

Progress on the definition of asymptotically flat and de Sitter spacetimes

Geoffrey Compère
Université Libre de Bruxelles

Simons Collaboration on Celestial Holography
New York, April 12, 2024

Outline

1. The “Puzzle piece” diagram for asymptotically flat spacetimes
2. BMS-covariant scattering (compatible with known infrared-physics)
3. What is the set of charges characterizing a non-radiative spacetime?
4. Memory effects and Λ -BMS in dS_4 . Why dS_4 is not described by a CFT_3 .

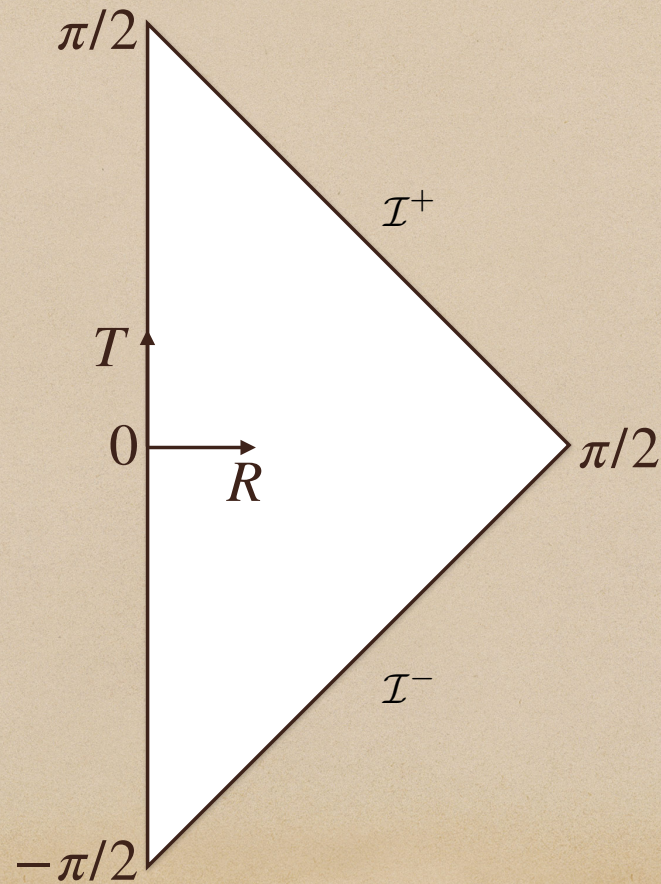
References

- ◆ “An asymptotic framework for gravitational scattering” with Samuel Gralla & Hongji Wei, 2303.17124
- ◆ “Metric reconstruction from celestial multipoles” with Roberto Oliveri & Ali Seraj, 2206.12597
- ◆ “Multipole expansion of gravitational waves: memory effects and Bondi aspects” with Luc Blanchet, Guillaume Faye, Ali Seraj & Roberto Oliveri, 2303.07732
- ◆ “Quadrupolar radiation in de Sitter: Displacement memory and Bondi metric” with Jahanur Hoque & Emine Kutluk, 2309.02081

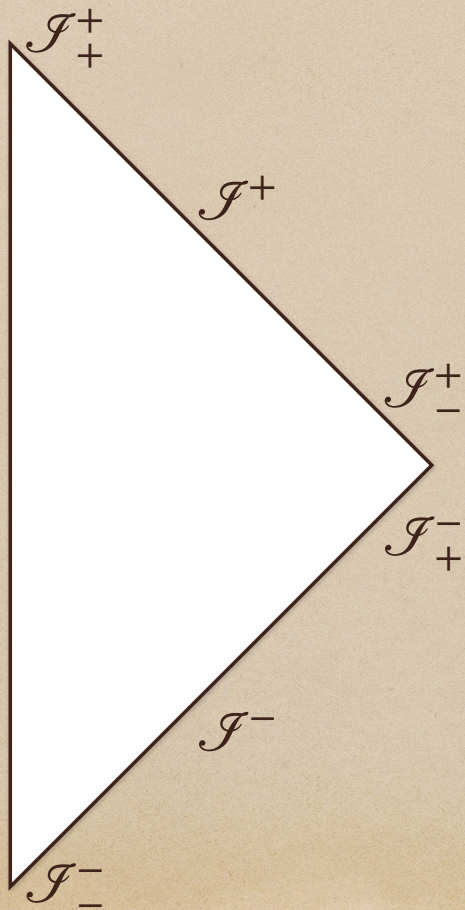
1. The "Puzzle piece" diagram for asymptotically flat spacetimes

The standard Penrose-Carter diagram

$$\begin{aligned} u = t - r = \tan U & & U = T - R \\ v = t + r = \tan V & & V = T + R \end{aligned}$$



Issues with the Penrose-Carter diagram



- ✓ Resolves $\mathcal{I}^+$ and $\mathcal{I}^-$
- ✗ No peeling
- ✗ No resolution of timelike and spatial infinity
- ✗ No intuition on the detector frame
- ✗ No clear AdS embedding

i^+ is EAdS_3

Massive 4-Momentum is mapped to (direction of motion $x^A = (\theta, \phi)$, rapidity $\rho = \text{arctanh}(r/t)$, proper time $\tau = \sqrt{t^2 - r^2}$)

The inverse transformation $t = \tau \cosh \rho$, $r = \tau \sinh \rho$ gives a unit hyperboloid (EAdS_3) foliation of Minkowski :

$$ds^2 = -d\tau^2 + \tau^2 h_{ab} dx^a dx^b, \quad h_{ab} dx^a dx^b \equiv d\rho^2 + \sinh^2 \rho \gamma_{AB} dx^A dx^B$$

where γ_{AB} is the unit metric over the sphere.

i^+ is EAdS_3

Massive 4-Momentum is mapped to (direction of motion $x^A = (\theta, \phi)$, rapidity $\rho = \text{arctanh}(r/t)$, proper time $\tau = \sqrt{t^2 - r^2}$)

The inverse transformation $t = \tau \cosh \rho$, $r = \tau \sinh \rho$ gives a unit hyperboloid (EAdS_3) foliation of Minkowski :

$$ds^2 = -d\tau^2 + \tau^2 h_{ab} dx^a dx^b, \quad h_{ab} dx^a dx^b \equiv d\rho^2 + \sinh^2 \rho \gamma_{AB} dx^A dx^B$$

where γ_{AB} is the unit metric over the sphere.

A spacetime is asymptotically flat at i^+ if it exists coordinates such that

$$ds^2 = -d\tau^2 + \tau^2 h_{ab} dx^a dx^b + O(\tau^{-1}) d\tau^2 + O(\tau^1) dx^a dx^b.$$

i^0 is dS_3

The relevant definitions are $\tau = \operatorname{arctanh}(t/r)$ and $\rho = \sqrt{r^2 - t^2}$. Starting from Minkowski spacetime and applying inverse transformations are $t = \rho \sinh \tau$, $r = \rho \cosh \tau$, we obtain

$$ds^2 = d\rho^2 + \rho^2 h_{ab}^0 dx^a dx^b, \quad h_{ab}^0 dx^a dx^b = -d\tau^2 + \cosh^2 \tau \gamma_{AB} dx^A dx^B$$

where now h_{ab}^0 is the metric on the unit timelike hyperboloid otherwise known as Lorentzian dS_3 .

