

Arithmetic holonomy bounds and Special values of L -functions

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On a new joint work with Frank Calegari and Yunqing Tang

Results

1. **Q**-linear independence of $1, \zeta(2) = \pi^2/6$, and
$$L(2, \chi_{-3}) = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right)$$
2. **Q**-linear independence of $1, \log\left(1 - \frac{1}{n}\right), \log\left(1 - \frac{1}{m}\right)$, and $\log\left(1 - \frac{1}{n}\right) \log\left(1 - \frac{1}{m}\right)$ for all integers (n, m) with $0 < \left|1 - \frac{m}{n}\right| < 10^{-6}$
3. An $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbf{Q}[[x]]$ belongs to the $\mathbf{Q}\left[x, \frac{1}{x}, \frac{1}{1-x}\right]$ -linear span of $\{1, \log(1-x)\}$ iff: (i) $f(x) \in \mathcal{O}(\mathbf{C} \setminus [1, \infty))$, and (ii) $[1, \dots, n]a_n \in M^{-1}\mathbf{Z}$
4. *Ditto* for $\{1, \log(1-x), \log^2(1-x)\}$ iff $f(x) \in \mathcal{O}(\mathbf{C} \setminus [1, \infty))$, extends holonomically to $\mathbf{CP}^1 \setminus \{0, \delta, 1, \infty\}$ for some $\delta \in (-\infty, 1)$, and has type $[1, \dots, n][1, \dots, n/2]$

Setting up an approach

Conjecture

The $\mathbf{Q}\left[x, \frac{1}{x}, \frac{1}{1-x}\right]$ -linear span of the five G -functions

$$1, \quad \operatorname{Li}_1(x) = \log(1-x), \quad \operatorname{Li}_{1,1}(x) = \log^2(1-x), \\ \operatorname{Li}_2(x), \quad \frac{1}{\sqrt{1-x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-t}} dt$$

ought probably to be characterized arithmetically as:

$\exists T \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ (finite) such that

$$f(x) \in \mathbf{Q}[[x]] \cap \mathcal{O}(\mathbf{C} \setminus [1, \infty))$$

has type $[1, \dots, n]^2$ and holomorphic analytic continuation along all paths in $\mathbf{CP}^1 \setminus \{0, 1, \infty\} \cup T$

Integral Solutions of Apéry-like Recurrence Equations

Don Zagier

ABSTRACT. In [4], Beukers studies the differential equation

$$(*) \quad \left((t^3 + at^2 + bt)F'(t) \right)' + (t - \lambda)F(t) = 0,$$

where a , b and λ are rational parameters, and asks for which values of these parameters this equation has a solution in $\mathbb{Z}[[t]]$, the motivating example being the Apéry sequence with $a = 11$, $b = -1$, $\lambda = -3$. We describe a search over a suitably chosen domain of 100 million triples (a, b, λ) . In this domain there are 36 triples yielding integral solutions of (*). These can be further subdivided into members of four infinite classes, two of which are degenerate special cases of the other two, and seven sporadic solutions. Of these solutions, twelve, including all the sporadic ones, have parametrizations of Beukers type in terms of modular forms and functions. These solutions are related to elliptic curves over $\mathbb{P}^1$ with four singular fibres.

Case	β	α	C	c_1	c_2	L
A	8	-1	$\frac{2}{\pi\sqrt{3}}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}\zeta(2) = 0.4112335\dots$
C	9	1	$\frac{3\sqrt{3}}{4\pi}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}L_{-3}(2) = 0.3906512\dots$
D	ϕ^5	$-\phi^{-5}$	$\frac{\phi^{5/2}}{2\pi\sqrt[4]{5}}$	$\frac{1}{\phi\sqrt{5}}$	$\frac{\phi}{\sqrt{5}}$	$\frac{1}{5}\zeta(2) = 0.3289868\dots$
E	8	4	$\frac{2}{\pi}$	0	1	$\frac{1}{2}L_{-4}(2) = 0.4579827\dots$
F	9	8	$\frac{3\sqrt{3}}{\pi}$	-2	3	$\frac{5}{8}L_{-3}(2) = 0.4883140\dots$

In each of the five cases of Table 5, formula (10) gives an interesting series of rational approximations for one of the numbers $\zeta(2)$, $L_{-3}(2)$ or $L_{-4}(2)$, but, as already mentioned, except in case **D** these do not converge quickly enough to yield the irrationality of the limit. Indeed, from the table we see that in cases **E** and **F** the quantity $v_n - Lu_n$ blows up exponentially like $4^n/n$ or $8^n/n$, respectively, and even in cases **A** and **C**, where $|\alpha| = 1$ and hence $v_n - Lu_n$ tends to zero like $O(1/n)$, this is not enough to give the irrationality of L because v_n itself has a denominator which blows up like e^{2n} . The speed of convergence is best in these two cases, with $v_n/u_n - L$ being of the order of $(\frac{1}{8})^n$ and $(\frac{1}{9})^n$, respectively, while the convergence in cases **E** (Catalan's constant) and **F** is only like $(\frac{1}{2})^n$ and $(\frac{8}{9})^n$, respectively. The approximations lead in each of the five cases to a simple infinite continued fraction

$L(2, \chi_{-3})$ and $\zeta(2)$ as periods of Eisenstein series in $M_3(\Gamma_0(6), \chi_{-3})$

Take these Eichler integrals of Eisenstein series:

$$B := \sum_{n=1}^{\infty} \left(\sum_{d|n} (-1)^{d-1} \chi_{-3}(n/d) d^2 \right) \frac{q^n}{n^2},$$

$$C := \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-3}(d) d^2 \right) \frac{q^n}{n^2} - \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-3}(d) d^2 \right) \frac{q^{2n}}{(2n)^2}$$

We did $f := \sum a_n q^n \rightsquigarrow \sum \frac{a_n}{n^{k-1}} q^n =: F$ starting from weight $k = 3$ because of Bol's identity:

$$\begin{aligned} \left(q \frac{d}{dq} \right)^{k-1} (F|_{2-k} \gamma) &= \left(\frac{d^{k-1}}{d\tau^{k-1}} F \right) \Big|_k \gamma = f|_k \gamma = f = \left(q \frac{d}{dq} \right)^{k-1} (F) \\ &\rightsquigarrow F|_{2-k} \gamma = F + [\text{period polynomial in } \tau \text{ of degree } \leq k-2] \end{aligned}$$

$L(2, \chi_{-3})/2$ is the w_6 -period of B

Choose $\gamma = w_6$, the involution exchanging the cusps $\tau = i\infty$ and $\tau = 0$ of $\Gamma_0(6)$:

$$\left(B(q) - \lim_{q \rightarrow 1} B(q) \right) \Big|_{-1} w_6 = B(q) - \lim_{q \rightarrow 1} B(q)$$

This Apéry limit is:

$$\lim_{q \rightarrow 1} B(q) = \lim_{q \rightarrow 1} \left\{ \sum_{n=1}^{\infty} \chi_{-3}(n) n^{-2} \frac{q^n}{1+q^n} \right\} = \frac{1}{2} L(2, \chi_{-3})$$

$\zeta(2)/4$ is the w_6 -period of C

Choose $\gamma = w_6$, the involution exchanging the cusps $\tau = i\infty$ and $\tau = 0$ of $\Gamma_0(6)$:

$$\left(C(q) - \lim_{q \rightarrow 1} C(q) \right) \Big|_{-1} w_6 = C(q) - \lim_{q \rightarrow 1} C(q)$$

This Apéry limit is:

$$\lim_{q \rightarrow 1} C(q) = \lim_{q \rightarrow 1} \left\{ \frac{1}{4} \sum_{n=1}^{\infty} \chi_{-3}(n) (4\mathrm{Li}_2(q^n) - \mathrm{Li}_2(q^{2n})) \right\} = \frac{1}{4} \zeta(2)$$

Simultaneous approximating forms to the two Eichler periods $L(2, \chi_{-3})/2$ and $\zeta(2)/4$

$$\begin{aligned}\left(B - \frac{1}{2}L(2, \chi_{-3})\right) \Big|_{-1} w_6 &= B - \frac{1}{2}L(2, \chi_{-3}), \\ \left(C - \frac{1}{4}\zeta(2)\right) \Big|_{-1} w_6 &= C - \frac{1}{4}\zeta(2)\end{aligned}$$

Kill the automorphy factor by multiplying by a modular form of the opposite weight $+1$.

