

Driving up superconducting T_c

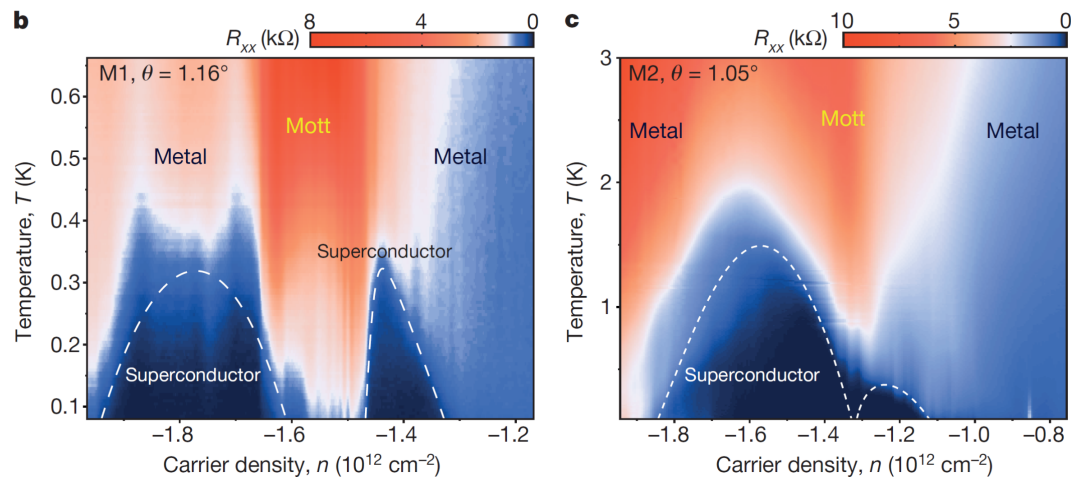
Ivar Martin

- I. Martin, arxiv (Eliashberg 90 contribution, arXiv:1911.05097)
- F. Wu, A. H. MacDonald and I. Martin, Phys. Rev. Lett. 121, 257001 (2018).
- I. Martin, D. Podolsky, S. Kivelson, PRB 72, 060502 (2005).
- M. Knap, M. Babadi, G. Refael, I. Martin, E. Demler, PRB 94, 214504 (2016)
- H. Kim, H. Dehghani, H. Aoki, I. Martin, M. Hafezi, arXiv:1912.13059



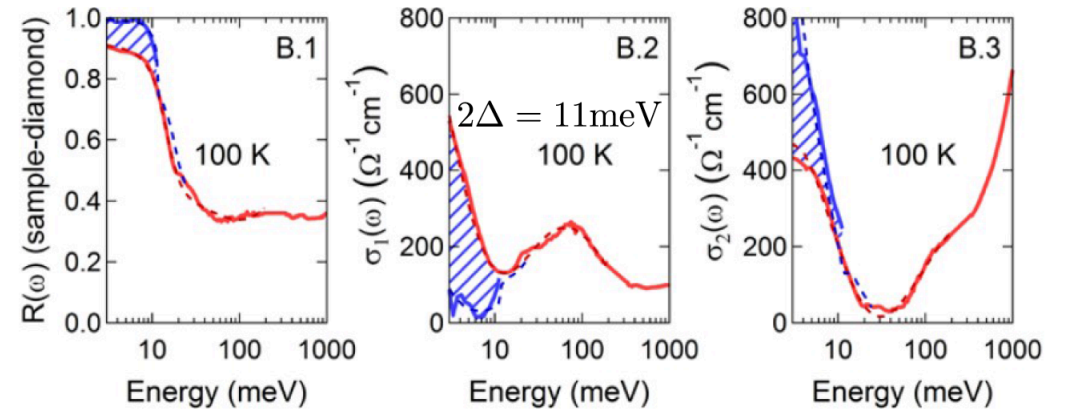
Inspirations

Superconductivity in twisted bilayer graphene



Y. Cao, *et al.*, Nature 556, 80 (2018)

Superconductivity in pumped fullerides



M. Mitrano, *et al.*, Nature (2015)

Outline

- Superconductivity
 - Holstein model → effective electronic model; BCS
 - Understanding McMillan
 - Wide vs narrow band superconductivity
- Spatial inhomogeneity
 - Why enhancement of T_c is generic
 - How it can help against Coulomb repulsion
- Temporal inhomogeneity
 - Modulations that works
 - Modulation that does not work
- Combining spatial and temporal modulations