A spacetime is asymptotically flat at i^0 if there exists coordinates such that

$$ds^2 = d\rho^2 + \rho^2 (-d\tau^2 + \cosh^2 \tau \gamma_{AB} dx^A dx^B) + O(\rho^{-1}) d\rho^2 + O(\rho) dx^a dx^b.$$

It exists an analytic continuation between i^+ and i^0 .

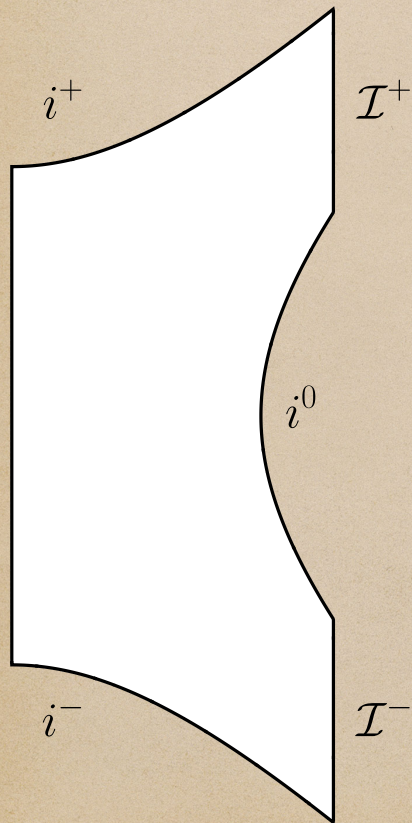
$\mathcal{I}^+$ is a singular limit of a timelike boundary

- Starting from Minkowski spacetime and applying $t = u + r$, we obtain $ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B$. The coordinates (u, θ, ϕ) span $\mathcal{I}^+$.
- At fixed large r the metric is timelike (u is the time). When $r \rightarrow \infty$, $\mathcal{I}^+$ has topology $\mathbb{R} \times S^2$ but has no metric.
- We can introduce a non-invertible metric $\gamma_{ab}dx^a dx^b = 0 du^2 + \gamma_{AB}dx^A dx^B$ of signature $(0, +, +)$ and the vector $n^a \partial_a = \partial_u$ such that $n^a \gamma_{ab} = 0$. The couple (γ_{ab}, n^a) forms a Carrollian structure.
- A spacetime is asymptotically flat at $\mathcal{I}^+$ if there exists coordinates such that

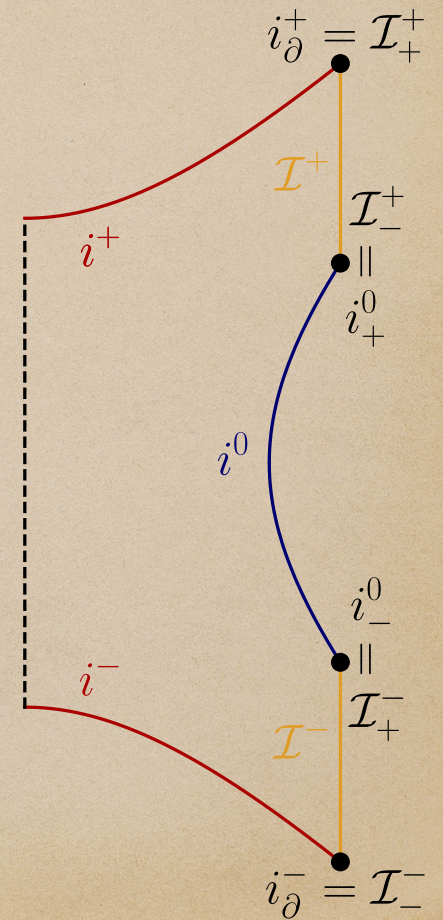
$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B + \left(\frac{2m}{r} + O(r^{-2})\right)du^2 + (rC_{AB} + O(r^0))dx^A dx^B \\ + \left(\nabla^B C_{AB} + \frac{4}{3r}(N_A + u\partial_A m - \frac{3}{32}\partial_A(C_{BC}C^{BC}))\right)dudx^A + O(r^{-2})dudr$$

- The field $C_{AB}(u, x^C)$ is the Bondi shear and its time derivative $N_{AB} \equiv \partial_u C_{AB}$ is the Bondi news.

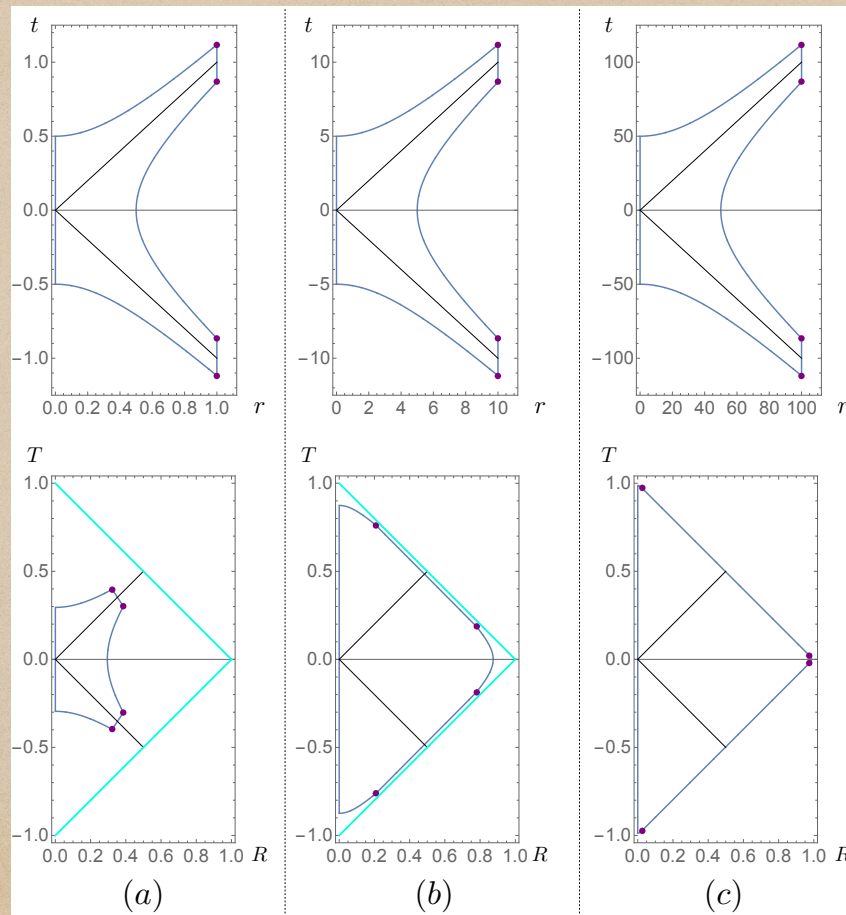
The “Puzzle piece” diagram for asymptotically flat spacetimes



- ✓ Resolves all 5 boundaries and the 4 corners.
- ✓ $\exists$ asymptotic solutions to Einstein equations
- ✓ Intuition at all boundaries
- ✓ Consistent with known infrared structure
- ✓ Embedding in AdS [see talk of Ana Raclariu]