We follow Zagier's choice:

$$\begin{aligned}\theta_{-3}(\tau) &:= \sum_{m,n \in \mathbf{Z}} q^{m^2 + mn + n^2} \in M_1(\Gamma_0(3), \chi_{-3}) \\ A &:= \frac{\theta_{-3}(\tau) + \theta_{-3}(2\tau)}{2} \in M_1(\Gamma_0(6), \chi_{-3})\end{aligned}$$

Simultaneous approximating forms to the two Eichler periods $L(2, \chi_{-3})/2$ and $\zeta(2)/4$

Now both $A \cdot (B - \frac{1}{2}L(2, \chi_{-3}))$ and $A \cdot (C - \frac{1}{4}\zeta(2))$ are regular in either of the cusps $\tau = i\infty$ and $\tau = 0$ of $Y_0(6)$

Apéry's key: $\mathbf{Z}[[q]] = \mathbf{Z}[[x]]$, turning modular forms into G -functions:

$$x = q \prod_{n=1}^{\infty} \frac{(1 - q^n)^4 (1 - q^{6n})^8}{(1 - q^{2n})^8 (1 - q^{3n})^4} = q - 4q^2 + 10q^3 + \dots$$

$$X_0(6) \setminus \{i\infty, 0, 1/3, 1/2\} = Y_0(6) = \mathbf{H}/\Gamma_0(6) \xrightarrow{\cong} \mathbf{P}^1 \setminus \{0, 1/9, 1, \infty\}$$

$$A = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \right) x^n \in \mathbf{Z}[[x]] \cap \text{Hol} \left(\mathbf{P}^1 \setminus \left\{ 0, \frac{1}{9}, 1, \infty \right\} \right)$$

The Picard–Fuchs equation

Explicitly:

$$L(A) = 0, \quad L(AB) = 1, \quad L(AC) = \frac{1}{1-x},$$

where

$$L := x(1-x)(1-9x)\frac{d^2}{dx^2} + (1-20x+27x^2)\frac{d}{dx} + (9x-3)$$

Beukers integral:

$$\begin{aligned} H &= \frac{1}{2}L(2, \chi_{-3})A - AB \\ &= \sum_{n=0}^{\infty} x^n \iint_{[0,1]^2} \frac{9^n s^n t^n (1-s^3)^n (1-t^3)^n}{(1+st+s^2t^2)^{2n+1}} ds dt \end{aligned}$$

Type $[1, \dots, n]^2$

$$H(\lambda(z/2)) \in \mathcal{O}(\mathbf{D}), \quad |\varphi'(0)| = 8 > e^2$$

The Apéry limits method

The holonomic construction

$AB - \frac{1}{2}L(2, \chi_{-3})A$ and $AC - \frac{1}{4}\zeta(2)A$ are regular not only at the singularity $x = 0$, but also at the next singularity $x = 1/9$.

$\rightsquigarrow$ so are all their $\mathbf{C}$ -linear combinations

$\rightsquigarrow$ a $\mathbf{Q}$ -linear dependency of the periods

$\Rightarrow H \in \mathbf{Q}[[x]]$ with the same overconvergence properties

The approach is to deny a certain form of G -function

$\nexists H(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, n]^2} \mathbf{Z} \cap \mathcal{O}(\mathbf{C} \setminus [1, \infty))$ satisfying a certain form of inhomogeneous 2nd order linear ODE $L(H) \in \mathbf{Q}(x)$ over $\mathbf{P}^1 \setminus \{0, \frac{1}{9}, -\frac{1}{8}; 1, \infty\}$

If H , then $9 = 5 + 4$ independent such functions over $\mathbf{Q}(x)$:

$$1, \quad \text{Li}_1(x) = \log(1-x), \quad \text{Li}_{1,1}(x) = \log^2(1-x),$$

$$\text{Li}_2(x), \quad \frac{1}{\sqrt{1-x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-t}} dt;$$

$$H(x), \quad H'(x), \quad H\left(\frac{x}{x-1}\right), \quad H'\left(\frac{x}{x-1}\right)$$

$$f\left(\frac{8(z+z^3)}{(1+z)^4}\right) \in \mathcal{O}(\mathbf{D}) \text{ for any such function } f(x)$$

An open question for layer $[1, \dots, n]^3, \{0, 1, \infty\}$

The question

Are there finitely many or infinitely many $\mathbf{Q}(x)$ -linearly independent G -functions of the type

$$f(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, n]^3} \mathbf{Z}$$

with singularities only at $\{0, 1, \infty\}$?

The largest derivative of a holomorphic mapping $\varphi : \mathbf{D} \rightarrow \mathbf{C} \setminus \{1\}$ subject to $\varphi^{-1}(0) = \{0\}$ is attained by the modular function
 $\lambda = 16q + \dots, q := e^{\pi i \tau} \qquad \varphi^* f \in \mathcal{O}(\mathbf{D})$

$$2 < \log 16 = 2.7725887 \dots < 3$$

Rational finiteness: semi-effective

Given a $\mathbf{Q}(x)$ -linearly independent m -tuple

$f_1, \dots, f_m \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, b_1 n] \cdots [1, \dots, b_r n]} \mathbf{Z}$; denominators growth rate
 $\tau(\mathbf{b}) = \sum_{i=1}^r b_i$;

$\mathcal{I}_D(L) := \{(Q_1, \dots, Q_m) \in \mathbf{Z}[x]_{<D} : \text{coeffs} \in [0, L] \cap \mathbf{Z}\}, \quad \# = L^{mD},$

$$\mathcal{I}_D(L) \subset \mathbf{Z}[x]_{<D}^{\oplus m} \hookrightarrow \mathbf{Q}[[x]], \quad (Q_1, \dots, Q_m) \mapsto \sum_{i=1}^m Q_i f_i \in \mathbf{Q}[[x]].$$

Let $\{0 \leq u(1) < u(2) < \dots < u(mD)\}$ be the possible $x = 0$ orders of an $F := \sum_{i=1}^m Q_i f_i$ in this evaluation module.

But if $\varphi(z), f_i(\varphi(z)) \in \mathcal{O}(\mathbf{D})$ for all i , the number of potentially admissible options for the $x^{u(p)}$ Taylor coefficient of $F := \sum_{i=1}^m Q_i f_i$ given all the previous coefficients is

$$\leq \gamma_p := 1 + L \exp \left(DT(\varphi) + (\tau(\mathbf{b}) - \log |\varphi'(0)|) u(p) + o(D + u(p)) \right)$$

$$\rightsquigarrow L^{mD} \leq \prod_{p=1}^{mD} \gamma_p \rightsquigarrow (\log |\varphi'(0)| - \tau(\mathbf{b})) \sum_{p=1}^{mD} u(p) \leq mD^2 T(\varphi)$$

Rational finiteness: semi-effective

This already proves the *rational holonomy bound*

$$m \leq \frac{2T(\varphi)}{\log |\varphi'(0)| - \tau(\mathbf{b})}, \quad T(\varphi) = \int_{\mathbf{T}} \log^+ |\varphi| \mu_{\text{Haar}}$$

(for $\varphi \in \mathcal{O}(\mathbf{D})$; for $\varphi \in \mathcal{M}(\mathbf{D})$, simply add in $T(\varphi)$ the Jensen–Nevanlinna polar counting function)

Following Bost and Charles (*Quasi-projective and formal-analytic arithmetic surfaces*, 2022), this is readily refined to

$$m \leq \frac{\iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w)}{\log |\varphi'(0)| - \tau(\mathbf{b})}.$$

The preceding construction for the d -th Cartesian power, $d \rightarrow \infty$, using the measure concentration property for the high dimension hypercube of auxiliary polynomial exponents (and the fact that the formation of Cartesian products commutes with the vanishing filtration jumps) yields also an elementary proof of a slightly cruder bound which is *almost* as good for all applications we encounter: $\iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w)$ replaced by $\iint_{\mathbf{T}^2} \log \max(|\varphi(z)|, |\varphi(w)|) \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w) = \int_0^1 2t \cdot (\log |\varphi(e^{2\pi i t})|)^* dt$

Rational finiteness: a convexity improvement

Let

$$\text{BC}(r, \varphi) := \iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(rw)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w).$$

A convex increasing function of $\log r \in (-\infty, 0]$.

