)

Step 1: local one-to-one correspondence

between $\mathbb{R}^2$ and $p = \text{const}$ near x_0

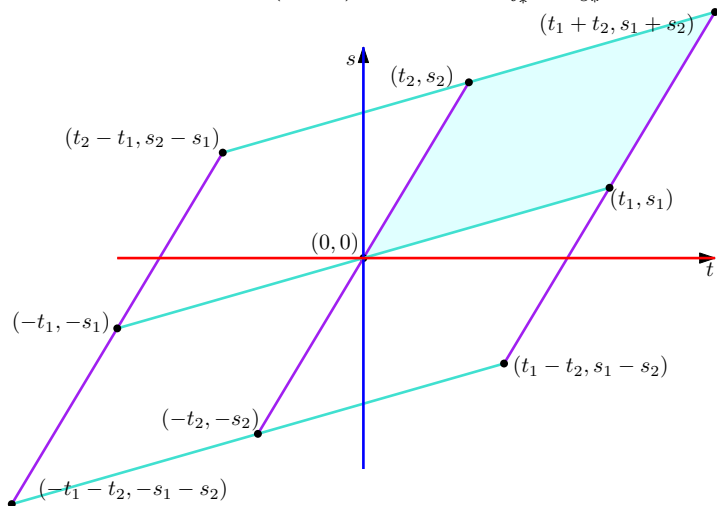


Note: $\Phi_t^j = \Phi_1^{tj}$ and $\Phi_s^B = \Phi_1^{sB}$ so that $\Phi_t^j \circ \Phi_s^B = \Phi_1^{tj+sB}$

Step 2: explore entire surface and record fixed points

Knowing that the surface is compact (closed and bounded in $\mathbb{R}^3$), there must be pairs

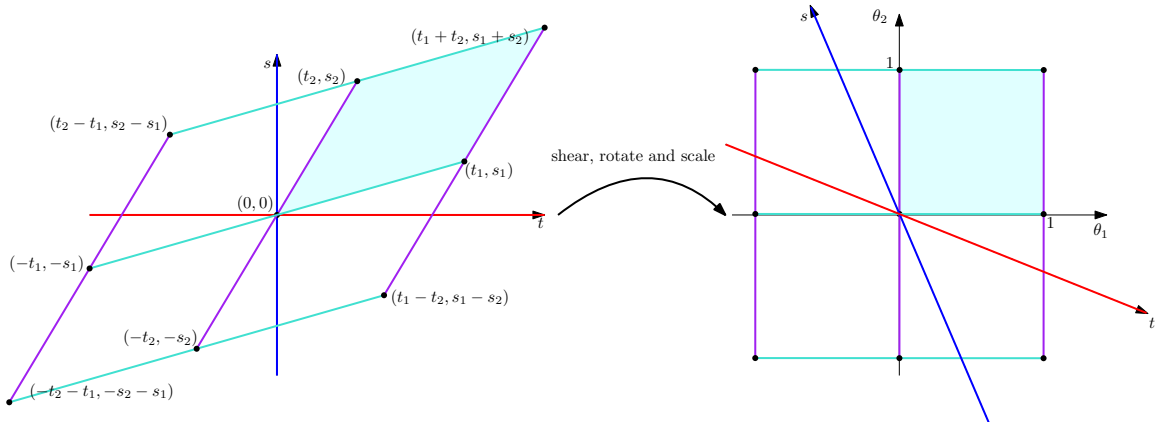
$$(t_*, s_*) \text{ such that } \Phi_{t_*}^j \circ \Phi_{s_*}^B = \text{id}$$



$$\Phi_1^{t_1 j + s_1 B}(x_0) = x_0$$

$$\Phi_1^{t_2 j + s_2 B}(x_0) = x_0$$

Step 3: shear, rotate and scale



hence

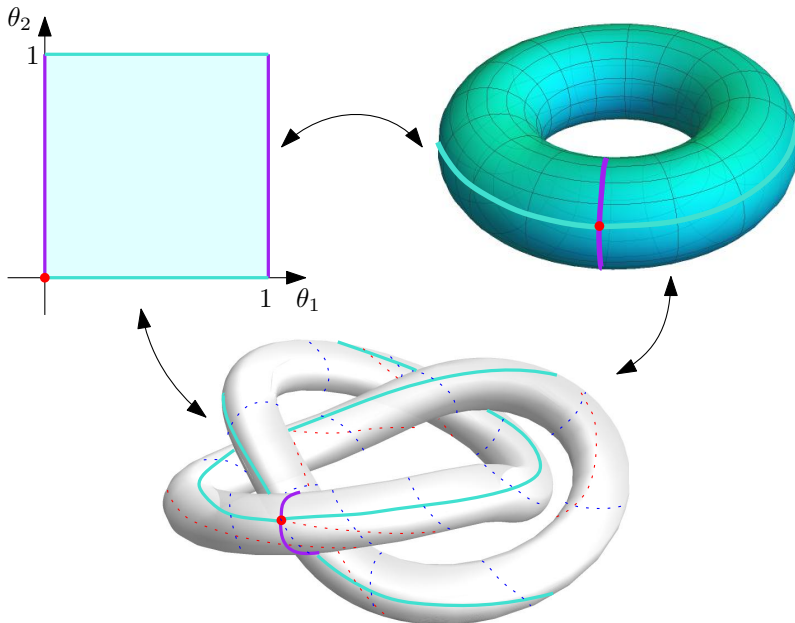
$$\text{because } \Phi_1^{t_k j + s_k B}(x_0) = x_0$$

$$\frac{\partial}{\partial \theta_k} = t_k j + s_k B$$

$$j = a_1 \frac{\partial}{\partial \theta_1} + a_2 \frac{\partial}{\partial \theta_2}$$

