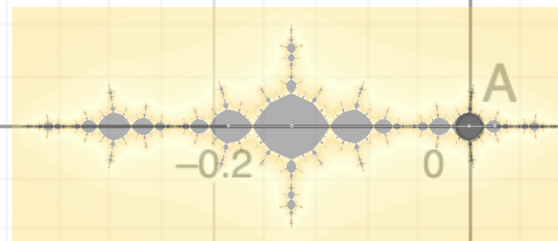
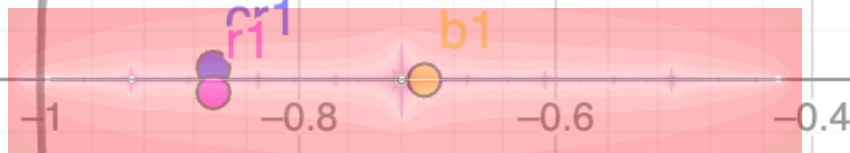
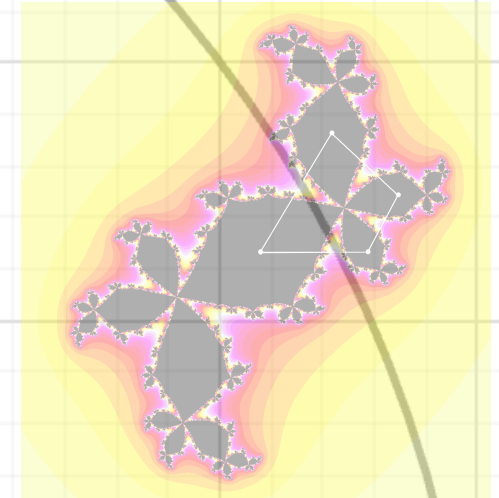


Complex dynamics without dynamics

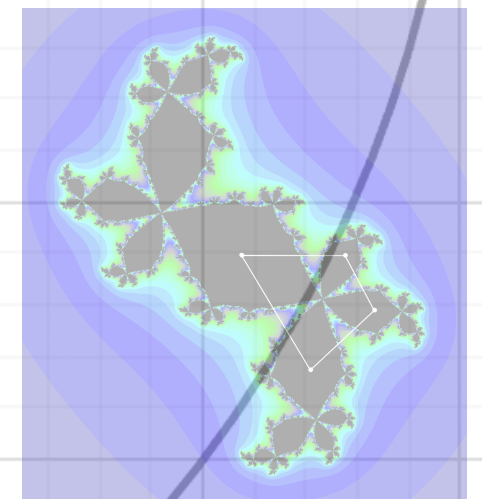
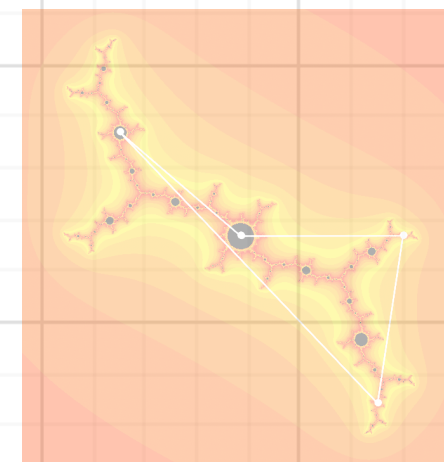
Sarah C. Koch