Puzzle piece versus Penrose-Carter diagram



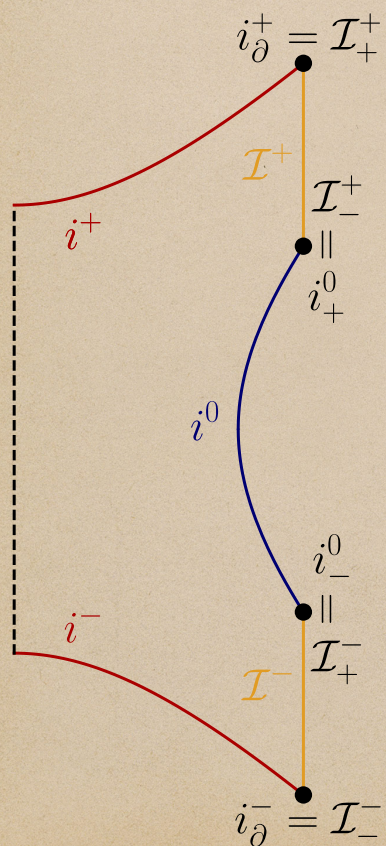
(a) $|t^2 - r^2| < 0.25, \quad r < 1$

(b) $|t^2 - r^2| < 2.5, \quad r < 10$

(c) $|t^2 - r^2| < 25, \quad r < 100$

2. BMS-covariant scattering (compatible with known infrared-physics)

Boundary conditions are required to match the four overlap regions



We prove that 4 asymptotically matched expansions at the corners

$$i_\partial^+ = \mathcal{I}_+^+, \quad \mathcal{I}_-^+ = i_+^0, \quad i_-^0 = \mathcal{I}_+^-, \quad i_\partial^- = \mathcal{I}_-^-$$

exist given the 4 (in fact 3) sets of boundary conditions at $\mathcal{I}^+$ and $\mathcal{I}^-$ as $u \rightarrow \pm \infty$:

$$m(u, x^A) = m^{(0)}(x^A) + m^{(1)}(x^A)u^{-1} + o(u^{-1})$$

$$C_{AB}(u, x^A) = C_{AB}^{(0)}(x^A) + C_{AB}^{(1)}(x^A)u^{-1} + o(u^{-1}) \quad \text{with } C_{AB}^{(0)} \text{ electric (parity even)}$$

$$N_A(u, x^A) = N_A^{\log}(x^A) \log u + N_A^{(0)}(x^A) + o(u^0)$$

These are consistent with the logarithmic correction to the subleading classical soft graviton theorem [Laddha-Sen, 2018].

Remarks on logarithmic divergences

The Laddha-Sen behaviour $N_{AB} \sim u^{-2}$ is consistent with and slightly more restrictive than Christodoulou-Klainerman falloff $N_{AB} = O(1 + |u|)^{-3/2}$.

$N_{AB} \sim u^{-2}$ and Einstein's equations imply $N_A \sim \log u$.

The logarithmic divergence prevents to define the super-Lorentz charges at spatial infinity as $\int d^2x \ Y^A N_A$.

The conjectured map $\Upsilon^* N_A|_{\mathcal{I}^+_-} = N_A|_{\mathcal{I}^+_-}$ [Hawking, Perry, Strominger, 2016] cannot be demonstrated.

However, we proved that $N_A^{(\ell=1)} \sim u^0$ as $u \rightarrow \pm \infty$ and $\Upsilon^* N_A^{(\ell=1)}|_{\mathcal{I}^+_-} = -N_A^{(\ell=1)}|_{\mathcal{I}^+_-}$. This implies that the total Lorentz charges are finite and obey a conservation law.

Frames in Special and General Relativity

- ◆ In Special Relativity, a Poincaré frame needs to be specified to set up an experiment.
- ◆ In General Relativity and in the presence of radiation, a BMS frame needs to be specified which include in addition to the Poincaré frame a **pure supertranslation frame**. Pure supertranslations are 4-dimensional spacelike transformations.
- ◆ At $\mathcal{I}^+$, the shear has two degrees of freedom and can be decomposed into electric (parity-even) and magnetic (parity-odd) parts as $C_{AB} = (-2\nabla_A\nabla_B + \gamma_{AB}\nabla^2)C + \epsilon_{C(A}\nabla_{B)}\nabla^C\Psi$. Only the $\ell \geq 2$ spherical harmonics of $C(u, \theta, \phi)$ are defined. Pure supertranslations $T(\theta, \phi)\partial_u + \dots$ act as $C \mapsto C + T(\theta, \phi)$ where $T(\theta, \phi)$ have $\ell \geq 2$ harmonics. For $u \rightarrow \pm\infty$, $C = C^{(0)}(\theta, \phi) + O(u^{-1})$.
- ◆ In General Relativity, in an asymptotic region where the Bondi news $N_{AB} = \partial_u C_{AB}$ asymptotically vanishes, one could require as a boundary condition that the electric part of the shear asymptotically vanishes, which selects a particular Poincaré subgroup of BMS. However, after the passage of radiation, the final and initial frames will differ by a supertranslation, encoding the displacement memory.

The BMS₄ algebra

Take the sphere with metric γ_{AB} , measure ϵ_{AB} and covariant derivative ∇_A .

We consider the arbitrary function $T(\theta, \phi)$ and the 2 functions $Y^A(\theta, \phi)$ such that Y^A obeys the conformal Killing equation over the sphere: $\nabla_A Y_B + \nabla_B Y_A = \gamma_{AB} \nabla_C Y^C$. There are 6 solutions to that equation.

The BMS algebra can be presented using such covariant 2-dimensional generators:

$$[T_1, T_2] = 0$$

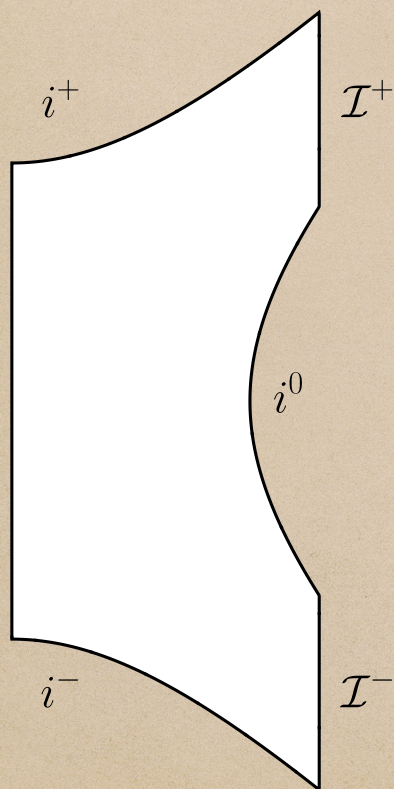
$$[Y, T] = Y(T) \quad \text{where} \quad Y(T) \equiv Y^A \partial_A T - \frac{1}{2} \nabla_A Y^A T$$

$$[Y_1, Y_2] = Y_1^A \partial_A Y_2 - Y_2^A \partial_A Y_1$$

The $\ell = 0, 1$ harmonics of T are called translations, the $\ell > 1$ harmonics are called the pure supertranslations. The generators Y^A are called the Lorentz transformations.

For each of the 5 boundaries, there is a map $2d \mapsto 3d \mapsto 4d$ that allows to reconstruct the 4 dimensional asymptotic BMS symmetries. The matching at the 4 junctions is performed as a match of 2-dimensional quantities.