Then,

$$m \leq \frac{\text{BC}(1, \varphi) - \frac{1}{m} \int_0^1 \left\{ r \frac{d}{dr} \text{BC}(r, \varphi) \right\}^2 \frac{dr}{r}}{\log |\varphi'(0)| - \tau(\mathbf{b})},$$

conditionally on the positive denominator.

If we define *ad hoc* (following Nevanlinna theory notation)

$$\mathring{A}(r, \varphi) := r \frac{d}{dr} \text{BC}(r, \varphi),$$

this bound can be rewritten as

$$m \leq \frac{\int_{\mathbf{T}} \log |\varphi| \mu_{\text{Haar}} + \int_0^1 \mathring{A}(r, \varphi) \frac{dr}{r} - \frac{1}{m} \int_0^1 \mathring{A}(r, \varphi)^2 \frac{dr}{r}}{\log |\varphi'(0)| - \tau(\mathbf{b})}$$

Bost–Charles finiteness: fully effective (case $\tau = 0$)

$$[\mathcal{M}(\tilde{\mathcal{V}}) : \mathbf{Q}(\varphi)] \leq \frac{\iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w)}{\log |\varphi'(0)|}$$

Theorem (Bost, Charles)

On a pseudoconcave f.a. arithmetic surface $\tilde{\mathcal{V}}$ over $\text{Spec } \mathcal{O}_K$, the ring of holomorphic functions $\mathcal{O}(\tilde{\mathcal{V}})$ constitutes a finitely generated $\mathcal{O}_K$ -algebra. This algebra can be effectively computed based on the arithmetic Hilbert–Samuel formula for arithmetic surfaces.

Furthermore, for any $x \in \mathcal{O}(\tilde{\mathcal{V}}) \setminus \mathcal{O}_K$, the fraction field $\mathcal{M}(\tilde{\mathcal{V}})$ of $\mathcal{O}(\tilde{\mathcal{V}})$ is a finite algebraic extension of $K(x)$ of degree at most

$$\left[\mathcal{M}(\tilde{\mathcal{V}}) : K(x) \right] \leq \frac{x_* \left(P, g_{\tilde{\mathcal{V}}_C} \right) \cdot x_* \left(P, g_{\tilde{\mathcal{V}}_C} \right)}{\widehat{\deg} \overline{N}_P(\tilde{\mathcal{V}})}.$$

Bost's *formal-analytic arithmetic surfaces*

Definition (Bost, *Theta Invariants of Euclidean Lattices and Infinite-dimensional Vector Bundles over Arithmetic Curves*; Bost–Charles, *Quasi-Projective and Formal-Analytic Arithmetic Surfaces* (2022))

A (smooth) **f.a. arithmetic surface** $\tilde{\mathcal{V}} = (\hat{\mathcal{V}}, (V, O), \iota)$ over $\mathrm{Spec} \mathbf{Z}$ is defined by the “gluing data”:

- ▶ a formal scheme $\hat{\mathcal{V}} \cong \mathrm{Spf} \mathbf{Z}[[x]]$
- ▶ a compact Riemann surface with nonempty boundary $(V, \partial V)$
- ▶ an antiholomorphic involution (real structure) on V , and point $O \in V(\mathbf{R})$
- ▶ $\iota: \tilde{\mathcal{V}}_{\mathbf{C}} \xrightarrow{\cong} \hat{\mathcal{V}}_O$ a “gluing isomorphism” compatible with the real structures

On writing $\hat{\mathcal{V}}_O \cong \mathrm{Spf} [[z]]$ in a local analytic coordinate near $O \in V$, the gluing datum amounts to a $\psi := \iota^* z \in \mathbf{R}[[X]]$ with $\psi(0) = 0$ and $\psi'(0) \neq 0$.

Prior example: Robinson's integral refinement of the rationality theorem of Pólya

Theorem (Fekete, Szegő, Pólya, Robinson)

Fix $\mathcal{K} \subset \mathbf{C}$ a compact with transfinite diameter < 1 . Define $R_{\mathcal{K}} \subset \mathbf{Z}[[1/X]]$ to be the ring of formal functions $f(X) \in \mathbf{Z}[[1/X]]$ holomorphic on $\widehat{\mathbf{C}} \setminus \mathcal{K}$. Then $R_{\mathcal{K}}$ is a finitely generated ring with field of fractions $\text{frac}(R_{\mathcal{K}}) = \mathbf{Q}(X)$.

This finitely generated $\mathbf{Z}$ -algebra has an effectively computable set of generators.

Momentary glimpse of the proof: Robinson's integral refinement of the rationality theorem of Pólya

By an arithmetic form of Riemann-Roch (Fekete and Szegő), $\exists h \in \mathbf{Z}[X] \setminus \{0\}$ with $\sup_{\mathcal{K}} |h| < 1$. We prove that $h(X)^n f(X)$ must be in $\mathbf{Z}[X]$ (a polynomial) once $n \gg 1$ is sufficiently big.

Its X^{-k} coefficient is an integer, on the one hand, but it is also the Cauchy integral $\frac{1}{2\pi i} \oint_{\Gamma} h(z)^n f(z) z^{k-1} dz$ for $\Gamma \subset \mathbb{C}$ any plane contour enclosing $\mathcal{K}$. Deform Γ to a contour going tightly around $\mathcal{K}$ (possibly reducible if $\mathcal{K}$ is disconnected). If $n \gg 1$ is sufficiently big, all these $k \in \{0, 1, \dots, \deg h\}$ integrals are < 1 in magnitude and are hence forced to vanish by integrality. But then, by dividing k by $\deg h$ with quotient and remainder, and then z^k mod the appropriate power of $h(z)$, all the subsequent X^{-k} coefficients, for $k > \deg h$ have to vanish as well: being linear combinations of the preceding ones $k \in \{0, 1, \dots, \deg h\}$ with the exponent n replaced by still larger values. $\square$

A simple nonunivalent example

A simple showcase problem could be: Compute the ring

$$R := \mathbf{Q}(\sqrt{1-12x}) \cap \mathbf{Z}[[x]] \cap \mathcal{O}(\mathbf{C} \setminus [\frac{1}{12}, \infty))$$

- ▶ $\mathbf{Z} \left[\sqrt{1-12x}, \frac{1}{x}, \frac{1}{1-12x} \right] \cap \mathbf{Z}[[x]]$?
- ▶ Not quite: a rational value $f(1/n) = p/q$
 $f(x) := \sqrt{1-12x} \in \mathbf{Z}[[x]]$ spawns this new function along with all its powers and derivatives:

$$q \frac{f(x) - f(1/n)}{1 - nx} \in R \setminus \mathbf{Z} \left[\sqrt{1-12x}, \frac{1}{x}, \frac{1}{1-12x} \right].$$

One readily finds the one such “exceptional value” $f\left(\frac{1}{16}\right) = \frac{1}{2}$.