University of Michigan

+
Math Corps

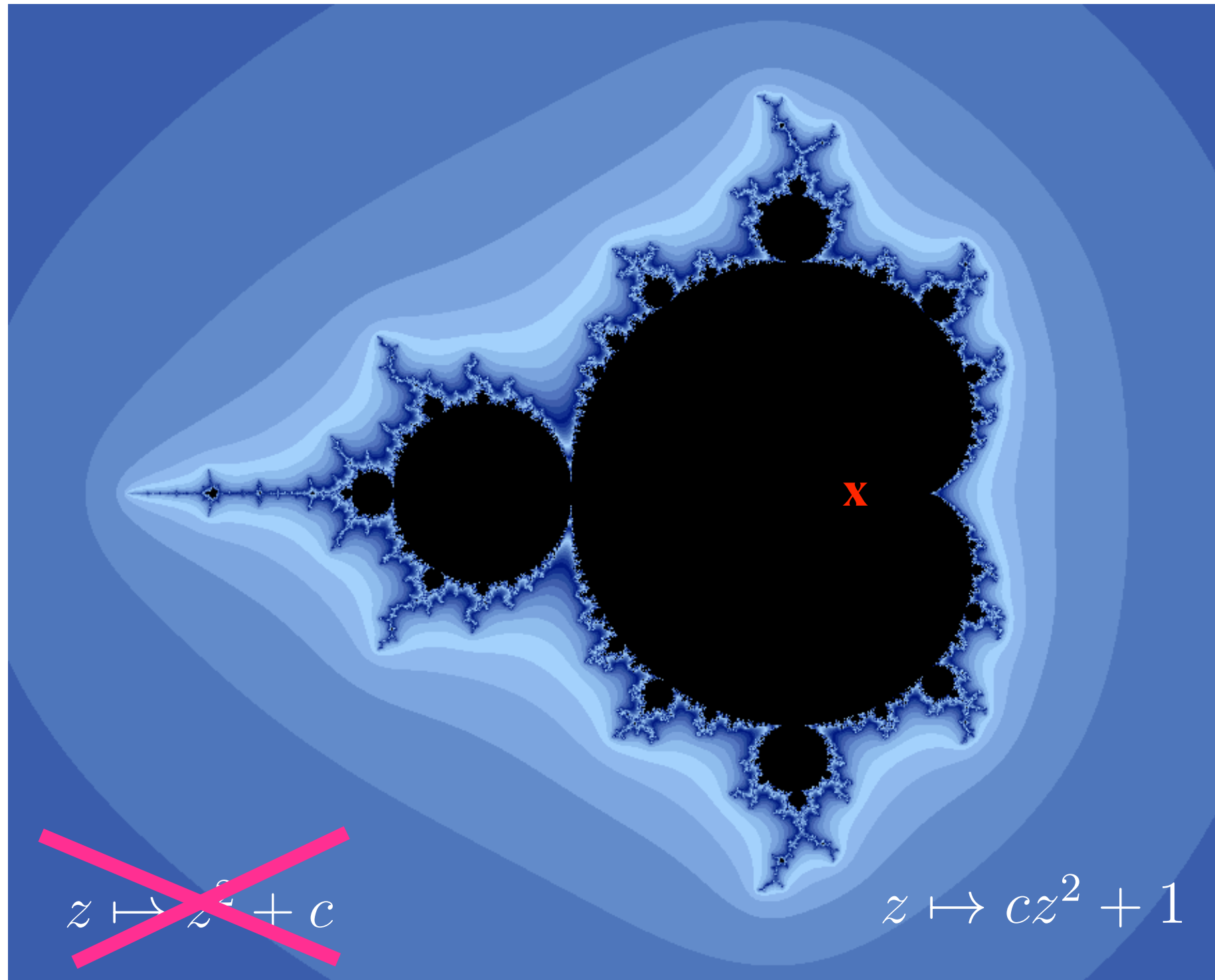


0.2 0.4 0.6 0.8



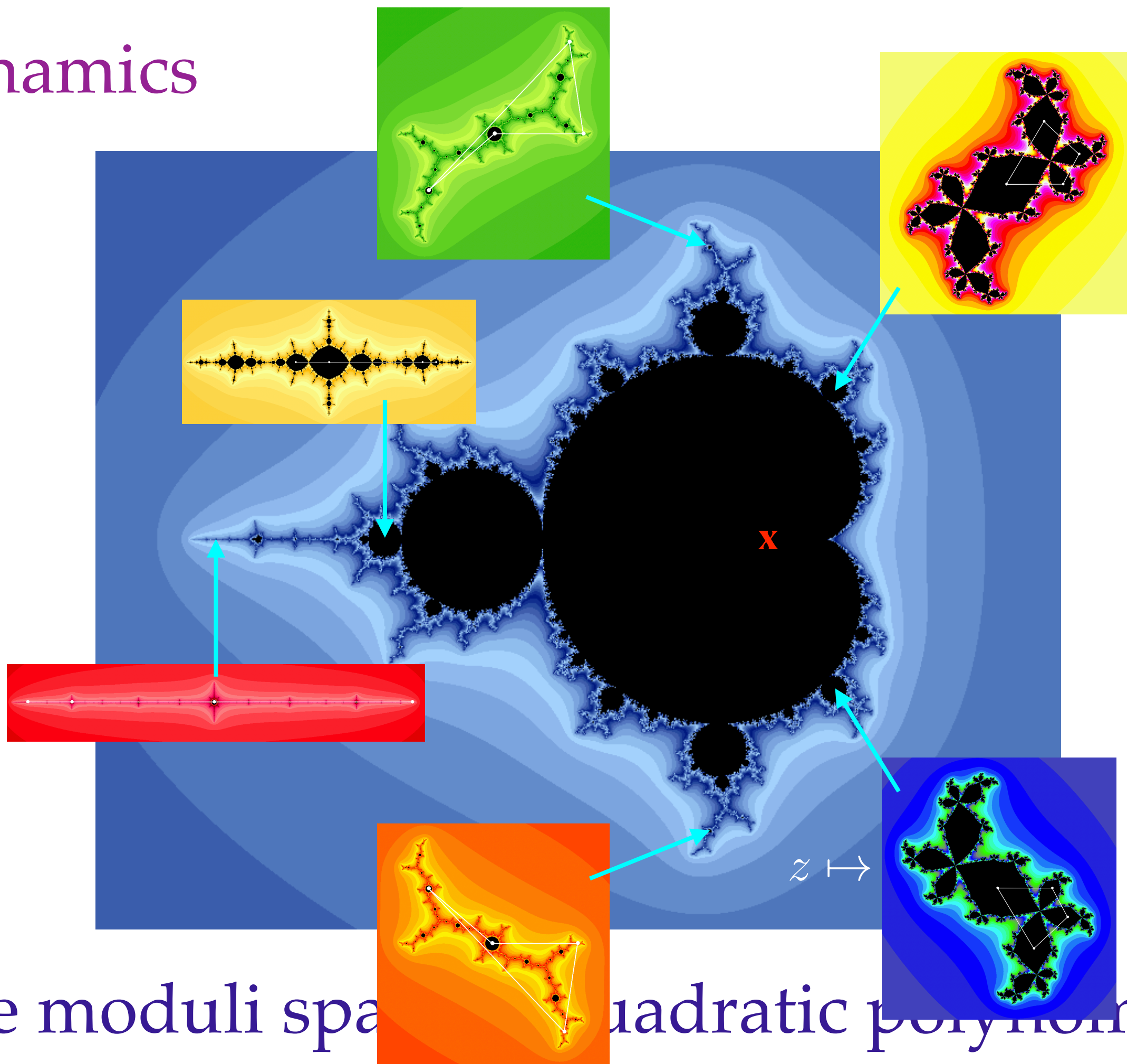
-0.6
cr2

Dynamics



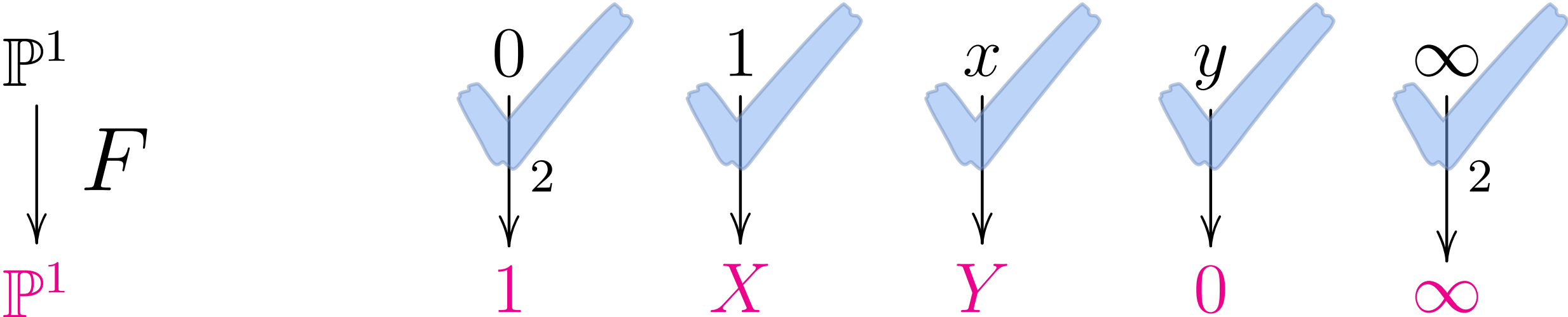
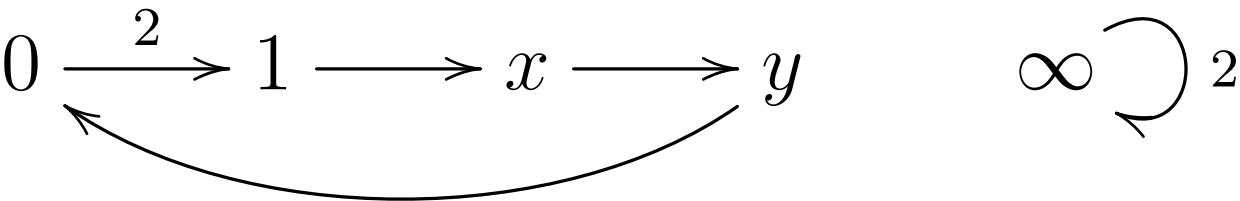
The moduli space of quadratic polynomials

Dynamics



The moduli space of quadratic polynomials

No dynamics



$$F(z) = \alpha z^2 + 1$$

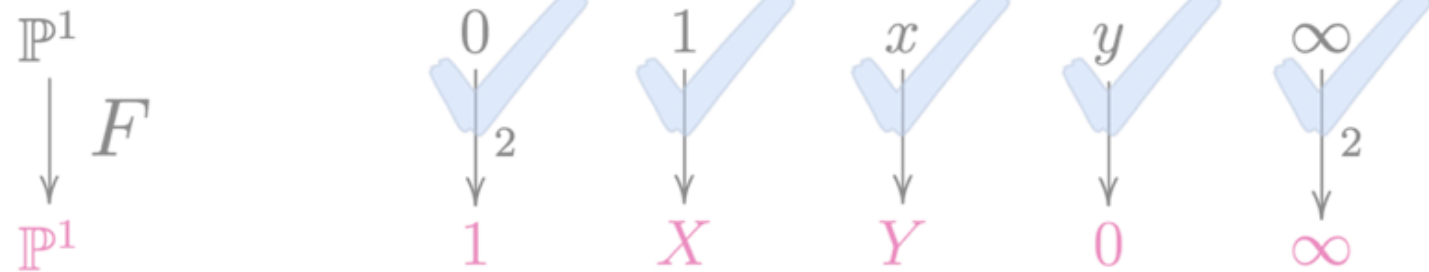
$$F(y) = 0 \implies \alpha = -\frac{1}{y^2}$$

$$F(1) = X \implies X = 1 - \frac{1}{y^2}$$

$$F(x) = Y \implies Y = 1 - \frac{x^2}{y^2}$$

No dynamics

$$0 \xrightarrow{2} 1 \longrightarrow x \longrightarrow y \quad \infty \curvearrowright_2$$



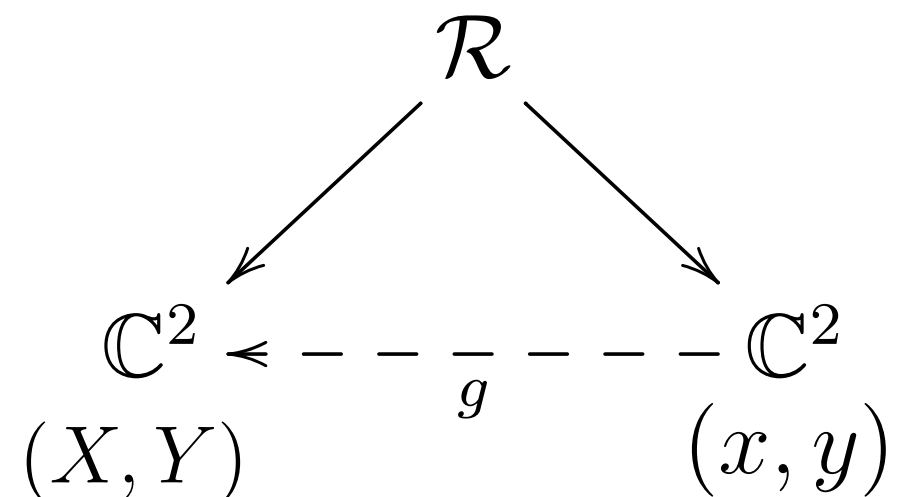
$$F(z) = \alpha z^2 + 1$$

$$F(y) = 0 \implies \alpha = -\frac{1}{y^2}$$

$$F(1) = X \implies X = 1 - \frac{1}{y^2}$$

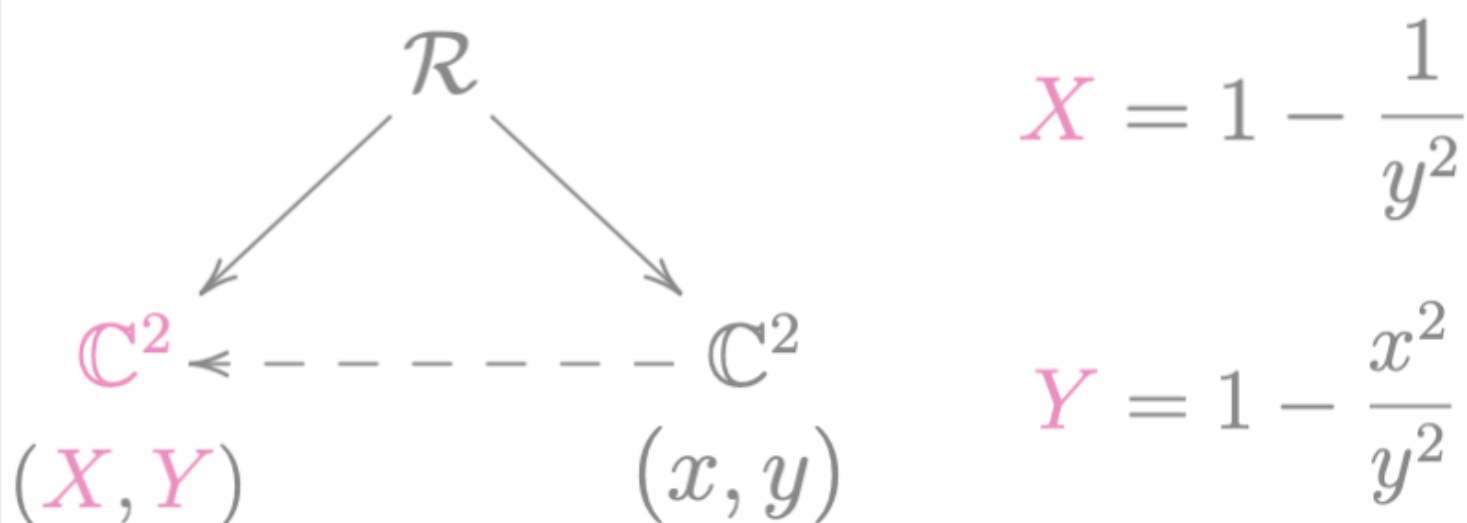
$$F(x) = Y \implies Y = 1 - \frac{x^2}{y^2}$$

Dynamics



fixed points:
six inside
one at $(0,1)$

$$\mathcal{R} = \{(F, (X, Y), (x, y)) \in \text{Rat}_2 \times \mathbb{C}^2 \times \mathbb{C}^2 \mid (F, (X, Y), (x, y)) \text{ realizes the data}\}$$



