

Bi- $\overline{\mathbb{Q}}$ -structures on Hermitian Symmetric Spaces and Quadratic Relations between CM Periods (work in progress with Andrei Yafaev and Ziyang Gao)

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Bi- $\overline{\mathbb{Q}}$ -structure

Definition

- ▶ Let V be a complex vector space. A **bi- $\overline{\mathbb{Q}}$ -structure** (W_1, W_2) on V is the data given by two $\overline{\mathbb{Q}}$ -structures W_1 and W_2 on V . A complex vector subspace V' of V is said to be *bi- $\overline{\mathbb{Q}}$ -rational* (or simply *bi- $\overline{\mathbb{Q}}$*) if V' is rational for both W_1 and W_2 .

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- ▶ If V' is a bi- $\overline{\mathbb{Q}}$ -subspace of dimension 1, then V' determines a complex number $\theta = \theta(V', W_1, W_2)$, well defined up to multiplication by an element of $\overline{\mathbb{Q}}^*$. Let $e_1 \in W_1$ and $e_2 \in W_2$ be some basis of V' defining the corresponding $\overline{\mathbb{Q}}$ -structures. Then $e_2 = \theta e_1$ for some $\theta \in \mathbb{C}/\overline{\mathbb{Q}}^*$. Such a θ will be called a **period** of V' for (W_1, W_2) .

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$$V = \bigoplus_j \mathbb{C}e_j = \bigoplus_j \mathbb{C}f_j$$

for a $\overline{\mathbb{Q}}$ -basis $\{e_1, \dots, e_n\}$ of W_1 and a $\overline{\mathbb{Q}}$ -basis $\{f_1, \dots, f_n\}$ of W_2 such that for each j , there exists $\theta_j \in \mathbb{C}^*/\overline{\mathbb{Q}}^*$ with $f_j = \theta_j e_j$.

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- ▶ We say that $\{\theta_1, \dots, \theta_n\}$ is a **complete set of periods** of V for (W_1, W_2) .

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Proposition

Let $V = (\mathbb{C}^n; \theta_1, \dots, \theta_n) := (\mathbb{C}; \theta_1) \oplus (\mathbb{C}; \theta_2) \oplus \dots \oplus (\mathbb{C}; \theta_n)$ be the split $\text{bi-}\overline{\mathbb{Q}}$ -structure with complete set of periods $(\theta_1, \dots, \theta_n)$.

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- ▶ If $\theta_i \neq \theta_j$ (for all i, j up to multiplication by $\overline{\mathbb{Q}}^*$), then V admits only finitely many bi- $\overline{\mathbb{Q}}$ vector subspaces.

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We can detect equalities between periods by knowing that there are infinitely many bi- $\overline{\mathbb{Q}}$ vector subspaces.

Bi- $\overline{\mathbb{Q}}$ -structure on the Tangent space of a CM abelian variety.

Let A be a complex CM abelian variety of dimension g . Then A is the extension to $\mathbb{C}$ of an abelian variety $A_{\overline{\mathbb{Q}}}$ defined over $\overline{\mathbb{Q}}$. The universal covering of A^{an} is $\mathbb{C}^g$. We endow $\mathbb{C}^g$ with a bi- $\overline{\mathbb{Q}}$ -structure :

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- ▶ The **geometric $\overline{\mathbb{Q}}$ -structure** on $\mathbb{C}^g$: It is known that there exists a matrix $\tau \in \text{Mat}_{g \times g}(\overline{\mathbb{Q}})$ such that $A(\mathbb{C}) \cong \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$. Hence $\overline{\mathbb{Q}} \cdot (\mathbb{Z}^g + \tau \mathbb{Z}^g) \subseteq \mathbb{C}^g$ gives a $\overline{\mathbb{Q}}$ -structure on $\mathbb{C}^g$, and we call it the geometric $\overline{\mathbb{Q}}$ -structure.

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- ▶ Set $\theta_j := \int_{\gamma} \omega_j$ for each $j \in \{1, \dots, g\}$. We call $\theta_1, \dots, \theta_g$ the holomorphic periods of A ; they are well-defined up to $\overline{\mathbb{Q}}$ and there exist $\gamma \in \Gamma$ such that $\theta_j \neq 0$.