- ▶ Revised guess: $\mathbf{Z} \left[\sqrt{1-12x}, \frac{1}{x}, \frac{1}{1-12x}, \frac{1}{1+2\sqrt{1-12x}} \right] \cap \mathbf{Z}[[x]]$?

Could be subtle! General point in case: each rational value $f(1/n) \in \mathbf{Q}$ leads to a new function to add. ← Certain effective cases of Siegel’s integral points on curves can be reduced to the Bost–Charles integral finiteness theorem

An analogy: André's take on the Siegel–Shidlovsky theorem on E -functions

Fourier–Laplace transform: $x \mapsto -\frac{d}{dx}, \frac{d}{dx} \mapsto x,$

$$F = \sum a_n \frac{x^n}{n!} \leftrightarrow f = \sum a_n x^n, \quad x \frac{\partial}{\partial x} F \leftrightarrow x \frac{d}{dx} f, \quad xF \leftrightarrow \left(x^2 \frac{d}{dx} + x\right) f$$

► *The Bezivin–Robba proof of Hermite–Lindemann–Weierstrass*

The theorem easily reduces to prove that if $F(x)$ is an exponential polynomial over $\mathbf{Q}$ that vanishes at $F(1) = 0$, then $G(x) := F(x)/(x-1)$ is also an exponential polynomial.

Fourier dual to an ODE with an irregular singularity at $x = 0$:

If $(x^2 \frac{d}{dx} + x - 1)g = f \in \mathbf{Q}(x)$ and g has a positive convergence radius at $x = 0$, then $g \in \mathbf{Q}(x)$.

- André's take: Chudnovsky's fundamental G -functions theorem to the order-2 linear differential system $\rightsquigarrow g \in \mathbf{Q}(x)$
(Honda–Katz since $x = 0$ is an irregular singular point here)

$$x^2 \frac{d}{dx} \begin{pmatrix} g \\ 1 \end{pmatrix} = \begin{pmatrix} 1-x & f \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g \\ 1 \end{pmatrix}$$

Yves André's *Transcendance sans transcendance*

$$x \mapsto -\frac{d}{dx}, \quad \frac{d}{dx} \mapsto x$$

The Fourier dual of “ $x = 0$ is a regular (Fuchsian) singular point” is just “absence of singularities outside $\{0, \infty\}$ ”

5. Autour de Laplace. Preuve de 4.3 et 4.6

5.1. *Pentes.* Soit $\varphi = \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} a_{i,j} z^j \frac{d^i}{dz^i} \in K[z, \frac{d}{dz}]$; son transformé de Fourier-Laplace est

$$\overline{\varphi} = \sum_{j=0}^{\nu} \sum_{i=0}^{\mu} (-1)^j a_{i,j} \frac{d^j}{dz^j} z^i = \sum_{j=0}^{\nu} \sum_{i=0}^{\mu} b_{j,i} z^i \frac{d^j}{dz^j}.$$

Tous les opérateurs différentiels que nous considérons sont de type exponentiel au sens de [M, XII], c'est-à-dire vérifient $a_{\mu,\nu} \neq 0$ (ce qui équivaut à $b_{\nu,\mu} \neq 0$). En effet nous ne considérons que des opérateurs différentiels réguliers en l'infini, ou facteurs de transformés de Fourier de tels.

Notons d'autre part que si φ est régulier en l'infini, on a alors $a_{i\nu} = 0$ pour $i < \mu$, donc $b_{\nu,i} = 0$ pour $i < \mu$, ce qui signifie que $\overline{\varphi}$ n'a de singularités qu'en 0 et ∞ . Ceci établit i).

André's proof and Beukers's refinement of the Siegel–Shidlovsky theorem on E -functions

All the qualitative results can be derived from this:

The ring of E -functions generates a free $\mathbf{Q} \left[x, \frac{1}{x} \right]$ -submodule of $\mathbf{Q}((x))$ (which, itself, is quite complicated as a module over $\mathbf{Q} \left[x, \frac{1}{x} \right]$!).

Concretely: if $f(1) \in \mathbf{Q}$ for an E -function (like in the trivial example of $f(x) = (x - 1)e^x$), then in fact the minimal order ODE of $(f(x) - f(1))/(x - 1)$ has a basis of holomorphic solutions at $x = 1$. All the relations resolve like this.

Beukers, *A refined version of the Siegel–Shidlovsky theorem* (2006)

Provisional hypothetical analog for G -functions

Consider now a pseudoconcave f. a. arithmetic surface with $V = \overline{\mathbf{D}}$, given by a gluing datum $\psi \in \mathbf{R}[[X]]$ with $\varphi := \psi^{-1}$ holomorphic on $\mathbf{D}$, thus having $|\varphi'(0)| > 1$. By convention, those $\tilde{\mathcal{V}}$ come with a distinguished nonconstant holomorphic function $x \in \mathcal{O}(\tilde{\mathcal{V}})$. Thus, by Bost–Charles, the ring of regular functions $\mathcal{O}(\tilde{\mathcal{V}})$ is a finitely generated $\mathbf{Z}$ -algebra of tr. deg. one and with computable generators.

Now suppose that, further, $|\varphi'(0)| > e^{b_1 + \dots + b_r}$ where we consider the $\mathcal{O}(\tilde{\mathcal{V}})$ -module $\mathcal{D}(\tilde{\mathcal{V}}; \mathbf{b})$ of $f(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, b_1 n] \cdots [1, \dots, b_r n]} \mathbf{Z}$ turning $f(\varphi(z)) \in \mathcal{O}(\mathbf{D})$.

Integral holonomy bounds: basic question

Freeness of the $\mathcal{O}(\tilde{\mathcal{V}})[1/x]$ -module $\mathcal{D}(\tilde{\mathcal{V}}; \mathbf{b}) \otimes_{\mathcal{O}(\tilde{\mathcal{V}})} \mathcal{O}(\tilde{\mathcal{V}})[1/x]$?

Simplest example. By our $\frac{d^k}{dx^k} \{(\text{Li}_2(x) - \text{Li}_2(1/n))/(1 - nx)\}$ discussion, this for instance would immediately imply irrationality proofs of $\text{Li}_2(1/n) \notin \mathbf{Q}$ for all $n \in \mathbf{Z}$, inc. outside the Hata–Viola range:

$\{-4, -3, -2, 2, 3, 4, 5\}$.

More loosely: Can one construct a finitely generated $\mathbf{Z}$ -algebra over which $\mathcal{D}(\tilde{\mathcal{V}}; \mathbf{b})$ is a finite module?

Integral finiteness: weakly semi-effective

Integrated types: $\prod_{h=1}^r [1, \dots, b_h n] \rightsquigarrow n \prod_{h=1}^r [1, \dots, b_h n]$

$$\mathbf{Z}[x]_{<D}^{\oplus m} \ni (Q_1, \dots, Q_m) \mapsto \sum_{i=1}^m Q_i f_i =: F \in \mathbf{Q}[[x]]$$

For the $x^n = x^{u(j)}$ evaluation map, the addition to the finite evaluation height upper bound is only an extra

$$\log [(n - D + 1)^+, \dots, n] = \left(\sum_{h=1}^{\lfloor n/D \rfloor - 1} \frac{1}{h} \right) D + \frac{n}{\lfloor n/D \rfloor} + o(n)$$

(Not the full $[1, \dots, n] = n + o(n)$)

Those extra denominators matter less and less if n/D is big; which in the aggregate is the case if m is big since—approximately speaking—the extrapolation range is an integral across $n/D \in [0, m]$.