BMS covariant framework



At $\mathcal{I}^+$, we have $C_{AB}^{(0)} = (-2 \nabla_A \nabla_B + \gamma_{AB} \nabla^2) C^{(0)}$ with

$$C^{(0)}|_{\mathcal{I}^+}(\theta, \phi), \quad C^{(0)}|_{\mathcal{I}^-}(\theta, \phi) = -Y^* C^{(0)}|_{\mathcal{I}^+}(\theta, \phi), \quad C^{(0)}|_{\mathcal{I}^-}(\theta, \phi)$$

(only $\ell \geq 2$ harmonics).

The antipodal map $Y : (\theta, \phi) \mapsto (\pi - \theta; \phi + \pi)$ at spatial infinity is a consequence of an evolution equation on dS_3 obtained from Einstein's equations. [Strominger, 2013]

We specify a unique BMS frame by specifying matching conditions at the four matching corners.

First subleading structure at i^+

The asymptotic metric reads as

$$ds^2 = (-1 - \frac{2\sigma}{\tau} + O(\tau^{-2}))d\tau^2 + O(\tau^{-1})d\tau dx^a + \tau^2(h_{ab} + \frac{k_{ab} - 2\sigma h_{ab}}{\tau} + O(\tau^{-2}))dx^a dx^b$$

Einstein's equations imply

$$(D^2 - 3)\sigma = \sum_{n=1}^N 4\pi M_n \frac{\delta^{(3)}(\phi - \phi_n)}{\sqrt{h}}.$$

$$\phi_n^a = (\rho_n, \theta_n, \phi_n)$$



Finite bodies are reduced to δ -function sources

σ

We impose the boundary condition: $\lim_{\rho \rightarrow \infty} \sigma = 0.$ (this fixes logarithmic translations)

Matching with $\mathcal{I}^+$ gives $\sigma = -2m^{(0)}e^{-3\rho} + o(e^{-4\rho})$. This matches the mass aspect between i^+ and $\mathcal{I}^+$.

We assume that k_{ab} is traceless and determined from a scalar Φ as

$$k_{ab} = -2(D_a D_b - h_{ab})\Phi$$

$$\Phi(\rho, \theta, \phi) = \sum_{\ell \geq 2, m} C_{\ell m}^{(0)} \psi_\ell^\mathcal{O}(\rho) Y_{\ell m}(\theta, \phi)$$

k_{ab}

Matching with $\mathcal{I}^+$ gives $k_{AB} = \frac{1}{2}e^\rho C_{AB}^{(0)} + o(e^{0\rho})$.

This matches the supertranslation frame between i^+ and $\mathcal{I}^+$.

Second subleading structure at i^+

The asymptotic metric reads as

$$ds^2 = \left(-1 - \frac{2\sigma}{\tau} - \frac{\sigma^2}{\tau^2} + o(\tau^{-2}) \right) d\tau^2 + o(\tau^{-2}) \tau d\tau d\phi^a \\ + \tau^2 \left(h_{ab} + \tau^{-1} (k_{ab} - 2\sigma h_{ab}) + \frac{\log \tau}{\tau^2} i_{ab} + \tau^{-2} j_{ab} + o(\tau^{-2}) \right) d\phi^a d\phi^b.$$

i_{ab}

Matching with $\mathcal{I}^+$ gives $i_{\rho A} = -4e^{-2\rho} N_A^{(\log)} + o(e^{-3\rho})$.

This matches the logarithmic corrections between i^+ and $\mathcal{I}^+$.

j_{ab}

Matching with $\mathcal{I}^+$ gives $j_{\rho A} = -\nabla^B C_{AB}^{(1)} + (-4N_A^{(0)} + \dots)e^{-2\rho} + o(e^{-3\rho})$.

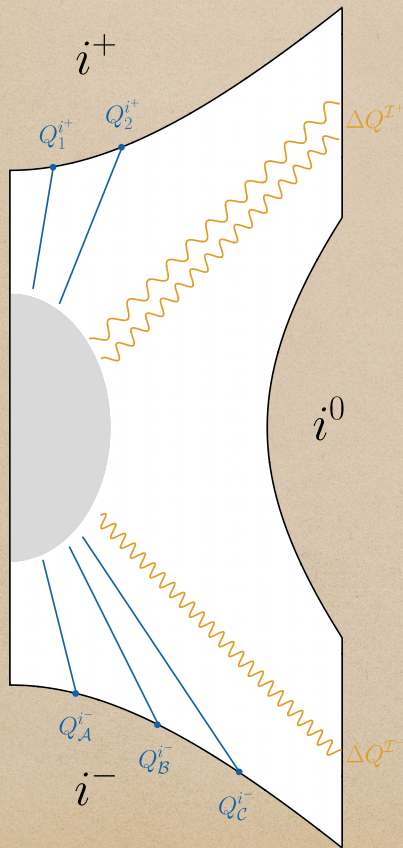
This matches the Lorentz charge aspect between i^+ and $\mathcal{I}^+$.

Lorentz charges are defined locally around each body as $Q_Y^i = -\frac{1}{8\pi} \oint_{C^i} \sqrt{q} d^2x (j_{ab} + \dots) r^a \xi_Y^b$
(the charge is finite only after a renormalization procedure)

Why is a BMS covariant framework useful?

- A. Formulate conservation laws
- B. Clarify the choice of BMS frame
- C. Define intrinsic spin of each massive body

A. Formulate conservation laws



The BMS group is the asymptotic symmetry group of all 5 infinities simultaneously.

There are therefore globally conserved BMS charges.

Let be N^- incoming bodies and N^+ outgoing bodies. The conservation laws are

$$\sum_{n=1}^{N^+} Q_n^{i^+} + \Delta Q^{\mathcal{T}^+} = Q^{i^0} = \sum_{n=1}^{N^-} Q_n^{i^-} + \Delta Q^{\mathcal{T}^-} \quad [\text{Strominger et al}].$$

for any charge associated with T and Y^A . This is a theorem under our assumptions.

Charge	name	generator
E	Energy	$T = 1$
P^i	Momentum	$T = n^i(\theta, \phi)$
$P_{\ell m}$	Supermomentum	$T = Y_{\ell m}(\theta, \phi)$
L^i	Angular Momentum	$Y^A = -\epsilon^{AB} \partial_B n^i(\theta, \phi)$
N^i	Mass Moment	$Y^A = \partial^A n^i(\theta, \phi)$

B. Clarify the choice of BMS frame

Even in a covariant theory, results are sometimes best derived in specific frames.

In special relativity:

$P^i|_{i_{\bar{0}}} = 0$ "initial center of momentum frame" (fixes boosts)

$N^i|_{i_{\bar{0}}} = P^i|_{i_{\bar{0}}} = 0$ "initial center of energy frame" (fixes boosts and spatial translations)

One could further fix time translations and rotations.

In general relativity:

$C|_{\mathcal{I}^-} = 0$ "good cut" (fixes pure supertranslations)

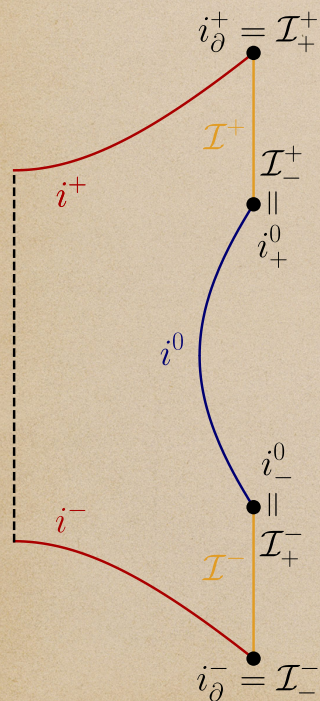
In the absence of incoming radiation, it is equivalent to $C|_{\mathcal{I}^+} = 0$ and $C|_{\mathcal{I}^\pm} = 0$

[Alternative (used in PM/NR): $\left(\frac{1}{4}\nabla^2(\nabla^2 + 2)C + m_{\ell \geq 2}\right)|_{\mathcal{I}^\pm} = 0$ "nice cut".