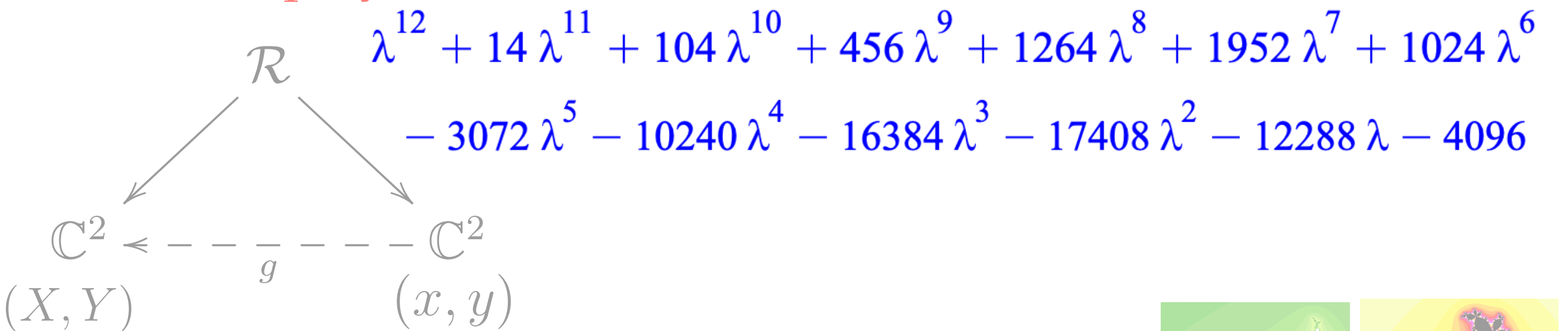
$$\mathbb{P}^2 \leftarrow \mathbb{P}^2$$

is a **postcritically finite**
endomorphism!

Dynamics

Characteristic polynomial: $\lambda^2 - \frac{2x^2\lambda}{y^3} + \frac{4x}{y^5}$

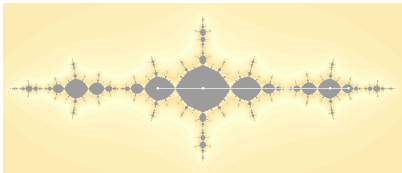
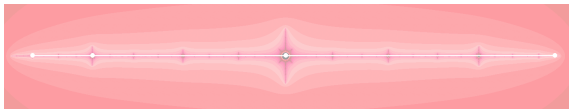
Minimal polynomial:



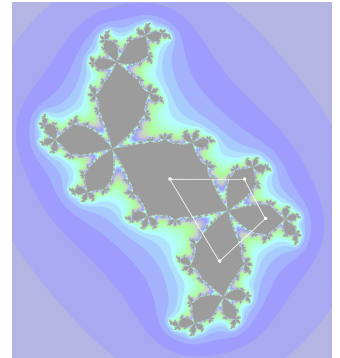
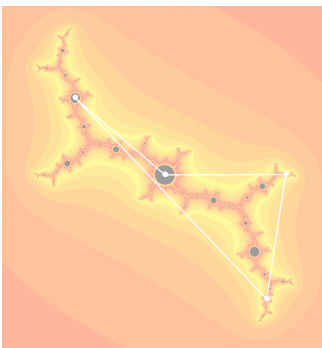
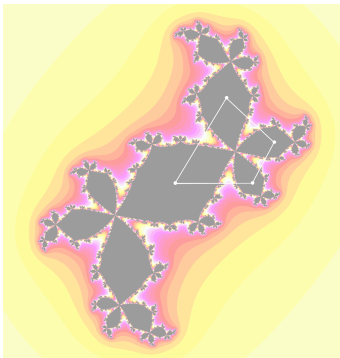
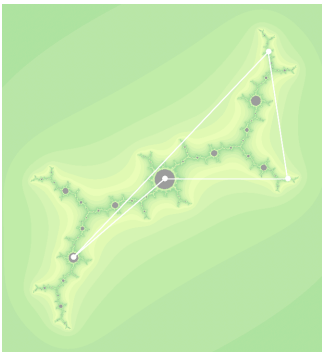
$$X = 1 - \frac{1}{y^2}$$

$$Y = 1 - \frac{x^2}{y^2}$$

$$0 \overset{2}{\longrightarrow} 1 \longrightarrow x \longrightarrow y \qquad \infty \curvearrowright 2$$



$$z \mapsto 1 - \frac{z^2}{y^2}$$



fixed points:
six inside

$$x = \frac{y^2 - 1}{y^2}$$

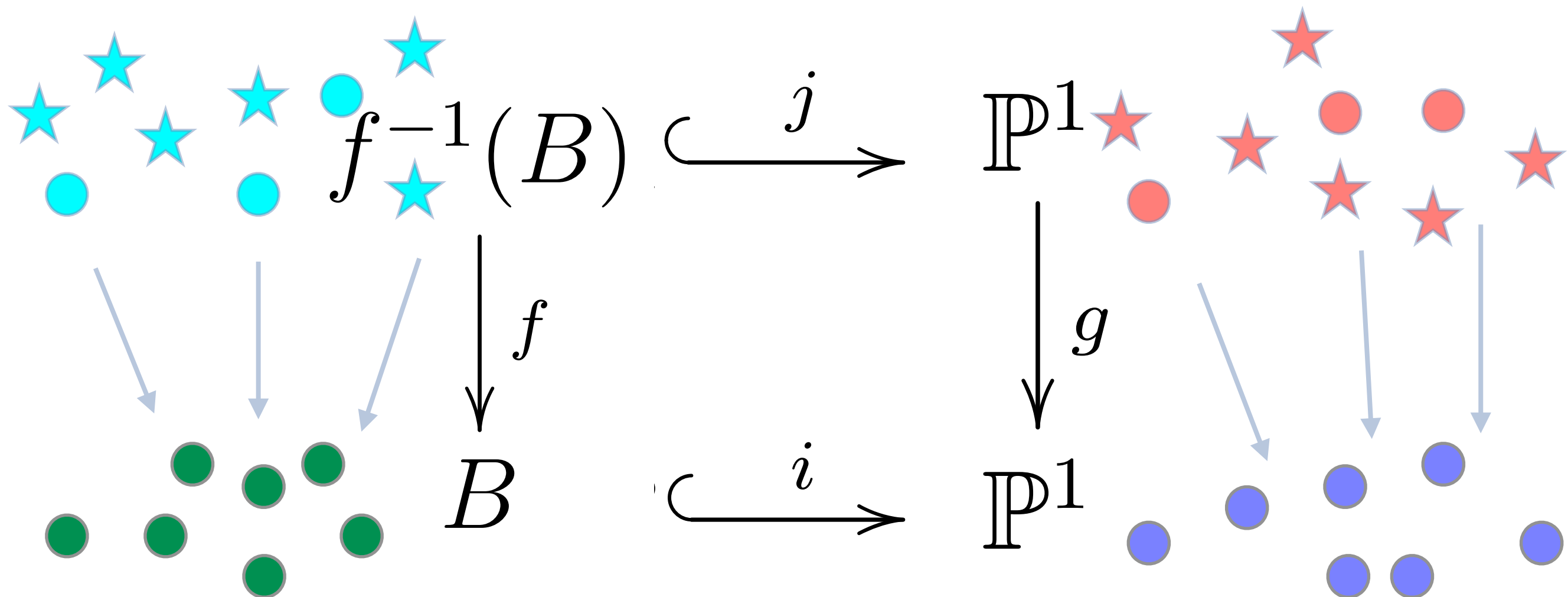
$$y^6 + y^3 + y^2 - y - 1 = 0$$

No dynamics

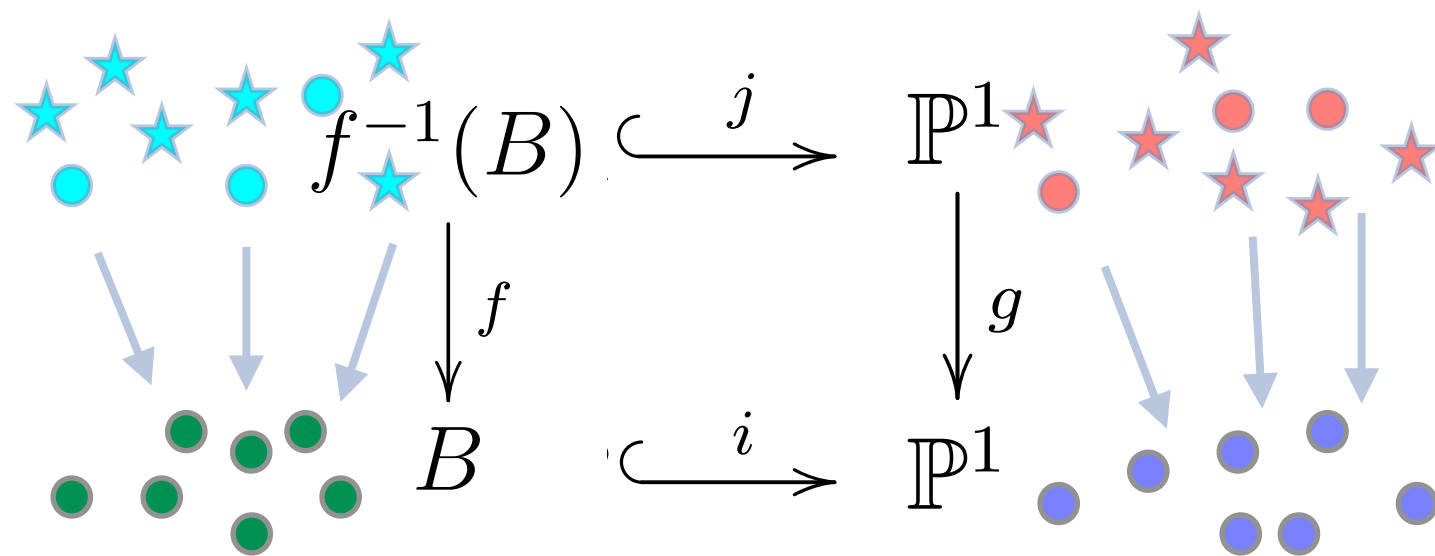
$f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ is rational map of degree $d \geq 2$ such that:

- B contains all critical values of f ,
- $A \subseteq f^{-1}(B)$, and
- $|A|, |B| \geq 3$.

same branch data



No dynamics



$$\deg(f|_x) = \deg(g|_{j(x)})$$

for all $x \in f^{-1}(B)$

$$(g, i, j|_A) \in \text{Rat}_d \times (\mathbb{P}^1)^B \times (\mathbb{P}^1)^A$$

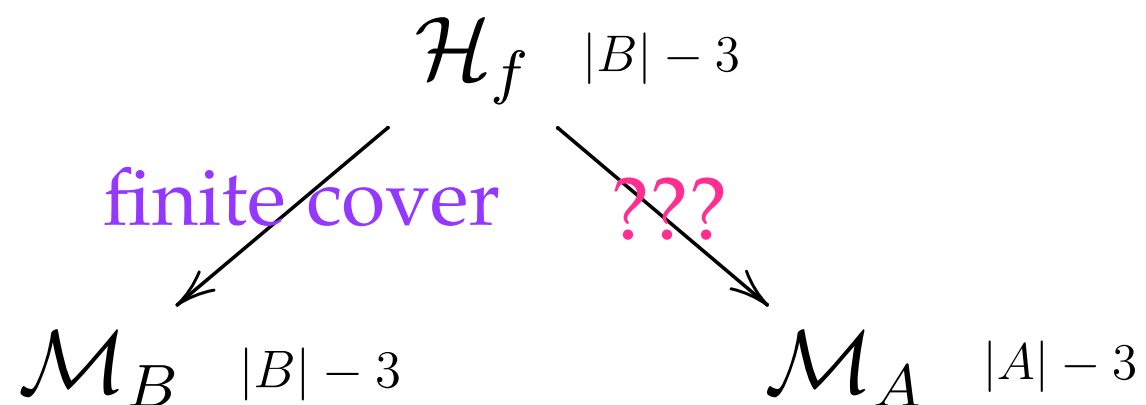
up to equivalence

$$(\nu \circ g \circ \mu^{-1}, \nu \circ i, \mu \circ j|_A), \quad \mu, \nu \in \text{Aut}(\mathbb{P}^1)$$

Let $\mathcal{H}_f$ be the irreducible component containing $(f, \text{id}_B, \text{id}_A)$.

smooth

quasiprojective

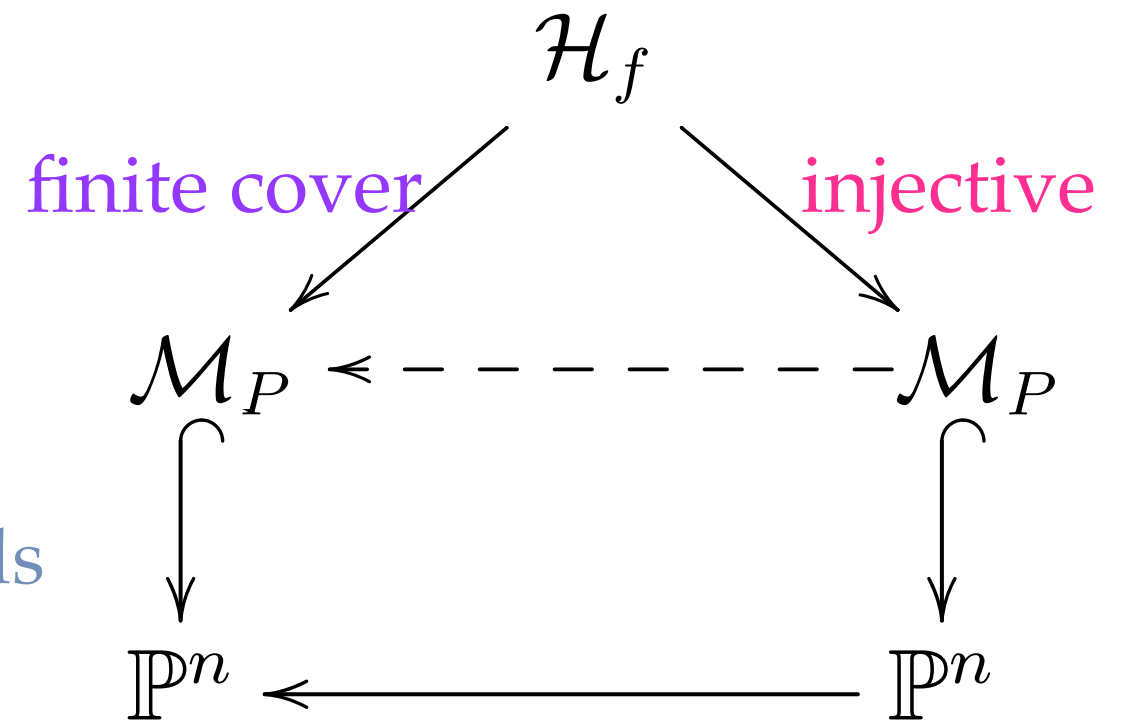


Moduli space correspondence

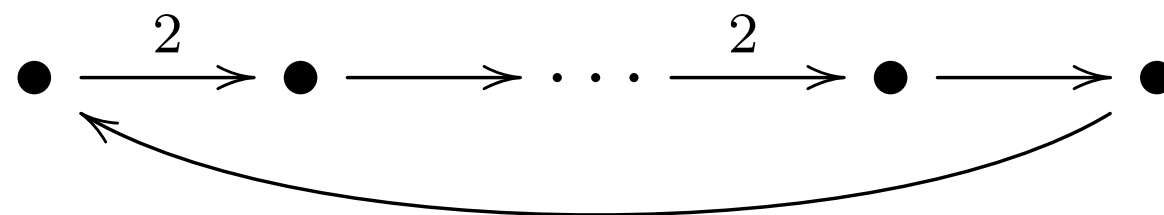
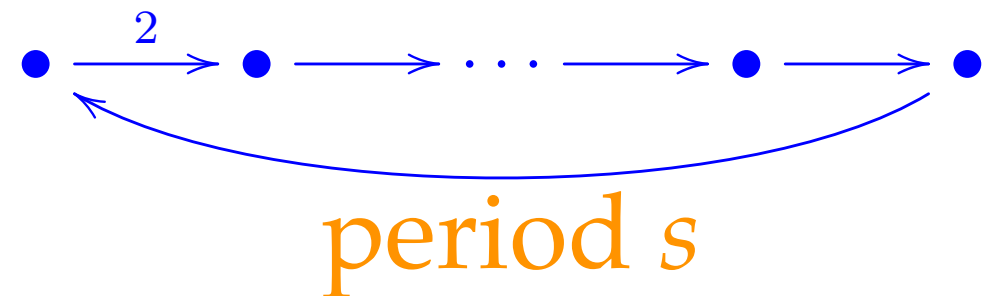
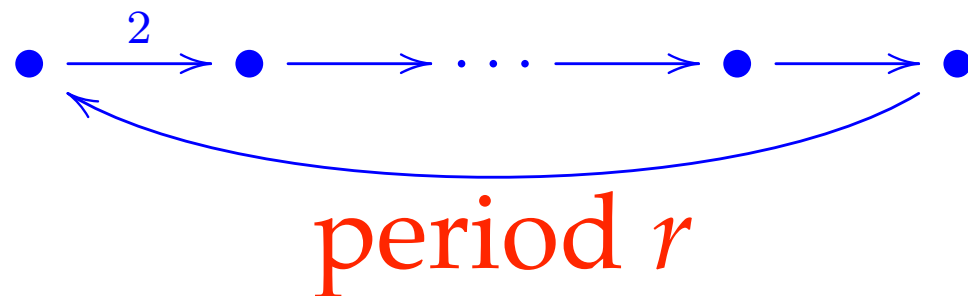
Dynamics

$f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ is PCF

In favorable cases, the moduli space correspondence is the graph of a map; in particularly favorable cases, the map extends to an endomorphism of projective space, necessarily **PCF**.



Example



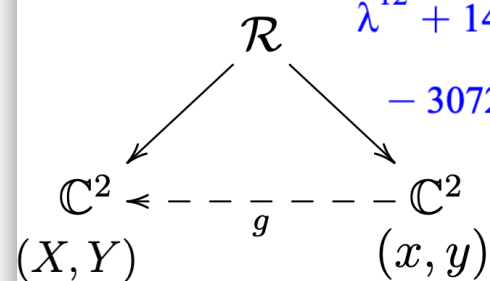
Dynamics

$f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ is PCF

Characteristic polynomial: $\lambda^2 - \frac{2x^2\lambda}{y^3} + \frac{4x}{y^5}$

Minimal polynomial:

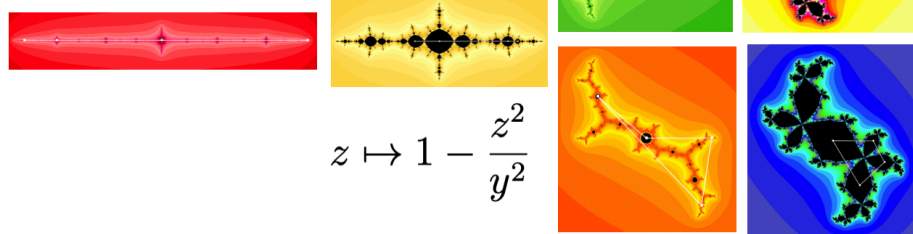
$$\lambda^{12} + 14\lambda^{11} + 104\lambda^{10} + 456\lambda^9 + 1264\lambda^8 + 1952\lambda^7 + 1024\lambda^6 - 3072\lambda^5 - 10240\lambda^4 - 16384\lambda^3 - 17408\lambda^2 - 12288\lambda - 4096$$



$$X = 1 - \frac{1}{y^2}$$

$$Y = 1 - \frac{x^2}{y^2}$$

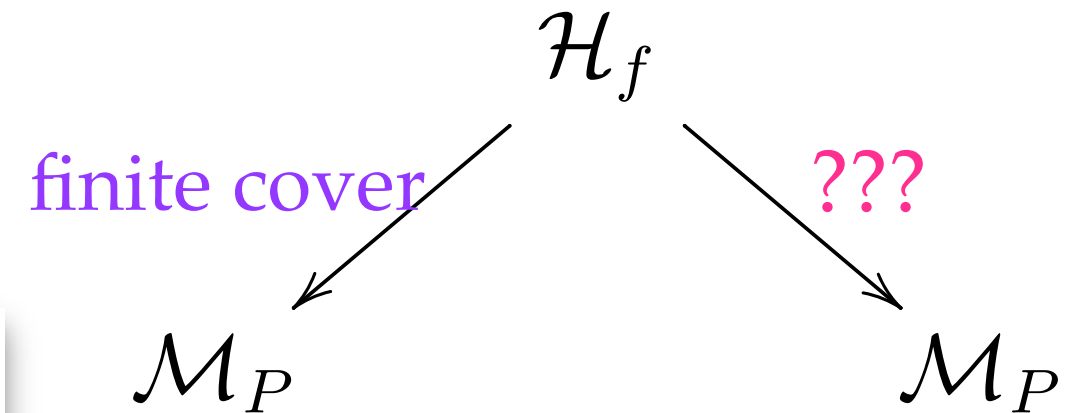
$$0 \xrightarrow{2} 1 \rightarrow x \rightarrow y \quad \infty \curvearrowright 2$$



$$z \mapsto 1 - \frac{z^2}{y^2}$$

fixed points:
six inside

$$x = \frac{y^2 - 1}{y^2} \quad y^6 + y^3 + y^2 - y - 1 = 0$$



Fixed points correspond to
PCF rational maps

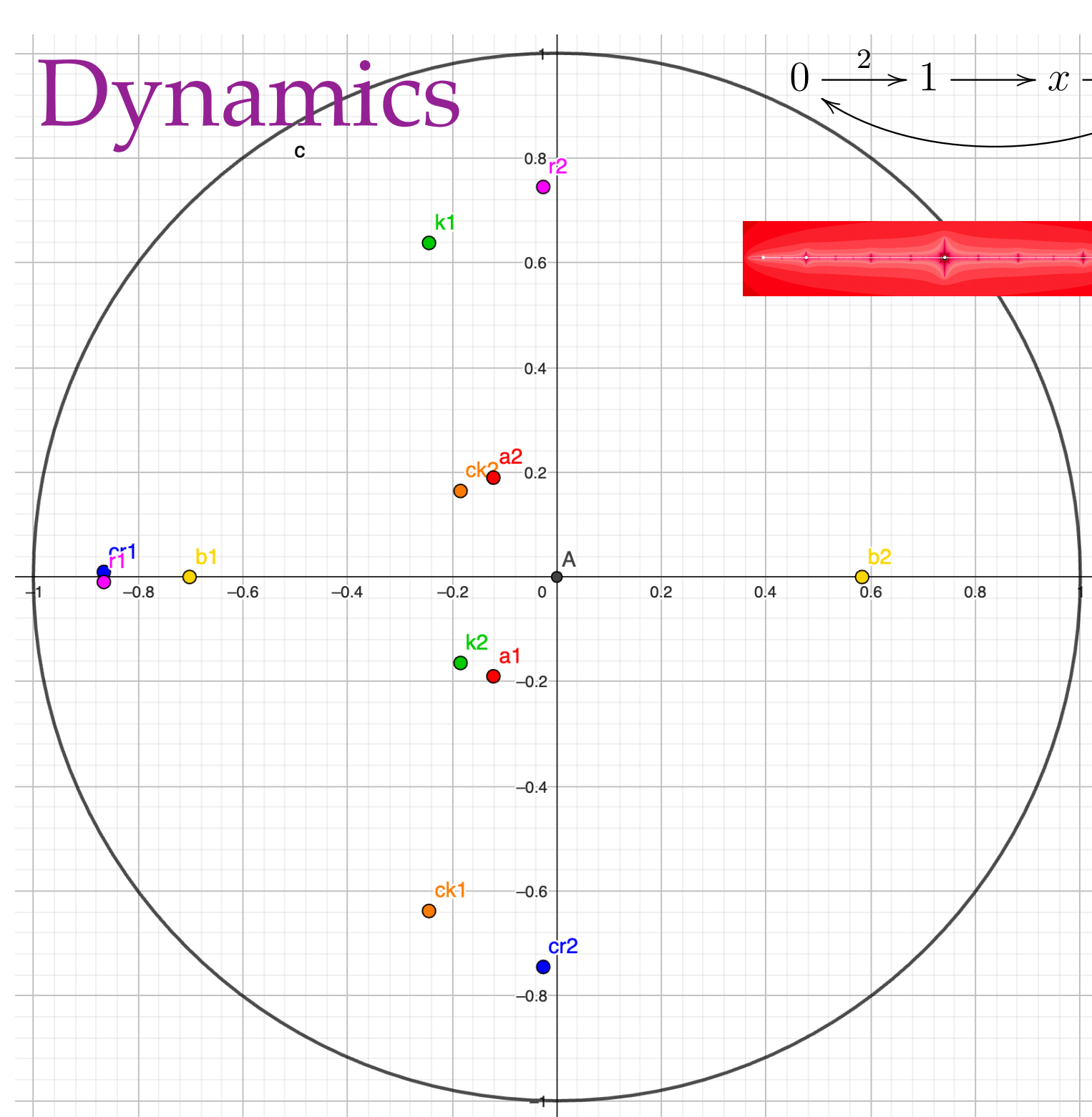
Compute the derivative at
the fixed points

What are the eigenvalues?

Minimal polynomial (correct direction):

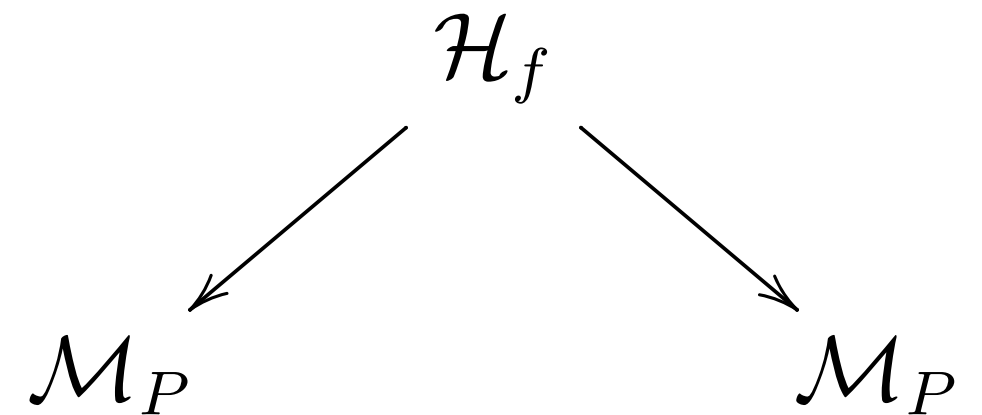
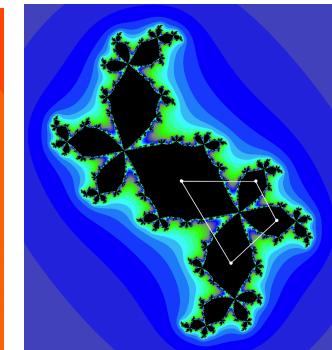
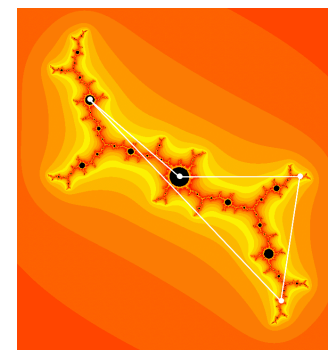
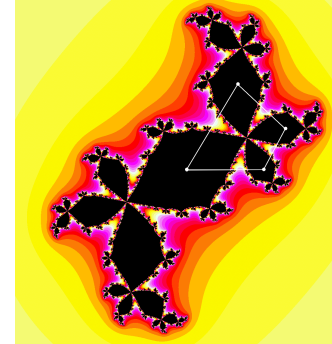
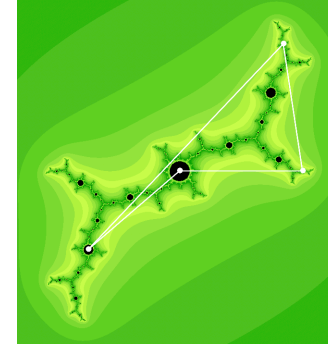
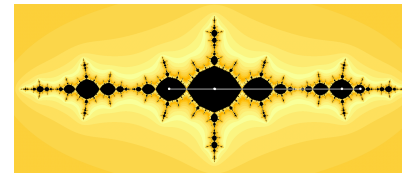
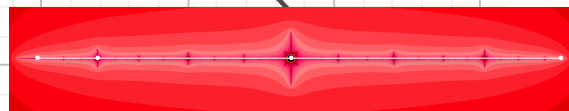
$$4096\lambda^{12} + 12288\lambda^{11} + 17408\lambda^{10} + 16384\lambda^9 + 10240\lambda^8 + 3072\lambda^7 - 1024\lambda^6 - 1952\lambda^5 - 1264\lambda^4 - 456\lambda^3 - 104\lambda^2 - 14\lambda - 1$$

Dynamics



$$0 \xrightarrow{2} 1 \longrightarrow x \longrightarrow y$$

$$\infty \curvearrowright 2$$



Compute the derivative at
the fixed points

What are the eigenvalues?

$$4096 \lambda^{12} + 12288 \lambda^{11} + 17408 \lambda^{10} + 16384 \lambda^9 + 10240 \lambda^8 + 3072 \lambda^7 \\ - 1024 \lambda^6 - 1952 \lambda^5 - 1264 \lambda^4 - 456 \lambda^3 - 104 \lambda^2 - 14 \lambda - 1$$