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Proposition

The bi- $\overline{\mathbb{Q}}$ -structure on $\mathbb{C}^g$ defined above is split, and is isomorphic to $(\mathbb{C}^g; \theta_1, \dots, \theta_g)$.

Properties of periods of CM Abelian varieties

Let A be a simple (or square free) CM abelian variety of dimension g . Let T be the Mumford-Tate group of A (an algebraic group). Let $(\theta_1, \dots, \theta_g)$ be the holomorphic periods of A .

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- ▶ We will study the possible quadratic relations between the θ_i . We expect that they should be only of the form $\theta_i \theta_j \simeq \theta_k \theta_l$

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- ▶ Let $u : X \rightarrow S$ be the uniformizing map. S is a quasi-projective variety and has a canonical model over $\overline{\mathbb{Q}}$. Let $[o] = u(o)$ be a CM point of S and

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- ▶ One can verify that $\mathfrak{g}_{\overline{\mathbb{Q}}} \cap \mathfrak{m}_{\mathbb{C}}^+$ defines a $\overline{\mathbb{Q}}$ -structure on $\mathfrak{m}_{\mathbb{C}}^+ = T_o(X)$. We call it the **Geometric Structure**

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- ▶ Every $m_{\mathbb{C}, \alpha}^+$ is one dimensional and bi- $\overline{\mathbb{Q}}$.

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- ▶ Let $\theta_1, \dots, \theta_g$ be its holomorphic periods. Let T be the Mumford-Tate group of A_o , T^m be a maximal torus containing T . Let $\Phi(T^m, \mathrm{GSp}(2g))$ be the associated root system of type C_g .

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Theorem

- ▶ *The periods of the (split) bi- $\overline{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}_g \simeq \mathrm{Sym}^2(T_0(A))$ are $\theta_i \theta_j / \pi$ with $1 \leq i \leq j \leq g$.*

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- ▶ Let $\mathbb{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g . The associated Shimura datum is $(\mathrm{GSp}_{2g}, \mathfrak{H}_g)$, where $\mathfrak{H}_g$ is the Siegel upper half space $\{\tau \in \mathrm{Mat}_{g \times g}(\mathbb{C}) : \tau = \tau^T, \mathrm{Im} \tau > 0\}$.
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- ▶ Let $\theta_1, \dots, \theta_g$ be its holomorphic periods. Let T be the Mumford-Tate group of A_o , T^m be a maximal torus containing T . Let $\Phi(T^m, \mathrm{GSp}(2g))$ be the associated root system of type C_g .

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- ▶ If Grothendieck's period conjecture is true β is a generic point of $MT(A)$ and there are no other relations of that form.

Wüstholz Analytic Subgroup Theorem (WAST) for CM Abelian Varieties

- ▶ Let $u : \mathbb{C}^g \longrightarrow A(\mathbb{C}) = \mathbb{C}^g/\Gamma$ be the uniformisation map of a CM abelian variety by the CM field E and

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Theorem (Wüstholz)

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Goal

As the complete set of periods for the bi- $\overline{\mathbb{Q}}$ -structure on the tangent space at a CM points of $\mathbb{A}_g$ are the $\frac{\theta_i \theta_j}{\pi}$ we would like to formulate an hyperbolic analogue of WAST and to investigate the possible quadratic relations between the θ_i .

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Hyperbolic Analytic Subspace conjecture

- ▶ Let S be a connected Shimura variety associated with the Shimura datum (G, X) , and let $u: X \rightarrow S^{\text{an}} = \Gamma \backslash X$ be the uniformisation map. Let $[o]$ be a CM point of S and

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- ▶ Let $\mathcal{D} \subset T_o(X)$ be the Harish-Chandra realisation of X as a bounded symmetric domain. If $G = \text{GL}(2)$, then $\mathcal{D}$ is the unit disc.
- ▶ Then the action of the maximal compact $K_o \subset G(\mathbb{R})$ on $\mathcal{D}$ is linear. Moreover any bounded symmetric subdomain $\mathcal{D}'$ of $\mathcal{D}$, containing o , is "linear" of the form $\mathcal{D}' = T_o(\mathcal{D}') \cap \mathcal{D}$ and bi- $\overline{\mathbb{Q}}$. We write $u : \mathcal{D} \longrightarrow S$ the uniformising map.