$\rightsquigarrow$ finite $\mathbf{Q}(x)$ -linear span of the integrals $\int f(x) dx$ of the $f(x) \in \mathcal{D}(\tilde{\mathcal{V}}; \mathbf{b}) \rightsquigarrow$ finite module generation over some finitely generated subring of $\mathbf{Q}(x)$

But this is a non-constructive finiteness argument

An example realizing $L(2, \chi_{-3}) = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right)$

$y := x^2/(x-1) = x \cdot x/(x-1) = x + x/(x-1)$, symmetrize

$$f(y) := \text{Sym}^-(\text{Li}_2)(y) := \left(x - \frac{x}{x-1} \right) \left(\text{Li}_2(x) - \text{Li}_2 \left(\frac{x}{x-1} \right) \right) \\ \in \bigoplus_{n=0}^{\infty} \frac{y^n}{[1, \dots, 2n]^2} \mathbf{Z},$$

holonomic on $Y_0(2) \setminus \{\text{elliptic point}\} = \mathbf{P}^1 \setminus \{0, 4, \infty\}$ (with $\mathbf{Z}/2$ local monodromy around the elliptic point $y = 4$)

- ▶ turned holomorphic on $z \in \mathbf{D}$ under the substitution $y := h(z) = \lambda(\sqrt{z})^2/(\lambda(\sqrt{z}) - 1) = -256z + \dots$
- ▶ $\varphi(z) := -\frac{64z(1+z)^2}{(1-z)^4}$ still has $|\varphi'(0)| = 64 > e^4$ and $f(\varphi(z)) \in \mathcal{O}(\mathbf{D})$, but now $\#\varphi^{-1}(1) = 1$
- ▶ $L(2, \chi_{-3}) = -\frac{2}{9}f(1)$

Symmetrization $Y(2) \rightarrow Y_0(2)$ and integrations

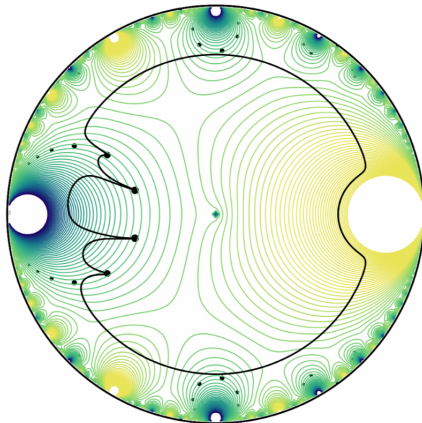
Idea: We do not know how to directly access the special value by an integral holonomy bound. Instead, we use the Apéry limit with *rational* holonomy bounds, but still make a gesture towards the integral finiteness argument that we described. This turns out to work well after taking a double quotient of our underlying f. a. arithmetic surface.

$$y := \frac{x^2}{x-1} \rightsquigarrow H(x) + H\left(\frac{x}{x-1}\right) \in \mathbf{Q}[[y]]$$

- ▶ the rational holonomy bounds are equivalent in the two settings (functoriality of the arithmetic intersection numbers in Bost–Charles)
- ▶ this translates the problem on the nine $[1, \dots, n]^2$, type $Y(2)$ (radius up to 16) functions into another problem on nine $[1, \dots, 2n]^2$, type $Y_0(2)$ (radius up to 256) functions: the symmetrizations.

But the integral holonomy bounds are no longer equivalent. In the latter setting, we get five added integrals of the type $n[1, \dots, 2n]^2$, for a total of 14 independent G -functions over $\mathbf{Q}(y)$. This pans out *much* better than eight added integrals of the type $n[1, \dots, n]^2$ but with half the log radius over $\mathbf{Q}(x)$.

A fortuitous contour choice on the $Y_0(2)$ domain



With all the improvements including convexity, the holonomy bound returns ≤ 13.75

Our G-functions extension of the rationality part

Theorem

For an $m \times r$ array $\mathbf{b} := (b_{i,j})_{1 \leq i \leq m, 1 \leq j \leq r}$, all of whose columns have the form $(0, \dots, 0; b, b, \dots, b)$ for some $b \geq 0$ (depending on the column), define $\sigma_1 \leq \dots \leq \sigma_m$ to be the row sums, and

$$\tau(\mathbf{b}) := \frac{1}{m^2} \sum_{i=1}^m (2i-1) \sigma_i \in [0, \sigma_m].$$

Assume now $|\varphi'(0)| > e^{\tau(\mathbf{b})}$ and that there exists an m -tuple of $\mathbf{Q}(x)$ -linearly independent set for formal functions

$$f_i(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, b_{i,1} \cdot n] \cdots [1, \dots, b_{i,r} \cdot n]} \mathbf{z}$$

turning $f_i(\varphi(z)) \in \mathcal{M}(\mathbf{D})$.

Our G -functions extension of the rationality part

Theorem

Assume now $|\varphi'(0)| > e^{\tau(\mathbf{b})}$ and that there exists an m -tuple of $\mathbf{Q}(x)$ -linearly independent set for formal functions

$$f_i(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, b_{i,1} \cdot n] \cdots [1, \dots, b_{i,r} \cdot n]} \mathbf{Z}$$

turning $f_i(\varphi(z)) \in \mathcal{M}(\mathbf{D})$.

Then,

$$m \leq \frac{\iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w)}{\log |\varphi'(0)| - \tau(\mathbf{b})}.$$

A slightly bigger term $\tau^{\sharp}(\mathbf{b})$ is calculated if the $\int f(x) dx$ are added. The bound then leads to the desired contradiction upon choosing $\varphi := h(V(z))$ with $V : (\mathbf{D}, 0) \hookrightarrow (\mathbf{D}, 0)$ the Riemann map of the contour drawn.

Additional slides that were not used in my presentation at the Simons Symposium

Application: Proof of the logarithm characterization

Choose $\varphi(z) := 4z/(1+z)^2$, and suppose for the contradiction that there is a third $\mathbf{Q}(x)$ -linearly independent function $f(x)$ also of the type $[1, \dots, n]$, and also holomorphic on $\mathbf{C} \setminus [1, \infty)$. We apply the theorem with $m = 3, r = 1$, and $\mathbf{b} = (0, 1, 1)^t$ with $\tau(\mathbf{b}) = 8/9$, and get the holonomy bound

$$3 = m \leq \frac{\log 4}{\log 4 - 8/9} = 2.787050 \dots < 3,$$

a contradiction!

For the integral enhancement: Once we know a $\mathbf{Q}(x)$ -basis of the solution space to be $\{1, \log(1-x)\}$, we can readily—in this simple case where the transcendence theory of $\exp$ is known—get the $\mathbf{Q} \left[x, \frac{1}{x}, \frac{1}{1-x} \right]$ refinement use the Hermite–Lindemann–Weierstrass theorem, and Mahler’s p -adic partial analog.