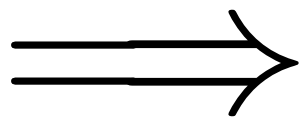
Dynamics

$f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ is PCF

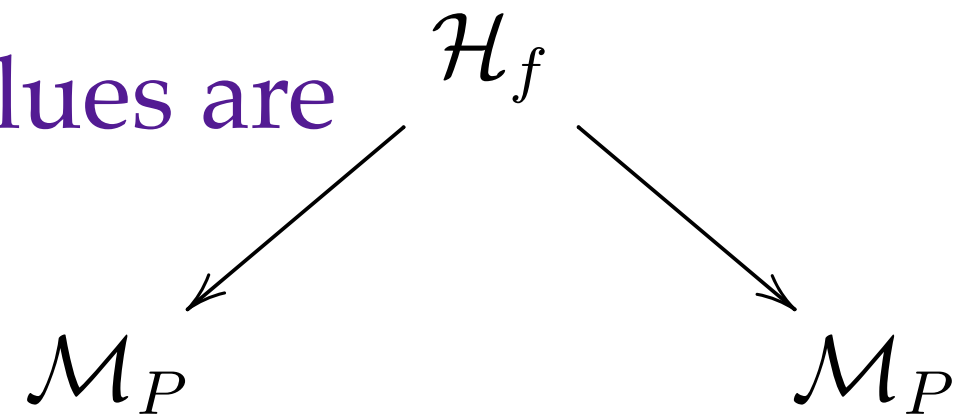
Thurston rigidity implies all eigenvalues are

★ algebraic (Galois conjugates)

★ strictly inside the unit disk*



★ algebraic integers only if they are zero*



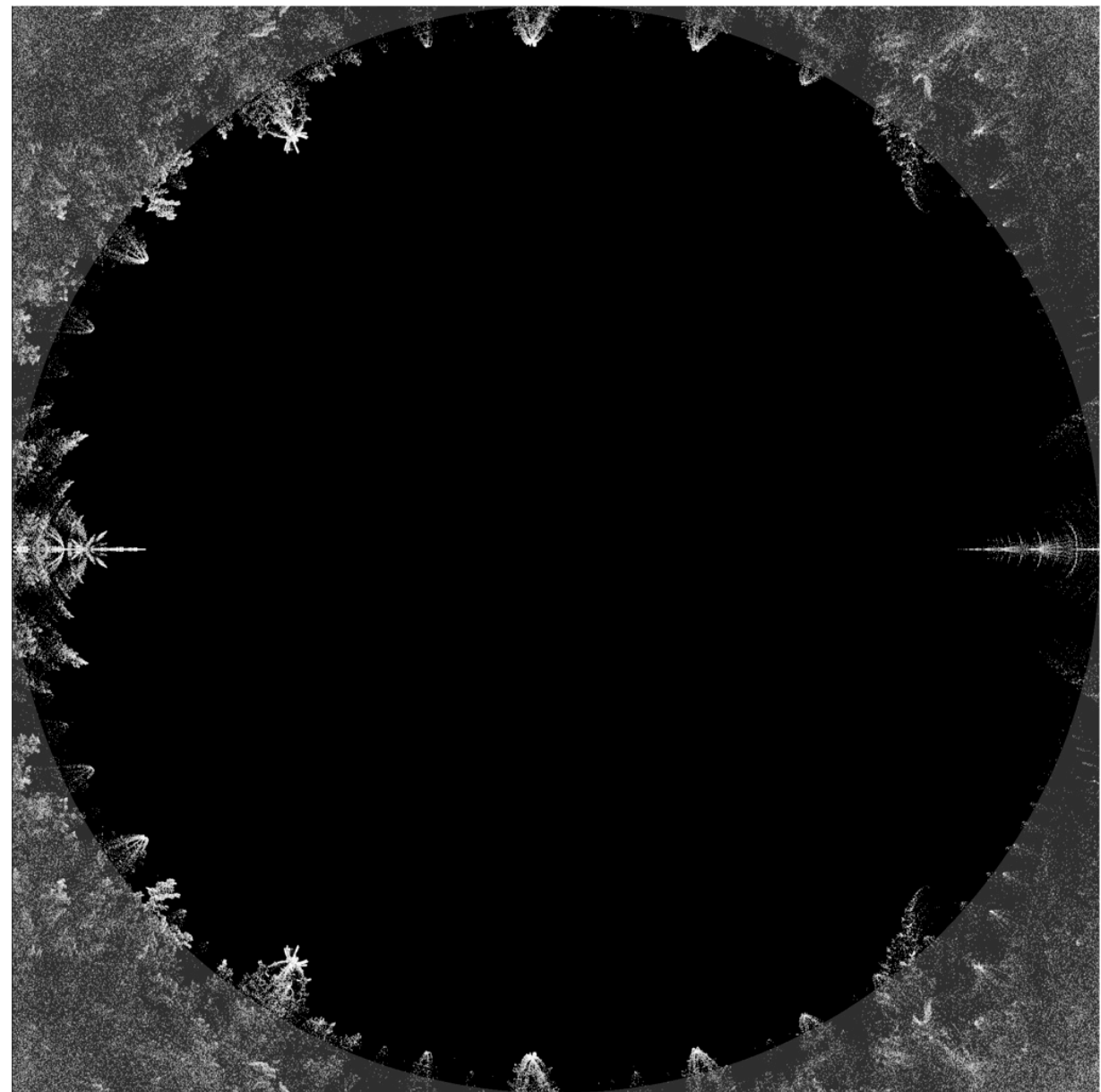
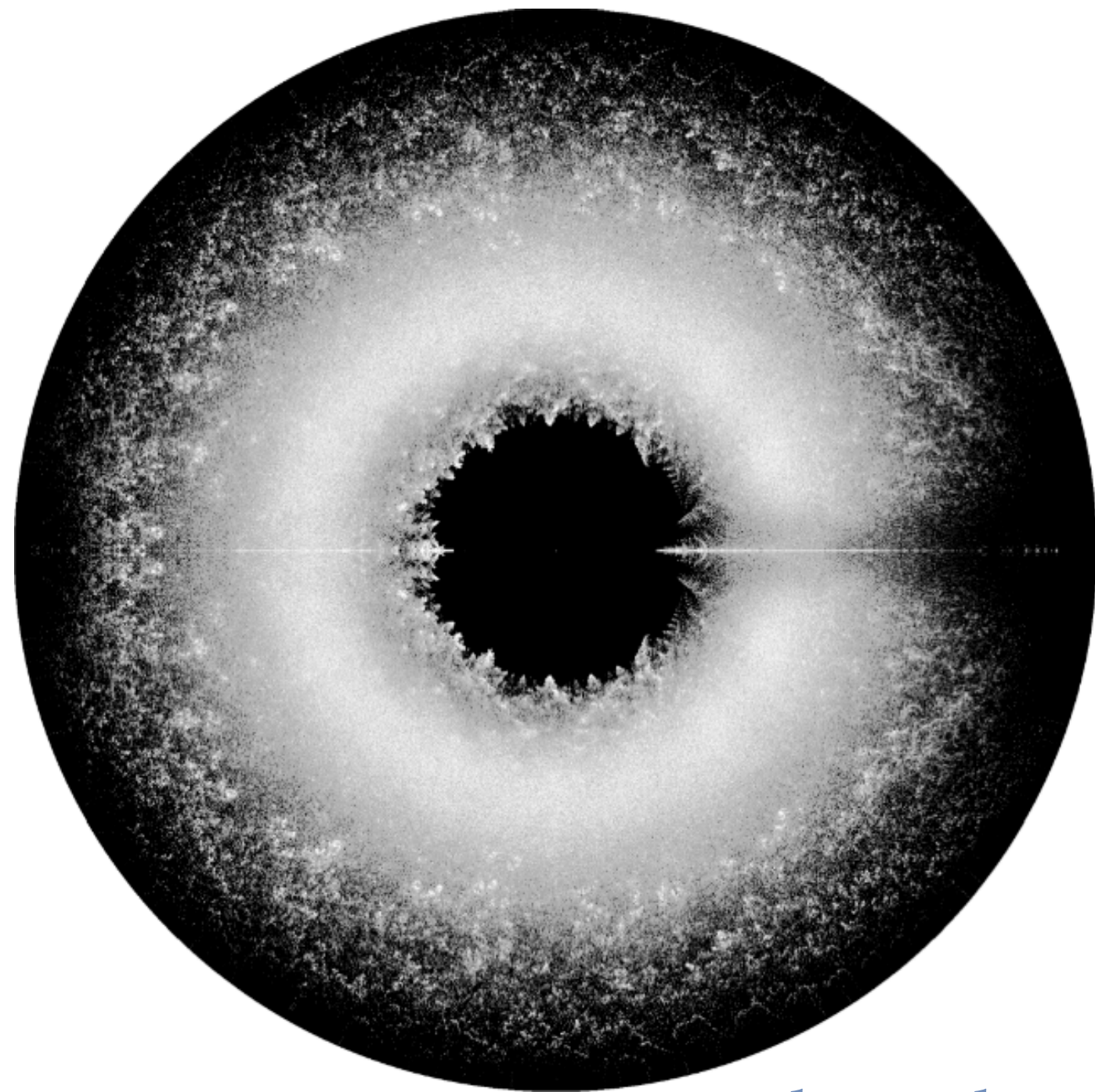
Lattès maps:

$\lambda = \pm 1$ or $\lambda \in S^1$ is an arbitrary quadratic number: if λ is an algebraic integer, λ is a root of unity of order 1, 2, 3, 4, or 6.