This is equivalent to cancelling the Moreschi supermomenta at $\mathcal{I}^\pm$.

However, $C|_{\mathcal{I}^+}(\theta, \phi) = -Y^* C|_{\mathcal{I}^-}(\theta, \phi)$ while $m|_{\mathcal{I}^+}(\theta, \phi) = +Y^* m|_{\mathcal{I}^-}(\theta, \phi)$.

Therefore the relationship $\left(\frac{1}{4}\nabla^2(\nabla^2 + 2)C + m_{\ell \geq 2}\right)|_{\mathcal{I}^\pm} = 0$ would not be simultaneously true.]



C. Define intrinsic spin of each massive body at i^+

In scattering problems, one needs to define the mass and (intrinsic) spin of individual bodies.

The mass of a body is defined as $M = \sqrt{E^2 - P^i P_i}$ which is fully BMS-invariant.

In special relativity, a body located at position x^i has (total) angular momentum $L^i = S^i + \epsilon^i_{jk} x^j P^k$ and mass moment $N^i = E x^i - P^i t$ where S^i is the spin. The formula for the spin in terms of the charges is thus

$$S^i = L^i - \frac{1}{E} \epsilon^i_{jk} N^j P^k.$$

In general relativity, we also define the spin as $S^i = L^i - \frac{1}{E} \epsilon^i_{jk} N^j P^k$. It is invariant under translations and it transforms in the expected way under rotations and boosts.

Using the representation of the BMS algebra on the conserved charges, we can prove that under a supertranslation T , the spin is invariant: $\delta_T S^i = 0$. This defines the supertranslation invariant spin in general relativity. The spin magnitude $S = \sqrt{S^i S_i}$ is BMS invariant.

3. What is the set of charges characterizing a non-radiative spacetime?

Multipolar post-Minkowskian formalism

Introduce a background Minkowski metric and define the non-linear field

$$h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$$

Simple algebra leads to

$$\partial_\mu h^{\alpha\mu} = \sqrt{-g} \square_g x^\alpha,$$

$$\square_g \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$$

Fix de Donder gauge, also known as harmonic gauge,

$$\partial_\mu h^{\alpha\mu} = 0. \quad \longleftrightarrow \quad \square_g x^\alpha = 0$$

The Einstein field equations then take the form

$$\square_{\eta} h^{\mu\nu} = \frac{16\pi G}{c^4} |g| T^{\mu\nu} + \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

$$\begin{aligned} \Lambda^{\alpha\beta} = & -h^{\mu\nu} \partial_{\mu\nu}^2 h^{\alpha\beta} + \partial_{\mu} h^{\alpha\nu} \partial_{\nu} h^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\mu\nu} \partial_{\lambda} h^{\mu\tau} \partial_{\tau} h^{\nu\lambda} \\ & - g^{\alpha\mu} g_{\nu\tau} \partial_{\lambda} h^{\beta\tau} \partial_{\mu} h^{\nu\lambda} - g^{\beta\mu} g_{\nu\tau} \partial_{\lambda} h^{\alpha\tau} \partial_{\mu} h^{\nu\lambda} + g_{\mu\nu} g^{\lambda\tau} \partial_{\lambda} h^{\alpha\mu} \partial_{\tau} h^{\beta\nu} \\ & + \frac{1}{8} (2g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu}) (2g_{\lambda\tau} g_{\epsilon\pi} - g_{\tau\epsilon} g_{\lambda\pi}) \partial_{\mu} h^{\lambda\pi} \partial_{\nu} h^{\tau\epsilon}. \end{aligned}$$

These equations are amenable to the post-Minkowskian (PM) expansion

$$h^{\mu\nu} = \sum_{n=1}^{+\infty} G^n h_n^{\mu\nu}.$$

In addition, a spherical harmonic decomposition or, equivalently, a decomposition in multipole moments is performed. This is motivated by the fact that GW emitted from compact sources mainly depend upon the lowest multipoles.

At linear order, $\square_\eta h_1^{\mu\nu} = 0$

We impose the boundary condition

no incoming radiation from $\mathcal{I}^-$

The most general solution (up to residual gauge transformations) is

$$\begin{aligned} h_1^{00} &= -4 \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \tilde{\partial}_L \left(\frac{M_L(\tilde{u})}{\tilde{r}} \right), \\ h_1^{0j} &= 4 \sum_{\ell=1}^{+\infty} \frac{(-)^\ell}{\ell!} \left[\tilde{\partial}_{L-1} \left(\frac{M_{jL-1}^{(1)}(\tilde{u})}{\tilde{r}} \right) + \frac{\ell}{\ell+1} \tilde{\partial}_{pL-1} \left(\frac{\varepsilon_{jpq} S_{qL-1}(\tilde{u})}{\tilde{r}} \right) \right], \\ h_1^{jk} &= -4 \sum_{\ell=2}^{+\infty} \frac{(-)^\ell}{\ell!} \left[\tilde{\partial}_{L-2} \left(\frac{M_{jkL-2}^{(2)}(\tilde{u})}{\tilde{r}} \right) + \frac{2\ell}{\ell+1} \tilde{\partial}_{pL-2} \left(\frac{\varepsilon_{pq(j} S_{k)qL-2}^{(1)}(\tilde{u})}{\tilde{r}} \right) \right], \end{aligned}$$

We defined: de Donder coordinates: $\tilde{x}^\mu = (\tilde{t}, \tilde{\mathbf{x}})$ or $(\tilde{t}, \tilde{r}, \tilde{\theta}^a)$.

$L = i_1 i_2 \dots i_\ell$ a multi-index made of ℓ spatial indices.

the multi-derivative operator $\partial_L = \partial_{i_1} \dots \partial_{i_\ell}$,

the product of vectors $n_L = n_{i_1} \dots n_{i_\ell}$

$x_L = x_{i_1} \dots x_{i_\ell} = r^\ell n_L$.

$M_L(u)$, $S_L(u)$ are the mass and current canonical multipole moments, respectively
They are STF (symmetric trace-free) tensors.