Application: a mixed periods example

Consider now with the bivalent map

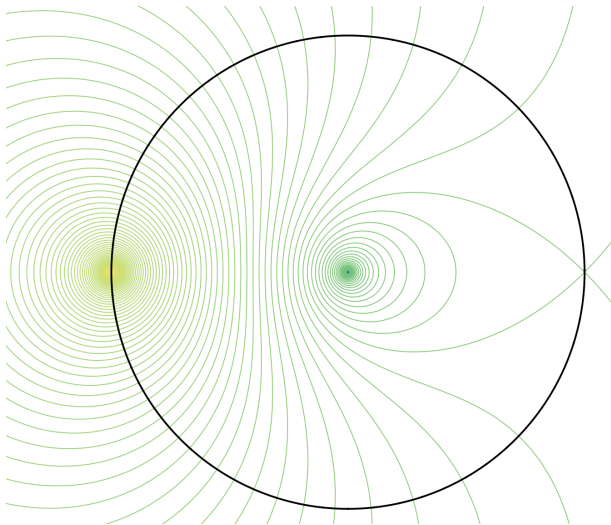
$$\varphi(z) := \frac{8(z + z^3)}{(1 + z)^4} = 1 - \left(\frac{1 - z}{1 + z} \right)^4, \quad \mathbf{D} \rightarrow \mathbf{C} \setminus \{1\}, \quad \varphi^{-1}(0) = \{0\}$$

(the next Landen iterative layer), which takes either connected component of $\mathbf{D} \setminus (-1, 1)$ conformally isomorphically onto $\mathbf{C} \setminus (-\infty, 1]$, but bijects

$$\varphi^{-1}(-\infty, 1] \longleftrightarrow (-1, 1).$$

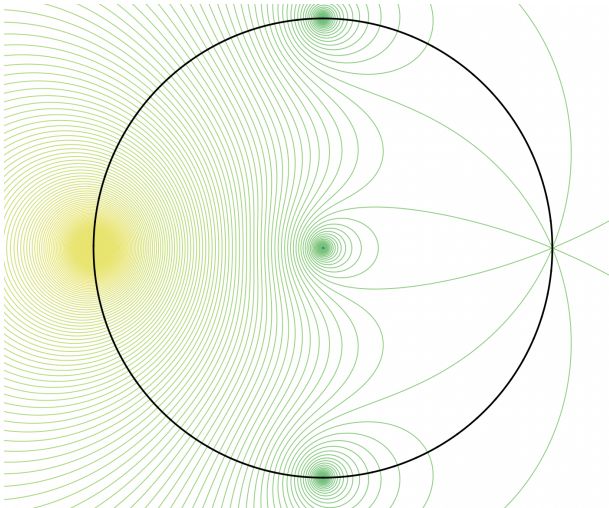
Start from the Riemann map $G : \mathbf{D} \xrightarrow{\sim} \mathbf{C} \setminus [1, \infty)$

$$\varphi_1(z) = G(z) := \frac{4z}{(1+z)^2} \in \mathbf{Z}[[x]]$$



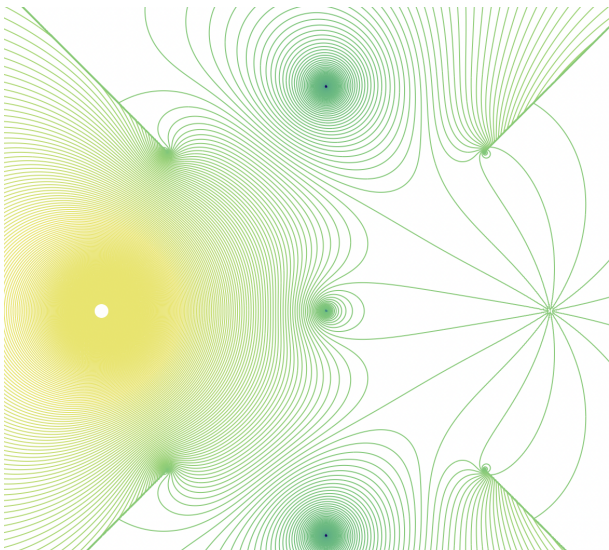
$$G\left(\sqrt{G(z^2)}\right) : \mathbf{D} \rightarrow \mathbf{C} \setminus \{1\}$$

$$\varphi_2(z) = G\left(\sqrt{G(z^2)}\right) := \frac{8(z+z^3)}{(1+z)^4} = 1 - \left(\frac{1-z}{1+z}\right)^4, \quad \varphi_2^{-1}(0) = \{0\}$$



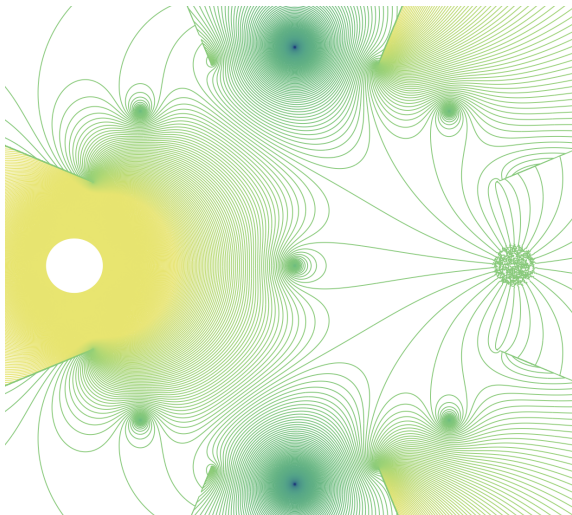
$$G \left(\sqrt{G \left(\sqrt{G(z^4)} \right)} \right) : \mathbf{D} \rightarrow \mathbf{C} \setminus \{1\}$$

$$\varphi_3(z) := 8\sqrt{2} \frac{z(1+z^2)^2\sqrt{1+z^4}}{(\sqrt{1+z^4}+z\sqrt{2})^4}, \quad \varphi_3^{-1}(0) = \{0\}$$



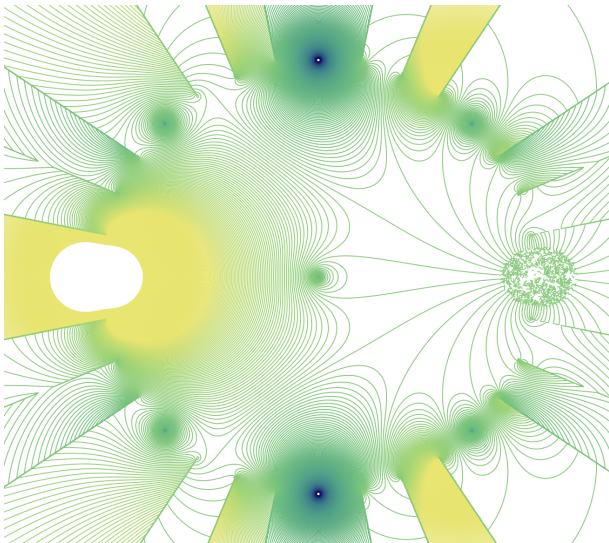
$$G \left(\sqrt{G \left(\sqrt{G \left(\sqrt{G(z^8)} \right)} \right)} \right) : \mathbf{D} \rightarrow \mathbf{C} \setminus \{1\}$$

$$\varphi_4(z) := 8 \cdot 2^{3/4} \frac{z(1+z^4)(1+z^8)^{1/4} (\sqrt{1+z^8+z^2} \sqrt{2})^2 (\sqrt{1+z^8+z^2} \sqrt{2})^2}{(1+z^8)^{3/4} (1+z^4+z(1+z^8)^{1/4} 2^{3/4})^4}$$



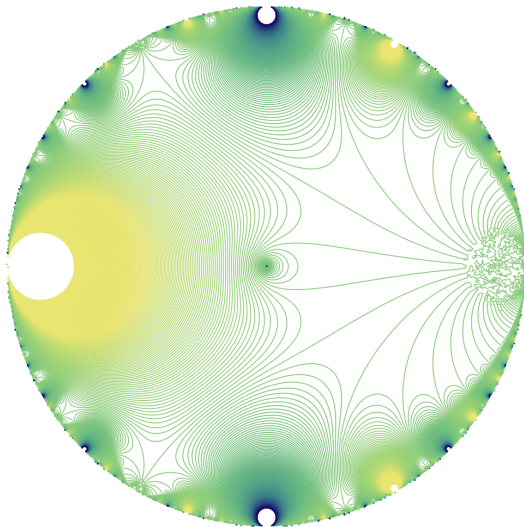
$$\varphi_5 : \mathbf{D} \rightarrow \mathbf{C} \setminus \{1\}$$