Dynamics

$$\frac{1}{8} < |\lambda| < 1$$

2λ is an invertible algebraic integer



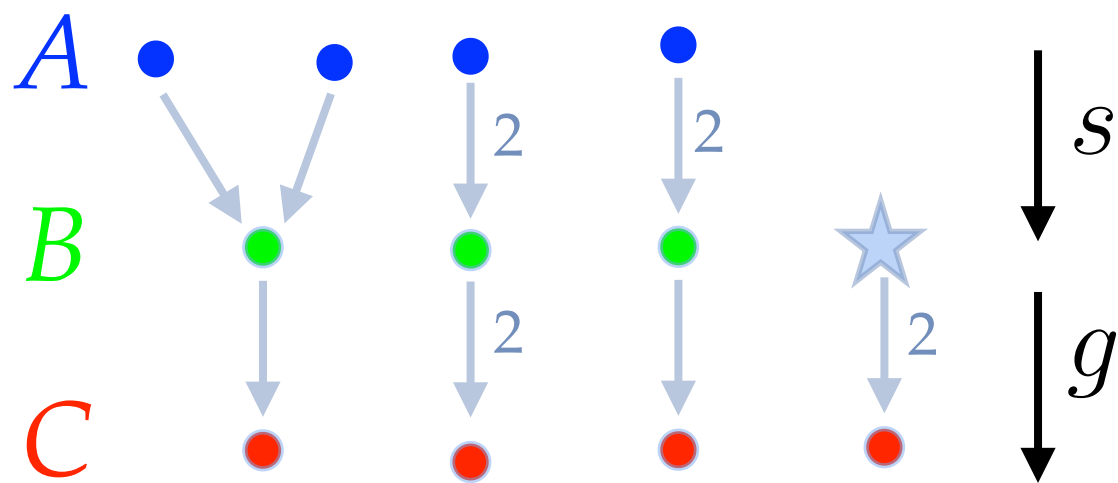
up through period 19

No dynamics

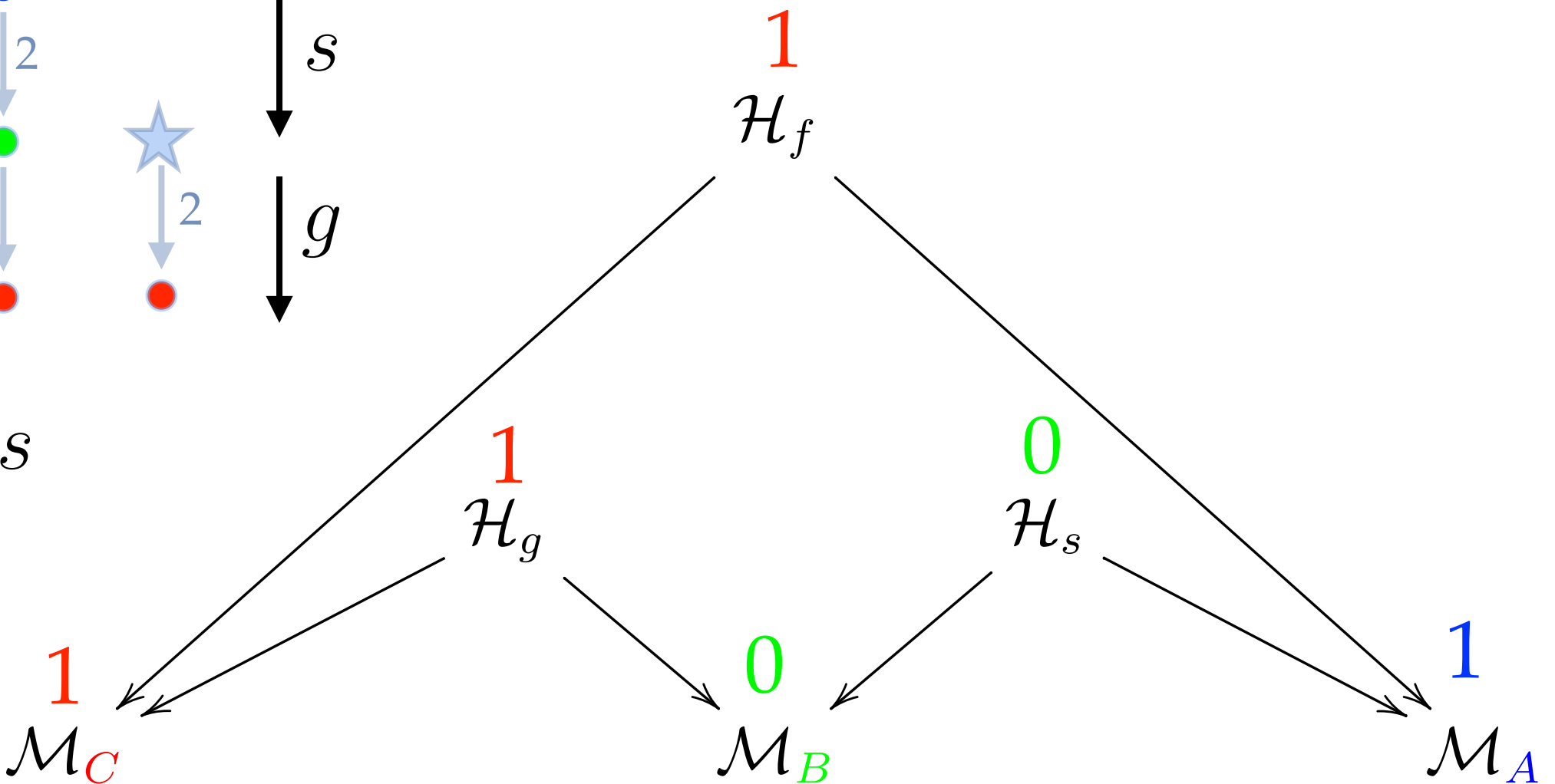
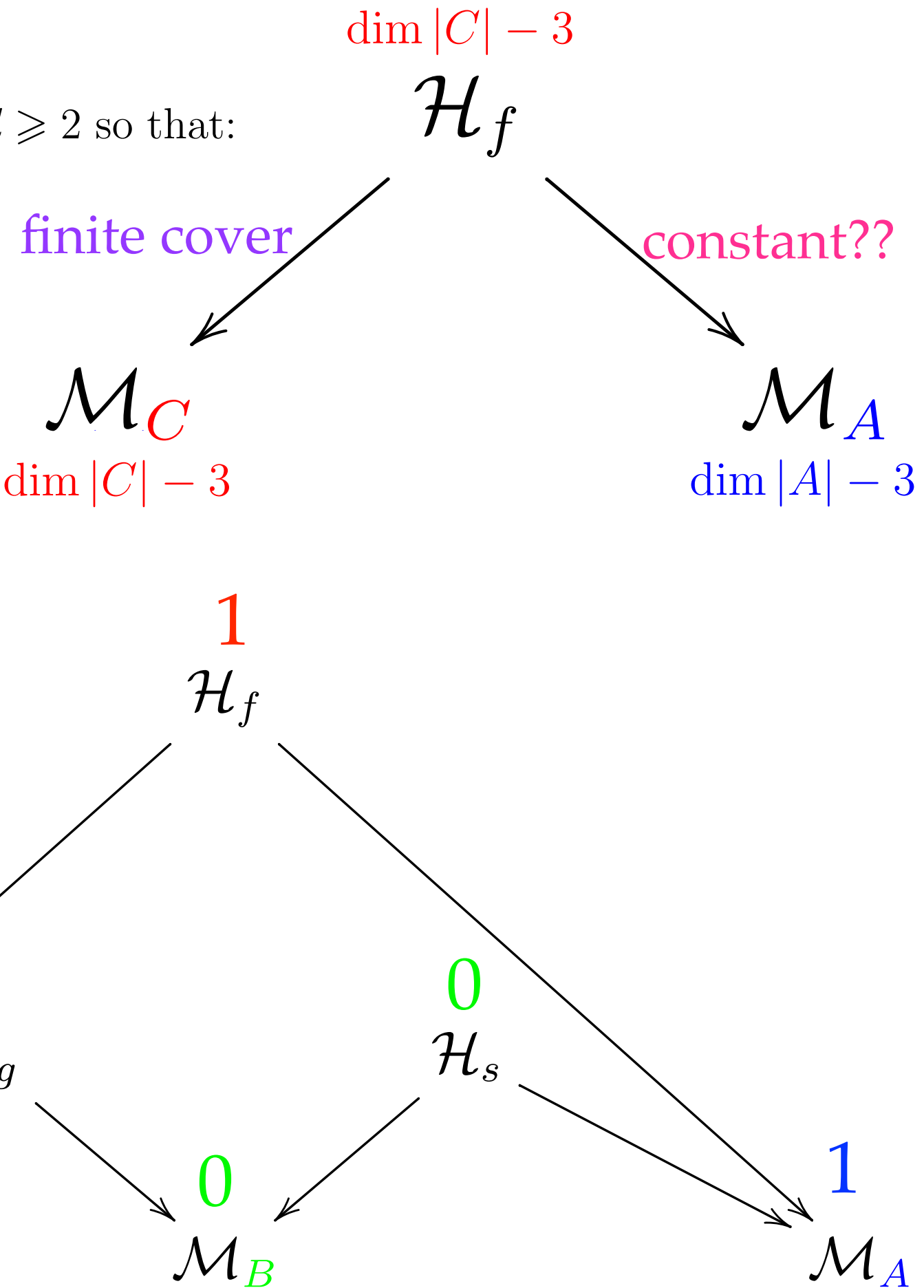
$f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, C)$ is rational map of degree $d \geq 2$ so that:

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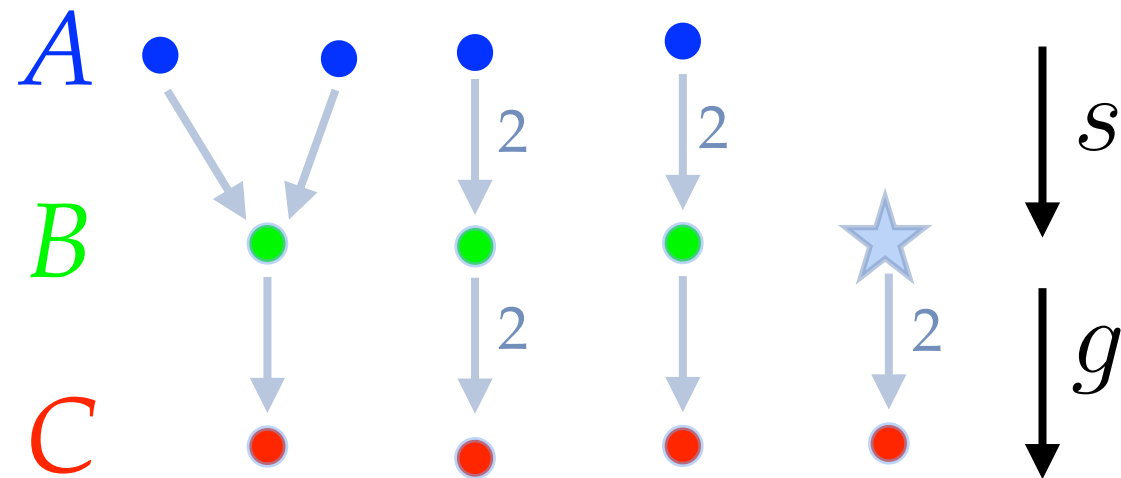
Example