We can perform a coordinate transformation to Bondi gauge and read off the Bondi data in terms of canonical multipole moments

We read :

$$G^{-1}m = \sum_{\ell=0}^{+\infty} \frac{(\ell+1)(\ell+2)}{2\ell!} n_L M_L^{(\ell)} + \mathcal{O}(G),$$

(NB: a, b are now indices over S^2)

$$G^{-1}N_a = e_a^i \sum_{\ell=1}^{+\infty} \frac{(\ell+1)(\ell+2)}{2(\ell-1)!} n_{L-1} \left[M_{iL-1}^{(\ell-1)} + \frac{2\ell}{\ell+1} \varepsilon_{ipq} n_p S_{qL-1}^{(\ell-1)} \right] + \mathcal{O}(G).$$

$$G^{-1}C_{ab} = e_{\langle a}^i e_{b \rangle}^j H_{\text{TT}}^{ij} - 2D_{\langle a} D_{b \rangle} f.$$

$$H_{\text{TT}}^{ij} = 4\perp_{\text{TT}}^{ijkl} \sum_{\ell=2}^{+\infty} \frac{n_{L-2}}{\ell!} \left[M_{klL-2}^{(\ell)} - \frac{2\ell}{\ell+1} \varepsilon_{kpq} n_p S_{lqL-2}^{(\ell)} \right] + \mathcal{O}(G),$$

ℓ derivatives of the canonical multipoles

Additional data: the “ $n \geq 2$ Bondi aspects”

$$g_{ab} = r^2 \gamma_{ab} + r C_{ab} + r^{-1} \sum_{n=2}^{\infty} r^{2-n} E_{ab(n)} + O(G^2)$$

[BMS, 1962] [Tambourino, Winicour, 1966] [...] [Barnich-Troessaert, 2011]

We read :

$$E_{ab(n)} = G e_{(a}^i e_{b)}^j 4 \frac{n-1}{n+1} \sum_{\ell=n}^{\infty} \frac{1}{\ell!} a_{n\ell} n_{L-2} \left[M_{ijL-2}^{(\ell-n)} + \frac{2\ell}{\ell+1} \epsilon_{ipq} n_p S_{jqL-2}^{(\ell-n)} \right] + O(G^2), \quad a_{n\ell} = \frac{(\ell+n)!}{2^n n! (\ell-n)!}$$

Less than ℓ derivatives of the canonical multipoles

[Blanchet, G.C., Faye, Oliveri, Seraj, 2013]

Non-radiative spacetimes (linear level)

$$N_{ab} = \partial_u C_{ab} = 0$$

$$C_{ab} = e^i_{\langle a} e^j_{b \rangle} H_{\text{TT}}^{ij} - 2D_{\langle a} D_{b \rangle} f.$$

$$H_{\text{TT}}^{ij} = 4\perp_{\text{TT}}^{ijkl} \sum_{\ell=2}^{+\infty} \frac{n_{L-2}}{\ell!} \left[M_{klL-2}^{(\ell)} - \frac{2\ell}{\ell+1} \varepsilon_{kpq} n_p S_{lqL-2}^{(\ell)} \right] + \mathcal{O}(G),$$

Equivalently, in terms of canonical multipole moments : $M_L^{(\ell+1)} = S_L^{(\ell+1)} = 0$

General solution in terms of conserved charges :

$$M_L(u) = \sum_{k=0}^{\ell} M_{L,k} u^k, \quad S_L(u) = \sum_{k=0}^{\ell} S_{L,k} u^k.$$

What are those conserved charges?

Summary so far

Any asymptotically flat spacetime which

(i) is described by a post-Minkowskian expansion

(ii) has no incoming radiation

(iii) is non-radiative

is algebraically determined by a set of conserved charges $M_{L,k}$, $S_{L,k}$, $0 \leq k \leq \ell$

Conserved charges are built from “dressed Bondi data”

$$n = 0 \quad m_{ab} \equiv m\gamma_{ab} + \frac{1}{2}D_{[a}D^c C_{b]c} = m\gamma_{ab} + m^-\epsilon_{ab}, \quad m^- \equiv \frac{1}{4}D_c D_d \tilde{C}^{cd}, \quad [\text{Godazgar, Godazgar, Pope, 2018}]$$

$$n = 1 \quad \mathcal{N}_a \equiv N_a - \frac{1}{4}C_{ab}D_c C^{bc} - \frac{1}{16}\partial_a(C_{bc}C^{bc}) - uD^b m_{ab}; \quad [\text{Hawking, Perry, Strominger, 2016}]$$

[Compère, Oliveri, Seraj, 2019]

$$n = 2 \quad \mathcal{E}_{ab} = E_{ab} - \frac{u}{2}C_{(a}^c m_{b)c} - \frac{u}{3}D_{\langle a}\mathcal{N}_{b\rangle} - \frac{u^2}{6}D_{\langle a}D^c m_{b\rangle c}, \quad [\text{Freidel, Pranzetti, 2021}]$$

$$n = 3 \quad \mathcal{E}_{ab} = E_{ab} - u \left\{ \mathcal{D}_0 E_{ab} + D^c \left[\left(\frac{1}{4}D_e C^{de} C_{d\langle a} - \frac{3}{32}D_{\langle a}C^2 + \frac{5}{32}C^2 D_{\langle a} - \frac{1}{3}N_{\langle a} \right) C_{b\rangle c} \right] \right\}$$

$$+ \frac{u^2}{2} \left[-\frac{1}{3}D^c (D^d m_{d\langle a} C_{b\rangle c}) + \frac{1}{2}\mathcal{D}_0 (m_{ac} C_b^c) + \frac{1}{3}\mathcal{D}_0 D_{\langle a} N_{b\rangle} \right] - \frac{u^3}{18}\mathcal{D}_0 D_{\langle a} D^c m_{b\rangle c}. \quad [\text{Grant, Nichols, 2022}]$$

[Blanchet, Compère, Faye, Oliveri, Seraj, 2022]

$$n \geq 4 \quad \dots \quad [\text{Grant, Nichols, 2022}]$$

The dressed quantities are conserved in the absence of news.

Uniqueness?

All local flux-balance laws at $\mathcal{I}^+$

$$n = 0 : \quad \frac{1}{4} D_b D_c N^{bc} = -\mathcal{F}(u) + \partial_u m,$$

$$n = 1 : \quad -\frac{u}{2} D_c D_{\langle a} D_{b \rangle} N^{bc} = -\mathcal{F}_a(u) + \partial_u \mathcal{N}_a,$$

$$n = 2 : \quad \frac{u^2}{12} \text{STF}_{ab} [D_a D_c D_{\langle b} D_{d \rangle} N^{cd}] = -\mathcal{F}_{ab}^{(2)}(u) + \partial_u \mathcal{E}_{ab}^{(2)},$$

$$n \geq 3 : \quad \frac{(-u)^n}{6 n!} \mathcal{D}_{n-3} \cdots \mathcal{D}_0 \text{STF}_{ab} [D_a D_c D_{\langle b} D_{d \rangle} N^{cd}] = -\mathcal{F}_{ab}^{(n)}(u) + \partial_u \mathcal{E}_{ab}^{(n)}.$$

$$\mathcal{F} \equiv -\frac{1}{8} N_{ab} N^{ab}$$

$$\mathcal{D}_n \equiv -\frac{n+2}{2(n+1)(n+4)} (D^a D_a + n^2 + 5n + 2).$$

[Grant, Nichols, 2021] [Freidel, Pranzetti, Raclariu, 2021]

Charges of non-radiative spacetimes

BMS Supermomenta and
BMS dual supermomenta

$$\mathcal{P}_L = \oint_S m \hat{n}_L$$

$$\mathcal{P}_L^- = \oint_S m^- \hat{n}_L = \frac{1}{2} \oint_S m_{ab} \epsilon^{ab} \hat{n}_L$$