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- ▶ If moreover $\text{MT}(o)$ is a maximal torus, then we can take $V' = T_{[o]}S'_{\overline{\mathbb{Q}}}$ for some Shimura subvariety S' of S which contains $[z]$ and $[o]$.
- ▶ In the Siegel case, $S = \mathbb{A}_g$,
 $\mathcal{D}_g = \{Z \in M_g(\mathbb{C}), Z^t = Z, Id - \overline{Z}^t Z > 0\}$, $[o] \in A(\overline{\mathbb{Q}})$. We can find some basis of $T_o(\mathcal{D}_g)$ and $T_{[o]}(\mathbb{A}_g)$ such that

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Let $(E^\vee, \Phi_E^\vee)$ be the reflex field and the reflex CM type of (E, Φ_E) and $A^\vee$ the associated reflex abelian variety. We say that $A^\vee$ is of anti-Weyl type if A is of Weyl type. If $A^\vee$ is anti-Weyl, then $[E^\vee : \mathbb{Q}] = 2^g$, $\dim(A^\vee) = 2^{g-1}$ and we have an isomorphism $MT(A^\vee) \simeq MT(A)$.

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- ▶ Moreover $\overline{\mathbb{Q}}(\Theta_I)_{I \subset \{1, \dots, g\}} = \overline{\mathbb{Q}}(\Theta_\emptyset, \Theta_{\{1\}}, \dots, \Theta_{\{g\}})$.

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- ▶ In degree 1 we find the relations $\Theta_I \Theta_{J^c} = 2i\pi$ and in degree 2 we find the relations $\Theta_I \Theta_J \simeq \Theta_K \Theta_L$ if and only if $I \cup J = K \cup L$ and $I \cap J = K \cap L$. Note that the Hodge conjecture is not known in this case but would predict all this relations are induced by some algebraic cycles. Grothendieck period conjecture would then predict that all the algebraic relations between periods are generated in this way.

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- ▶ Hyperbolic Analytic Subspace Conjecture implies that all quadratic relations are elementary. It could be quite difficult to prove, but maybe easier than Grothendieck's period conjecture.

Bi- $\overline{\mathbb{Q}}$ -geometric proof

Second proof : Geometry governs periods

- ▶ Let $[o]$ be the CM point of $\mathbb{A}_{2g-1}$ corresponding to $A^\vee$. For any (I, J, K, L) such that $I \cup J = K \cup L$ and $I \cap J = K \cap L$ we construct a Shimura variety $Z_{I,J,K,L}$ in $\mathbb{A}_{m2g-1}$ containing the point $[\underline{o}] := [o] \times [o] \times \cdots \times [o]$.

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- ▶ The construction is done by an explicit root space computation.

Bi- $\overline{\mathbb{Q}}$ -geometric proof

Second proof : Geometry governs periods

- ▶ Let $[o]$ be the CM point of $\mathbb{A}_{2g-1}$ corresponding to $A^\vee$. For any (I, J, K, L) such that $I \cup J = K \cup L$ and $I \cap J = K \cap L$ we construct a Shimura variety $Z_{I,J,K,L}$ in $\mathbb{A}_{m2g-1}$ containing the point $[\underline{o}] := [o] \times [o] \times \cdots \times [o]$.
- ▶ The bi- $\overline{\mathbb{Q}}$ -sub vector space given by the tangent space of $Z_{I,J,K,L}$ at $[\underline{o}]$ forces the equality $\Theta_I \Theta_J \simeq \Theta_K \Theta_L$.
- ▶ The construction is done by an explicit root space computation.
- ▶ We believe that the results for the anti-Weyl CM abelian varieties can be extended for any CM abelian varieties (the relations between holomorphic periods of A should be generated in degree 2 and 1 by elementary relations and these relations should be explained by the existence of sub-Shimura varieties on some appropriate Siegel space.)

Happy Birthday Yuri !

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Thanks to the organizers of the conference !