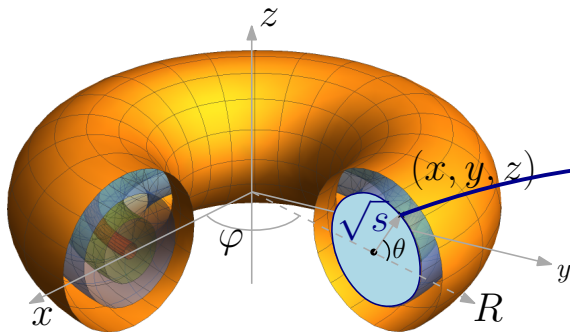
$$B = b_1 \frac{\partial}{\partial \theta_1} + b_2 \frac{\partial}{\partial \theta_2}$$

Step 4: quotient space $\mathbb{R}^2/\mathbb{Z}^2$ is a torus

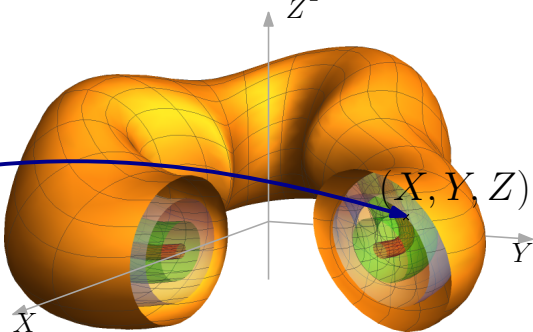


Is it easy to generate nested flux-surfaces ?

standard torus



deformed space



smooth map

$$\mathbf{a} = \Phi_e s \nabla \theta + \Psi(s) \nabla \varphi$$

$$\mathbf{b} = \Phi_e \nabla s \times \nabla \theta + \iota(s) \nabla s \times \nabla \varphi$$

$$s = (\sqrt{x^2 + y^2} - R_0)^2 + z^2$$

$$F(x, y, z) = (X, Y, Z)$$

$$(x, y, z) = F^{-1}(X, Y, Z)$$

$$\mathbf{A} = (A_X, A_Y, A_Z)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$A_i = \frac{\partial F^{-1j}}{\partial X^i} a_j$$

“Find a divergence-free field B which commutes with its curl $\nabla \times B$ ”

Points of view:

- ① nested flux-surfaces **easy** $\longrightarrow$ MHD force-balance **hard**
- ② nested flux-surfaces **hard** $\longleftarrow$ MHD force-balance **easy**

Only states with the same invariants can possibly be transformed into each other by a motion. We should therefore not minimize W among all states, but only among states with the same values of the invariants as the initial state.

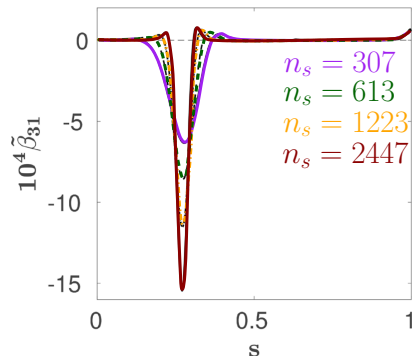
[Kruskal and Kulsrud, 1958]

VMEC approach

$$\text{Minimise } W_{\text{MHD}} = \int \frac{B^2}{2} dv = \frac{1}{2} \|B\|_{L^2}^2$$

over all F -related magnetic fields.
where F is volume-preserving map

- spectral method $\Rightarrow$ global convergence
- local error on force-balance



singular behaviour of $\beta_{m=3,n=1}$
increasing with radial resolution

[Mikhailov et al., 2019]

Grad-Shafranov 2D boundary-value problem (BVP)

$$\begin{aligned} -\Delta^* \Psi &= \mu_0 R^2 p' + \frac{1}{2} (F^2)', & \text{on } M_\epsilon \\ \Psi &= 0, & \text{on } \partial M_\epsilon \end{aligned}$$

with $p'(\Psi) = \text{const}$ and $F = \text{const}$

- smooth deformation of domain M_ϵ as sequence of Cassini ovals

$$((R - R_0)^2 + (Z - \epsilon a)^2)((R - R_0)^2 + (Z + \epsilon a)^2) \leq a^4$$

- well-posed elliptic BVP $\Rightarrow$ unique solution
- change of topology beyond $\epsilon \gtrsim 0.87$
 $\Rightarrow$ configuration *not accessible via ideal motion*