M_L, S_L

Generalized
BMS

Super-Lorentz
charges

$$-\mathcal{J}_L = \frac{1}{2} \oint_S \epsilon^{ab} \partial_b \hat{n}_L \mathcal{N}_a$$

$$\mathcal{K}_L = \frac{1}{2} \oint_S \partial^a \hat{n}_L \mathcal{N}_a$$

$$M_L = m_{L,\ell} u^\ell + m_{L,\ell-1} u^{\ell-1} + \cdots + m_{L,1} u^1 + m_{L,0} u^0$$

$$n \geq 2$$

Bondi charges

$$\mathcal{Q}_{n,L}^+(u) \equiv \oint_S \mathcal{E}_{(n)}^{ab} D_a D_b \hat{n}_L,$$

Non-stationary
moments

Stationary
Geroch-Hansen
moments

[G.C., Oliveri, Seraj, 2022]

Charges of non-radiative spacetimes

BMS Supermomenta and
BMS dual supermomenta

$$\mathcal{P}_L = \oint_S m \hat{n}_L$$

$$\mathcal{P}_L^- = \oint_S m^- \hat{n}_L = \frac{1}{2} \oint_S m_{ab} \epsilon^{ab} \hat{n}_L$$

Generalized

BMS

Super-Lorentz
charges

$$-\mathcal{J}_L = \frac{1}{2} \oint_S \epsilon^{ab} \partial_b \hat{n}_L \mathcal{N}_a$$

$$\mathcal{K}_L = \frac{1}{2} \oint_S \partial^a \hat{n}_L \mathcal{N}_a$$

$$n \geq 2$$

Bondi charges

$$\mathcal{Q}_{n,L}^+(u) \equiv \oint_S \mathcal{E}_{(n)}^{ab} D_a D_b \hat{n}_L,$$

Non-stationary
moments

Stationary
Geroch-Hansen
moments

M_L, S_L

$$M_L = m_{L,\ell} u^\ell + m_{L,\ell-1} u^{\ell-1} + \cdots + m_{L,1} u^1 + m_{L,0} u^0$$

These charges can be expressed in terms of the real part of the $Lw_{1+\infty}$ basis

[G.C., Oliveri, Seraj, 2022]

[Freidel, Raclariu, Pranzetti, 2021]

$$\mathcal{Q}_{n,L}^\pm = \text{Re}(Q_n(\tau_L^\pm))$$

$$Q_s(\tau) \equiv \oint_S \tau_{-s}(x^a) \hat{q}_s(u, x^a), \quad \hat{q}_s(u, x^a) \equiv \sum_{n=0}^s \frac{(-u)^{s-n}}{(s-n)!} \partial^{s-n} q_n.$$

Remark: While the $Q_n(\tau_L^\pm)$ obey the $Lw_{1+\infty}$ algebra, the $\mathcal{Q}_{n,L}^\pm$ do not.

[Geiller, 2024]

4. Memory effects and Λ -BMS in dS_4 . Why dS_4 is not described by a CFT_3 .

The asymptotic structure of $\mathcal{I}^+$ for $\Lambda > 0$ in a nutshell

$$\begin{array}{c} \mathcal{I}^+ \\ \hline \tau = 0 \\ \uparrow \tau \end{array}$$

$$H = \sqrt{\frac{\Lambda}{3}}$$

Starobinsky / Fefferman-Graham gauge :

$$ds^2 = -d\tau^2 + \tau^2(g_{ab}^{(0)}(x^c) + \dots + \tau^{-3}T_{ab}(x^c) + \dots)dx^a dx^b \quad T_a^a = 0, \quad D_{(0)}^a T_{ab} = 0$$

The residual gauge transformations consist in 4 functions of x^a ("integration constants" after gauge fixing)

We can further gauge fix the boundary metric :

$$g_{ab}^{(0)}dx^a dx^b = H^2 du + q_{AB}(u, x^C)dx^A dx^B \quad \det(q_{AB}) = \det(\dot{q}_{AB})$$

The residual gauge transformations are spanned by 3 functions of $x^A = (\theta, \phi)$.

They form the Λ -BMS algebroid whose structure constants depend upon the phase space field q_{AB} .

In the presence of radiation, an observer located close to $\mathcal{I}^+$ cannot gauge fix the diffeomorphism group any further. The Λ -BMS symmetries reflect the freedom at setting up a detector at $\mathcal{I}^+$ in asymptotically de Sitter. (Same results in Bondi gauge)

The asymptotic structure of $\mathcal{I}^+$ for $\Lambda > 0$ in a nutshell

“2d” presentation of the Λ -BMS generators :

$$\begin{aligned}\xi^u &= U(u, x^A) \\ \xi^A &= Y^A(u, x^A) + O(r^{-1})\end{aligned}$$

$$\begin{aligned}\partial_u U &= -\frac{1}{2}D_A Y^A \\ \partial_u Y^A &= -H^2 q^{AB} \partial_B U\end{aligned}$$

Algebroid :

$$[(U, Y^A), (U', Y'^A)] = (U'', Y''^A)$$

$$\begin{aligned}U'' &= Y^A \partial_A U' + \frac{1}{2} U D_A Y'^A - ((\leftrightarrow)) \\ Y''^A &= Y^B \partial_B Y'^A - H^2 U q^{AB} \partial_B U' - ((\leftrightarrow))\end{aligned}$$

In the flat limit, the algebra reduces to the generalized BMS algebra $\text{diff}(S^2) + \text{vect}(S^2)$

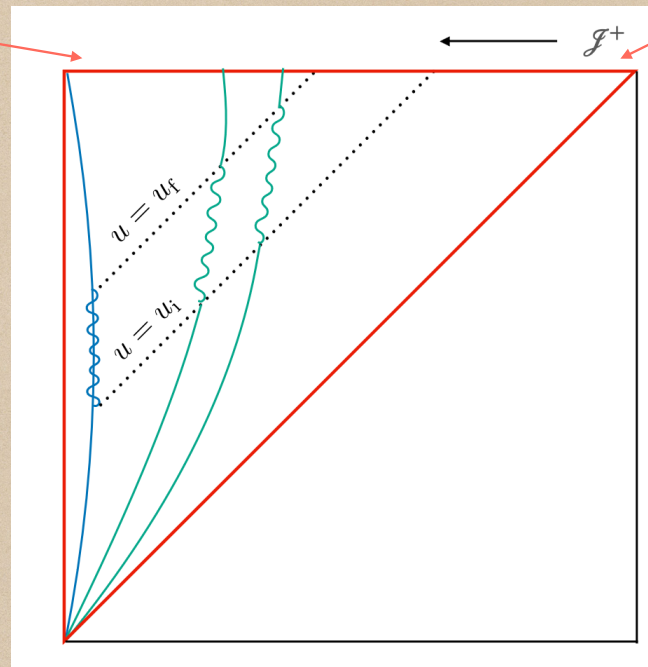
[GC, Fiorucci, Ruzziconi, 2019]
[Barnich, Troessaert, 2010]
[Campiglia, Laddha, 2015]

When $q_{AB}(u, x^A) = \dot{q}_{AB}(x^A)$, the Λ -BMS algebroid becomes the Λ -BMS algebra that contains the $SO(4,1)$ algebra of exact symmetries of de Sitter.

What is the structure at $\mathcal{I}^+$ generated by a localized event?

No radiation at $\mathcal{I}_+^+$

No radiation at $\mathcal{I}_-^+$



Note: The topology at $\mathcal{I}^+$ is S^3 minus 2 points : $\mathbb{R} \times S^2$.