Next layer $\varphi_5^{-1}(0) = \{0\}$



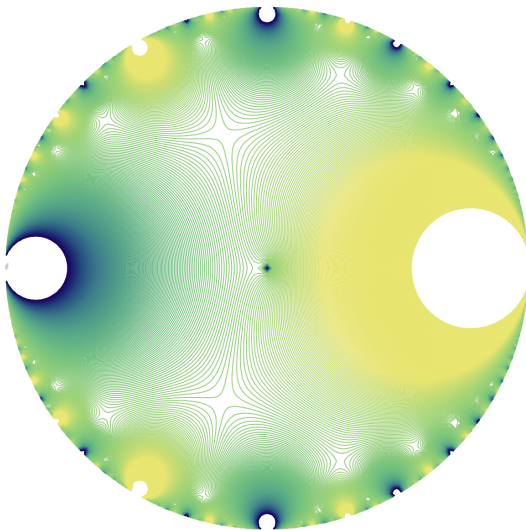
The convergence to the modular λ map

$$\varphi_n(q) \rightarrow \lambda(q) = \left(\sum_{n \in 1+2\mathbb{Z}} q^{n^2/4} \right)^4 / \left(\sum_{n \in 2\mathbb{Z}} q^{n^2/4} \right)^4$$



The $X_0(2)$ coordinate

$$h(q) := \lambda(\sqrt{q}) + \lambda(\sqrt{q})/(\lambda(\sqrt{q}) - 1) = -\lambda(\sqrt{q})^2/(\lambda(\sqrt{q}) - 1) = -256 \frac{\Delta(2\tau)}{\Delta(\tau)} = -256q \prod_{n=1}^{\infty} (1 + q^n)^{24}$$



Application: a mixed periods example

$\varphi(z) := \frac{8(z+z^3)}{(1+z)^4}$: bivalent, but still simple enough to have its Bost–Charles integral computed exactly.

Essentially following Smyth's calculations of the bivariate Mahler measure $m(1+x+y-xy) = 2G/\pi$, where $G = L(2, \chi_{-4})$ is the Catalan constant:

Lemma

$$\iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w) = \log 8 + \frac{4G}{\pi}$$

$$\frac{\varphi(z) - \varphi(w)}{z - w} = 8 \frac{(1 - zw)(1 + ix - iy - xy)(1 - ix + iy - xy)}{(1 + z)^4(1 + w)^4}.$$

Application: the mixed periods example

This choice gives us completely similarly the “ δ is a nonapparent singularity” case of:

Theorem

Suppose $f(x) \in \bigoplus_{n=0}^{\infty} \frac{x^n}{[1, \dots, n][1, \dots, n/2]} \mathbf{Z}$ is holomorphic in $\mathbf{C} \setminus [1, \infty)$, and is analytically continuable as a holomorphic function along all paths in $\mathbb{P}^1 \setminus \{0, \delta, 1, \infty\}$, for some $\delta \in (-\infty, -1)$.

Then,

$$f(x) = Q_0(x) + Q_1(x) \log(1-x) + Q_2(x) \log^2(1-x)$$

for some rational functions $Q_0, Q_1, Q_2 \in \left[x, \frac{1}{x}, \frac{1}{1-x}\right] \subset \mathbf{Q}(x)$.

Observe that $f\left(\frac{8(z+z^3)}{(1+z)^4}\right) \in \mathcal{O}(\mathbf{D})$.

Application: the mixed periods example

Proof of the “non- $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ case”:

$$1, \quad \log(1-x), \quad \log^2(1-x), \quad f(x), \quad f\left(\frac{x}{x-1}\right) \\ \rightsquigarrow \mathbf{b} := \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}^t, \quad \tau(\mathbf{b}) = \frac{69}{50} = 1.38.$$

The rational holonomy bounds reads

$$5 = m \leq \frac{\log 8 + \frac{4G}{\pi}}{\log 8 - 69/50} = 4.640395\dots,$$

a contradiction.

Application: the mixed periods example

$$H_A(x) := \frac{1}{\sqrt{1-4x}} \in \mathbf{Z}[[x]],$$

$$H_B(x) := \frac{1}{\sqrt{1-4x}} \int_0^x \frac{1}{1-t} \frac{1}{\sqrt{1-4t}} dt \in \mathbf{Q}[[x]],$$

$$H_C(x) := \frac{1}{\sqrt{1-4x}} \int_0^x \frac{\log(1-t)}{t\sqrt{1-4t}} dt \in \mathbf{Q}[[x]],$$

$$H_D(x) := \frac{1}{\sqrt{1-4x}} \int_0^x \frac{\log(1-t)}{1-t} \frac{1}{\sqrt{1-4t}} dt \in \mathbf{Q}[[x]].$$

But the $x = 1/4$ local monodromy operator is lower-triangular with period matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -2L(1, \chi_{-3}) & 1 & 0 & 0 \\ \frac{\pi^2}{9} & 0 & 1 & 0 \\ -2(L(1, \chi_{-3}) \log 3 - L(2, \chi_{-3})) & 0 & 0 & 1 \end{pmatrix}$$

Application: irrationality proof for a Mahler measure

Corollary

$$m\left(\frac{(1+x+y)^4}{3}\right) = \frac{3L(2, \chi_{-2}) - \frac{\pi}{\sqrt{3}} \log 3}{\pi/\sqrt{3}} \notin \mathbf{Q}.$$

The classical Hermite–Padé approximants to $\log(1 - x)$

$$\begin{aligned} & 2 \sum_{j=0}^k \binom{k}{j}^2 (H_{n-k} - H_j) (1-y)^j + \log(1-y) \sum_{j=0}^k \binom{k}{j}^2 (1-y)^j \\ &= y^{2k+1} \int_0^1 \frac{t^k (t-1)^k}{(ty-1)^{k+1}} dt = -\frac{y^{2k+1}}{(2k+1) \binom{2k}{k}} + O(y^{2k+2}), \end{aligned}$$

- ▶ Set $y := 1/n$ and take the x^k generating function:

$$\text{holonomic}^+ \text{ on } \mathbf{C} \setminus \left\{ \left(\frac{1 \pm \sqrt{1-1/n}}{1/n} \right)^2 \right\}$$

- ▶ Set $y := 1/m$ and take the x^k generating function:

$$\text{holonomic}^+ \text{ on } \mathbf{C} \setminus \left\{ \left(\frac{1 \pm \sqrt{1-1/m}}{1/m} \right)^2 \right\}$$

- ▶ Form the Hadamard product construction

The log * log result

$$\sum_{k=0}^{\infty} (b_k - a_k \operatorname{Li}_1(1/n)) x^k, \quad \sum_{k=0}^{\infty} (v_k - u_k \operatorname{Li}_1(1/m)) x^k$$
$$\rightsquigarrow \sum_{k=0}^{\infty} (b_k v_k - a_k u_k \operatorname{Li}_1(1/n) \operatorname{Li}_1(1/m)) (nm x)^k$$

- ▶ the type would be $[1, \dots, k]^2$
- ▶ the ODE singularities would be (say, $m, n > 0$):

$$0; \quad nm \left(1 - \sqrt{1 - 1/n}\right)^2 \left(1 - \sqrt{1 - 1/m}\right)^2 \quad (\text{overconvergent!});$$
$$nm \left(1 \pm \sqrt{1 - 1/n}\right)^2 \left(1 \mp \sqrt{1 - 1/m}\right)^2 = 1 + o_{|n/m| \rightarrow 1}(1);$$
$$nm \left(1 + \sqrt{1 - 1/n}\right)^2 \left(1 + \sqrt{1 - 1/m}\right)^2 \rightarrow \infty$$

Prévost's interpretation of Apéry's function

$[2k/2k]$ Hermite–Padé approximants to $\zeta(3, 1 + 1/y)$:

$$\left\{ \left(\sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \left(\frac{1}{y} \right) \binom{k+1/y}{k} \right) \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{1/y}{m} \binom{m+1/y}{m}} \right. \\ \left. - \left(\sum_{k=1}^{\infty} \frac{1}{(k+1/y)^3} \right) \left(\sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \left(\frac{1}{y} \right) \binom{k+1/y}{k} \right) \right\} y^{2n} \\ = O(y^{4n+1})$$

Setting $y := 1/n$, then the x^n generating function recovers the *Apéry overconvergent function*

$$G(x) = B(x) - \zeta(3)A(x), \quad \text{type } [1, \dots, n]^3$$

This writes down an $\{0, (\sqrt{2}-1)^4\}$ -overconvergent branch of a holonomic function on $\mathbf{C} \setminus \{0, (\sqrt{2} \pm 1)^4\}$.