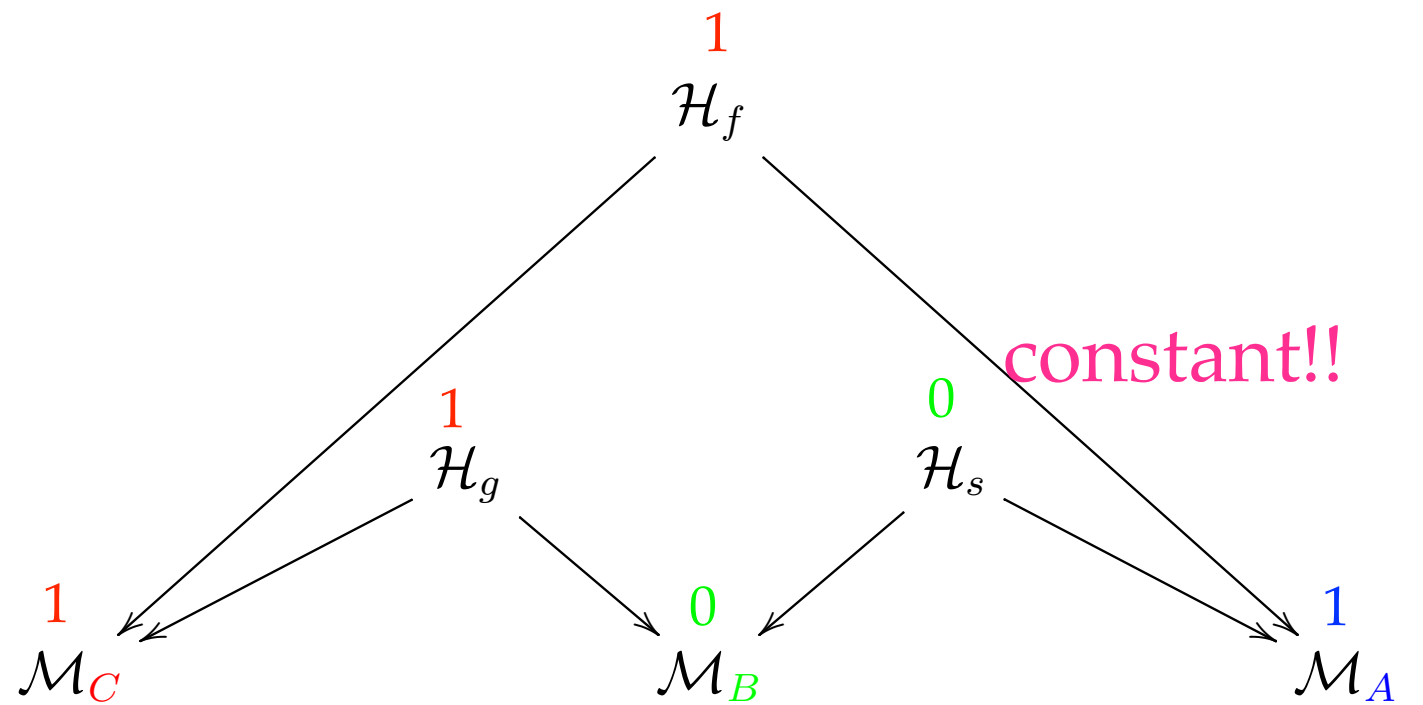
$$f = g \circ s$$



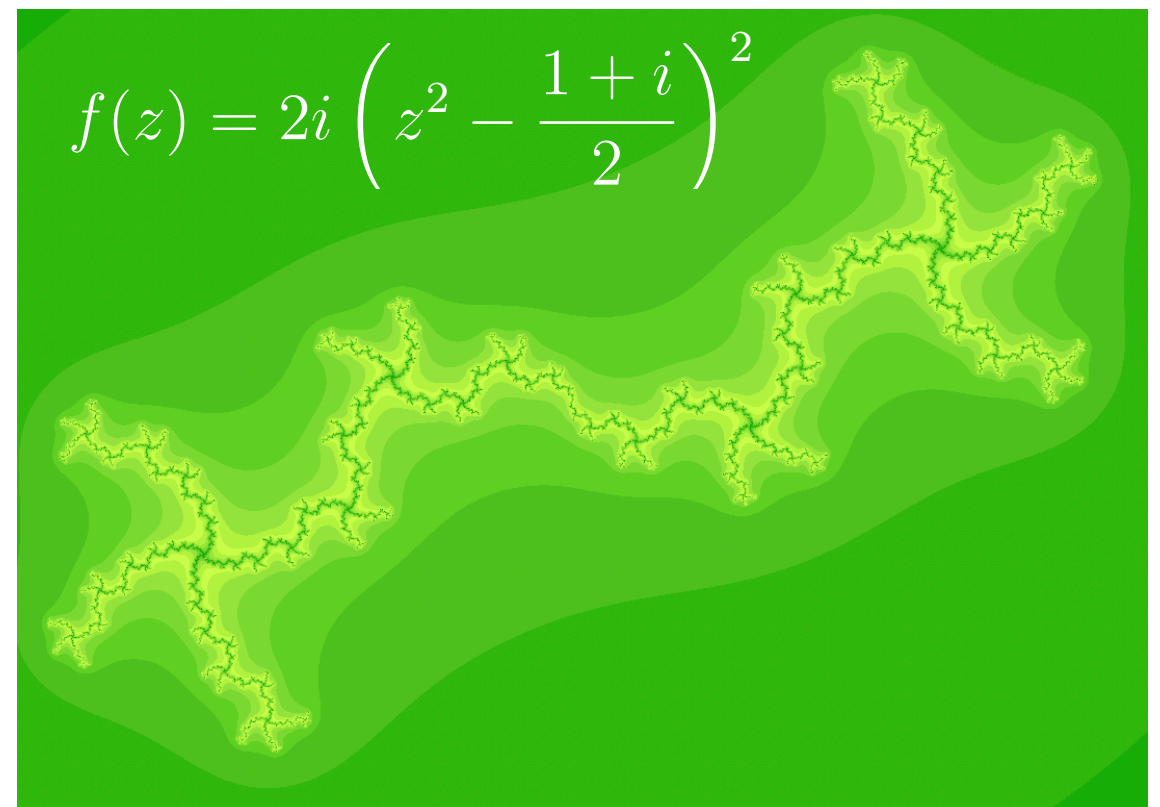
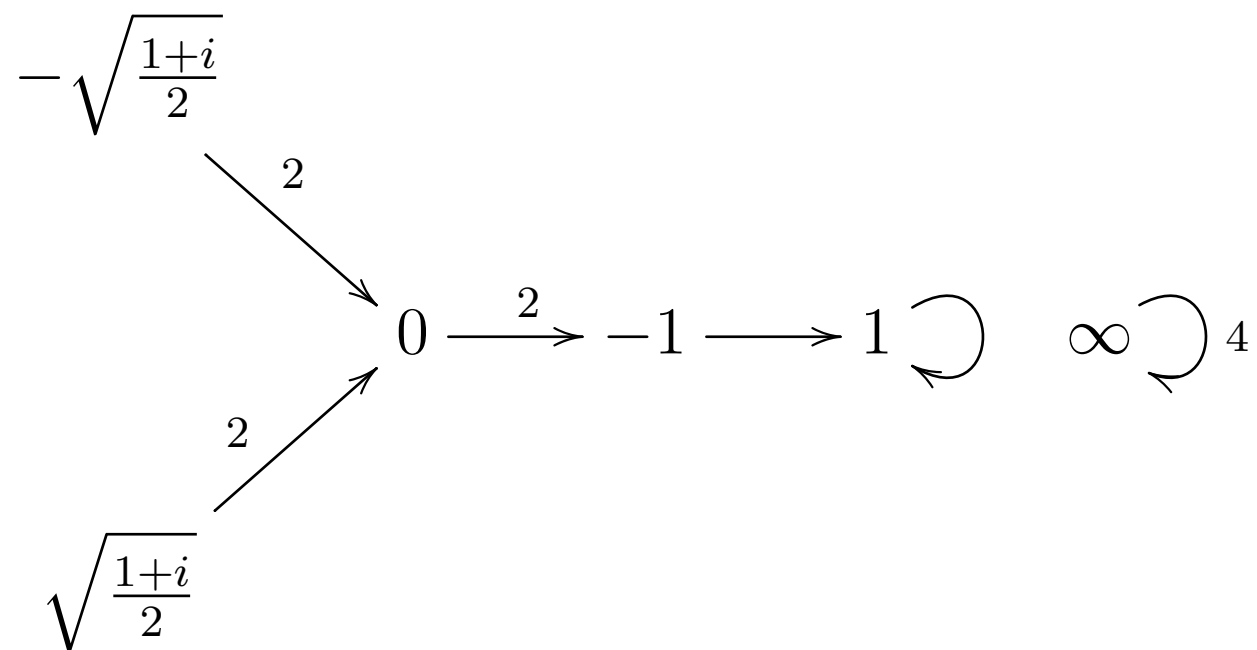
No dynamics



$$f = g \circ s$$



Dynamics



Dynamics

Question (Pilgrim): Are there examples of PCF rational maps for which the Thurston pullback map is constant (on Teichmüller space)?

Answer (McMullen): Yes. implications for Thurston maps

Are there any examples in prime degrees?
($d=2,3$ settled)

All known examples have composite degrees and are exceptional.