What is the metric $q_{AB}(u, x^A)$ resulting of a localized event below the Hubble scale?
Is there a Λ -BMS group transition after the passage of the gravitational wave strain?

Linear spin 2 field on de Sitter

Linear equations of motion
(in generalized harmonic gauge)

$$\begin{aligned}\square\left(\frac{\hat{\chi}}{\eta}\right) &= -\frac{16\pi G\hat{T}}{\eta}, \\ \square\left(\frac{\chi_{0i}}{\eta}\right) &= -\frac{16\pi GT_{0i}}{\eta}, \\ \left(\square + \frac{2}{\eta^2}\right)\left(\frac{\chi_{ij}}{\eta}\right) &= -\frac{16\pi G}{\eta}T_{ij},\end{aligned}$$

$$\hat{T} := T_{00} + T^i_i.$$

$$\square = -\partial_\eta^2 + \partial_i^2$$

Scalar and vectors modes are **similar** to flat space.

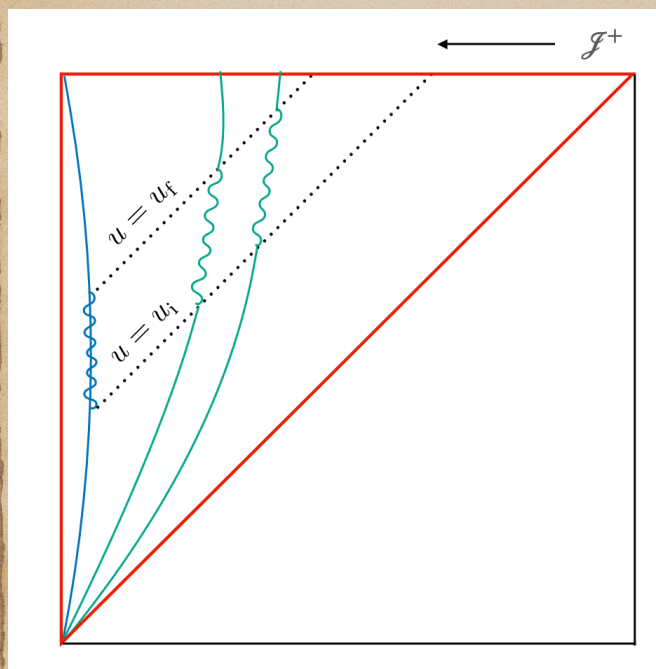
Tensor mode depends upon the de Sitter potential. There is propagation **inside the lightcone**.

We solve the equations in the quadrupolar truncation:

$$\int d^3x a^{\ell+1} T_{\mu\nu} x^L = 0, \quad \forall \ell > 2.$$

(Distinct from [Ashtekar, Bonga, Kesavan, 2015]
[Bunster, Perez, Bonga, 2023])

Result of the linear analysis up to quadrupolar order



We define the even parity and odd parity quadrupolar moments of the stress-energy tensor as

$$Q_{ij}^{(\rho+p)}(\eta) \equiv \int d^3x a^3(\eta) (T_{00} + T_{kk}) x_i x_j \quad K_{ij}(\eta) \equiv \frac{4}{3} \int d^3x a^3(\eta) \epsilon_{kl(i} T_{j)k} x_l$$

We assume for simplicity staticity in addition to non-radiative boundary conditions at early and late times.

The boundary metric at $\mathcal{I}^+$ of the linear perturbation is given by

$$g_{ab}^{(0)} dx^a dx^b = H^2 du^2 + q_{AB} dx^A dx^B$$

$$q_{AB} = \dot{q}_{AB} + 2\dot{q}_{C\langle A} \dot{D}_{B\rangle} \dot{\xi}^C + e_{\langle A}^i e_{B\rangle}^j \left(\partial_u \zeta_{ij} + 2H^2 \partial_u Q_{ij}^{(\rho+p)} + 2H^2 \epsilon_{ikl} n_k (K_{jl} + H \int^u du' K_{jl}(u')) \right),$$

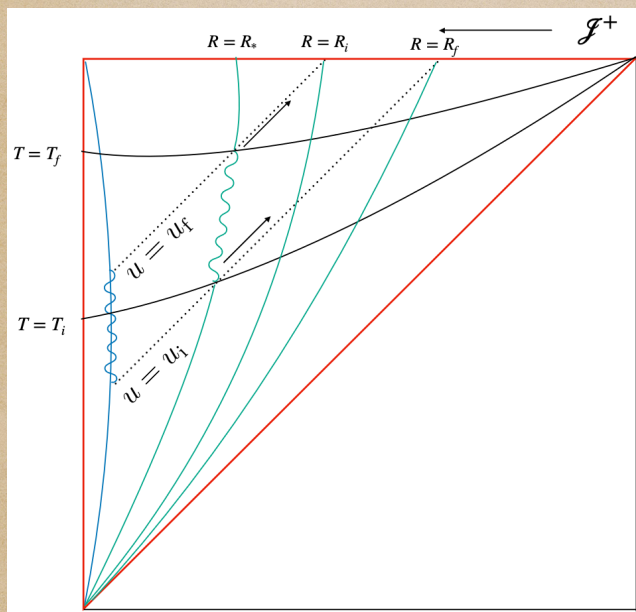
where $\partial_u^2 \zeta_{ij} - 3H^2 \zeta_{ij} = -2H^4 Q_{ij}^{(\rho+p)}$

(Here, the multipoles are evaluated at $\mathcal{I}^+ : \eta = -H^{-1}e^{-Hu}$.)

[G.C., Hoque, Kutluk, 23]

(See also [Ashtekar, Bonga, Kesavan, 2015]
[Bunster, Perez, Bonga, 2023])

Cosmological displacement memory effect



In the even sector, a fixed quadrupole can be re-absorbed at either $u = u_f$ or $u = u_i$ as a $\Lambda - BMS$ transformation.

In the even sector, the finite difference of the quadrupole between $u = u_f$ or $u = u_i$ leads to a finite displacement memory, which is gauge invariant.

In the odd sector, a fixed quadrupole cannot be absorbed into a residual gauge transformation. There also again a finite displacement memory.

Contrary to the flat case, the displacement memory is at leading order. In a sense it also arises from a flux-balance law, $\partial_u q_{AB} = H C_{AB}$, which becomes trivial in the flat limit. There are also subleading memory effects (which match the flat case).

Similarly to the flat case, there is a distinction between even and odd sectors with respect to memory.

Consequences for holography

Localized sources in dS_4 lead to a non-trivial boundary metric.

Dirichlet boundary conditions are therefore generically violated. Even in the absence of radiation, in the presence of a static odd quadrupole.

(Sending advanced signals from the past cosmological horizon would induce observable non-linear interferences)

This breaks the conformal asymptotic symmetry group.

Therefore, dynamical gravity in dS_4 cannot be modelled by a CFT_3 .

Summary

1. The “Puzzle piece” diagram allows to match all 5 boundaries of Minkowski
2. A BMS-covariant formalism for scattering can be formulated. It lead to a new definition of intrinsic spin at i^+ .
3. A transition from an initial non-radiative region to final non-radiative region in asymptotically flat spacetimes can be characterized by a transition of the (real) celestial charges.
4. dS_4 admits memory effects (even sector: Λ -BMS transitions). The conformal group is broken.