*(which lives more naturally on $X_1(6)^+$)

Appendix: Reserve slides added for a possible discussion

Bost–Charles: the analogy and the integral finiteness theorem

- *The analogy.* Consider $C^\circ = C \setminus \Sigma$ an *affine* complex algebraic curve, $\pi : \mathcal{V} \rightarrow C$ a complex-analytic surface germ along a section $\varepsilon : C \rightarrow \mathcal{V}$, enhanced “at infinity” with a reduced compact connected curve $\varepsilon(\sigma) \in F_\sigma \subsetneq \pi^{-1}(\sigma)$ over each $\sigma \in \Sigma$.

Intersection pairing on the connected compact curve complex $\varepsilon(C) + \sum_{\sigma \in \Sigma} F_\sigma$, signature $(+, -, -, \dots, -)$ if not negative semidefinite.

On $\bigsqcup_{\sigma \in \Sigma} F_\sigma$ there is supported a unique **equilibrium divisor** V , characterized by the requirement that $D := \varepsilon(C) + V$ should be orthogonal to every F_σ . The *nef and big* condition $D.D = \varepsilon(C).D > 0$ is the analog of Bost and Charles’s $|\psi'(0)| < 1$ or our $|\varphi'(0)| = 1/|\psi'(0)| > 1$.

We call such $\mathcal{V}$ pseudoconcave.

Bost–Charles: the analogy and the integral finiteness theorem

- *The rational holonomy bound.* For $\alpha : \mathcal{V} \rightarrow X$ any nonconstant π -compatible holomorphic map from a *pseudoconcave* $\mathcal{V}$ (iff!) to any connected regular projective algebraic surface with fibration $\pi_X : X \rightarrow C$, the Hodge index formula on the surface X implies (Nori's argument) that the meromorphic function field $\mathbf{C}(\mathcal{V})$ is a finite extension of $\alpha^*\mathbf{C}(X)$ of degree at most

$$[\mathbf{C}(\mathcal{V}) : \alpha^*\mathbf{C}(X)] \leq \frac{\alpha_* D \cdot \alpha_* D}{D \cdot D}.$$

Nori's basic lemma

For $f : Y \rightarrow X$ a morphism of integral normal projective surfaces over an algebraically closed field, if D is any divisor on Y with $D \cdot D > 0$, then $\deg f \leq \frac{f_* D \cdot f_* D}{D \cdot D}$. (Hodge to $f^* f_* D - (\deg f) D$)

Bost–Charles: the analogy and the integral finiteness theorem

Consequently, if $\mathcal{V}$ is pseudoconcave and $\mathcal{M}(\mathcal{V}) \neq \mathbf{C}(C)$, there exists—by finite generation of transcendence degree one—a universal meromorphic map $\alpha^{\text{uni}} : \mathcal{V} \dashrightarrow \mathcal{V}^{\text{alg}}$ to a connected regular projective algebraic surface over C . Using blow-up modifications, we may still define a big and nef divisor $\Gamma_{\alpha_*^{\text{uni}}} D$ “ $:= \alpha_*^{\text{uni}} D$ ” as in the holomorphic (but here meromorphic) case.

- ▶ *The integral holonomy bound.* Thus $\Gamma_{\alpha_*^{\text{uni}}} D \cdot C > 0$ for all the integral curves $C \subset \mathcal{V}^{\text{alg}}$ outside of a finite union of curves E (found by effective versions of the Riemann–Roch or Hilbert–Samuel theorems).

Bost–Charles: the analogy and the integral finiteness theorem

- *The integral holonomy bound.* Zariski proved the $n = 2$ case of Hilbert's 14th problem (to which Nagata eventually constructed a counterexample in dimensions $n \geq 32$):

For any line bundle L over a regular projective surface X/k , the k -algebra

$$\bigoplus_{k=0}^{\infty} \Gamma(X, L^{\otimes k})$$

is finitely generated.

Applied to $X = \mathcal{V}^{\text{alg}}$ and $L = \mathcal{O}(E \cup \pi_{\mathcal{V}^{\text{alg}}}^{-1}(\Sigma))$ this gives:

Theorem (Bost, Charles—geometric protospace)

On a pseudoconcave $\mathcal{V}$, the ring of holomorphic functions $\mathcal{O}(\mathcal{V})$ is a $\mathbf{C}[C^\circ]$ -algebra of finite type.

Bost–Charles: the analogy and the integral finiteness theorem

Transplanting the arguments to the arithmetic case:

Theorem (Bost, Charles)

On a pseudoconcave f.a. arithmetic surface $\tilde{\mathcal{V}}$ over $\mathrm{Spec} \mathbf{Z}$, the ring of holomorphic functions $\mathcal{O}(\tilde{\mathcal{V}})$ constitutes a finitely generated $\mathbf{Z}$ -algebra. It can be effectively computed based on the arithmetic Hilbert–Samuel formula for arithmetic surfaces.

Furthermore, for any $x \in \mathcal{O}(\tilde{\mathcal{V}}) \setminus \mathbf{Z}$, the fraction field $\mathcal{M}(\tilde{\mathcal{V}})$ of $\mathcal{O}(\tilde{\mathcal{V}})$ is a finite algebraic extension of $\mathbf{Q}(x)$ of degree at most

$$\left[\mathcal{M}(\tilde{\mathcal{V}}) : \mathbf{Q}(x) \right] \leq \frac{x_* \left(P, g_{\tilde{\mathcal{V}}_C} \right) \cdot x_* \left(P, g_{\tilde{\mathcal{V}}_C} \right)}{\widehat{\deg} \bar{N}_P(\tilde{\mathcal{V}})}.$$

$(P : \mathrm{Spec} \mathbf{Z} \rightarrow \mathrm{Spf} \mathbf{Z}[[x]])$ the section analogous to ϵ , and $g_{\tilde{\mathcal{V}}_C}$ the equilibrium potential analogous to D)

Concretely in the case $V = \overline{\mathbf{D}}$

In our setting, with $\psi := \varphi^{-1}$ constructed from $\varphi \in \mathcal{O}(\tilde{\mathcal{V}}) \setminus \mathbf{Z}$.
The rational+integral holonomy bound amounts to the following.

Theorem (Bost, Charles)

Given the analytic datum $\varphi : (\overline{\mathbf{D}}, 0) \rightarrow (\mathbf{C}, 0)$ of $|\varphi'(0)| > 1$, the $f \in \mathbf{Z}[[x]]$ formal functions turning $f(\varphi(z)) \in \mathcal{M}(\mathbf{D})$ span over $\mathbf{Q}(x)$ a vector space of the finite dimension

$$\leq \frac{\int \int_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \mu_{\text{Haar}}(z) \mu_{\text{Haar}}(w)}{\log |\varphi'(0)|},$$

and the subring of $f \in \mathbf{Z}[[x]]$ turning $f(\varphi(z)) \in \mathcal{O}(\mathbf{D})$ holomorphic make a (finite) free module over $\mathbf{Z}[x, h]$ for some effectively constructible rational function $h \in \mathbf{Q}(x)$.

How may the integrality part be extended in the presence of $\prod_{i=1}^r [1, \dots, b_i n]$ denominators?

Thank you for your attention

—and for the wonderful and
mathematically enriching event!