Superconducting drosophila

Holstein Model

$$H = H_e + H_{ph} + \alpha \sum_{j,\sigma} n_{j\sigma} x_j,$$

$$H_{ph} = \sum_j \omega_0 (b_j^\dagger b_j + 1/2),$$

$$H_{e-ph} = \frac{\alpha \ell_0}{\sqrt{2}} \sum_{j,\sigma} n_{j\sigma} (b_j^\dagger + b_j).$$

$$1 = U \int_0^{\min(\omega_0, E_F)} d\epsilon_p \frac{N(\epsilon_p)}{\sqrt{\epsilon^2 + \Delta^2}} \left[1 - 2n_F(\sqrt{\epsilon^2 + \Delta^2}) \right]$$

$$\text{BCS: } T_c = \min(\omega_0, E_F) \times e^{-1/\lambda}$$

$$\lambda = \nu_{\epsilon_F} U$$

Effective Hamiltonian, weak coupling

$$\mathcal{U}(t) = e^{-iHt} = e^{-iH_0 t} \underbrace{\mathcal{T} e^{-i \int_0^t dt' \hat{H}_{e-ph}(t')}}_{\text{Dyson series}}$$

$$1 - i \int_0^t dt' \hat{H}_{e-ph}(t') - \int_0^t dt' \int_0^{t'} dt'' \hat{H}_{e-ph}(t') \hat{H}_{e-ph}(t'').$$

$$\mathcal{U}_{eff}(t) = \langle 0_{ph} | \mathcal{U}(t) | 0_{ph} \rangle \equiv e^{-iH_{eff}t}$$

$$\int_{-\infty}^0 dt \langle 0_{ph} | b^\dagger(0) b(t) + b(0) b^\dagger(t) | 0_{ph} \rangle \approx \frac{i\omega_0}{\omega_0} = i$$

$$H_{eff} = H_e - \frac{\ell_0^2 \alpha^2}{2\omega_0} \sum_j n_j^2 = H_e - U \sum_j n_j^2 \quad (@ \omega < \omega_0)$$

Beyond: I. Esterlis, et al, [Phys. Rev. B 97, 140501 \(2018\)](#).

Beyond BCS

- Use Migdal-Eliashberg equations:

$$T_c \approx \Theta \exp \left[-\frac{1 + \lambda}{\lambda - \mu^*(1 + 0.6\lambda)} \right].$$

McMillan formula

Dissecting McMillan



$$\begin{aligned}\Sigma_{el}(q) &= \alpha^2 \ell_0^2 \int_p D(p) G(p+q) \\ &= \alpha^2 \ell_0^2 \nu_{\epsilon_F} \log \frac{\omega_0 - \omega}{\omega_0 + \omega} \\ &\approx \lambda \omega\end{aligned}$$

$$T_c \approx \Theta \exp \left[- \frac{1 + \lambda}{\lambda - \mu^* (1 + 0.6\lambda)} \right].$$



$$\Sigma_{ph}(q) = \alpha^2 \ell_0^2 \int_p G(p+q) G(p) = \alpha^2 \ell_0^2 \Pi(q).$$

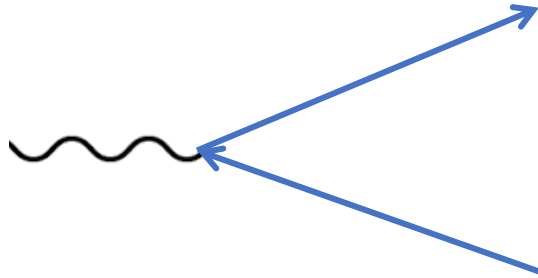
$$\underline{\epsilon_F > \omega_0} \quad \Pi(q) \approx \nu_{\epsilon_F}$$

$$\omega_0 \rightarrow \omega_0(1 - \lambda).$$

$$\underline{\epsilon_F < \omega_0} \quad \Pi(q) \approx \nu_{\epsilon_F} \times (\epsilon_F / \omega_0)$$

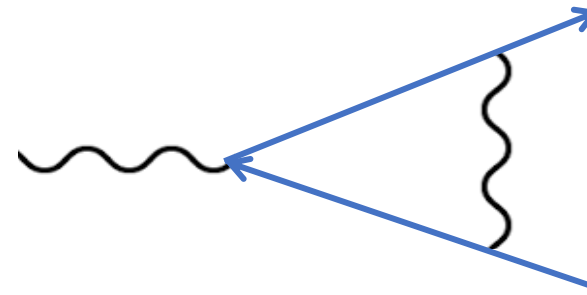
$$\omega_0 \rightarrow \omega_0(1 - \frac{\epsilon_F}{\omega_0} \lambda).$$

Migdal criterion



Bare vertex

$$\Gamma_0 = 1.$$



First vertex correction

$$\Gamma_1(q) = \alpha^2 \ell_0^2 \int_{p'} D(p - p') G(p' + q) G(p')$$

$$G = 1/(i\omega - \epsilon_{\vec{k}}) \text{ and } D = 2\omega_0/(\omega^2 + \omega_0^2)$$

$$\underline{\epsilon_F > \omega_0} \quad D \sim \epsilon_F^{-1} \quad \Gamma_1 \sim \lambda(\omega_0/\epsilon_F)$$

$$\underline{\epsilon_F < \omega_0} \quad D \sim \omega_0^{-1} \quad \Gamma_1 \sim \lambda$$

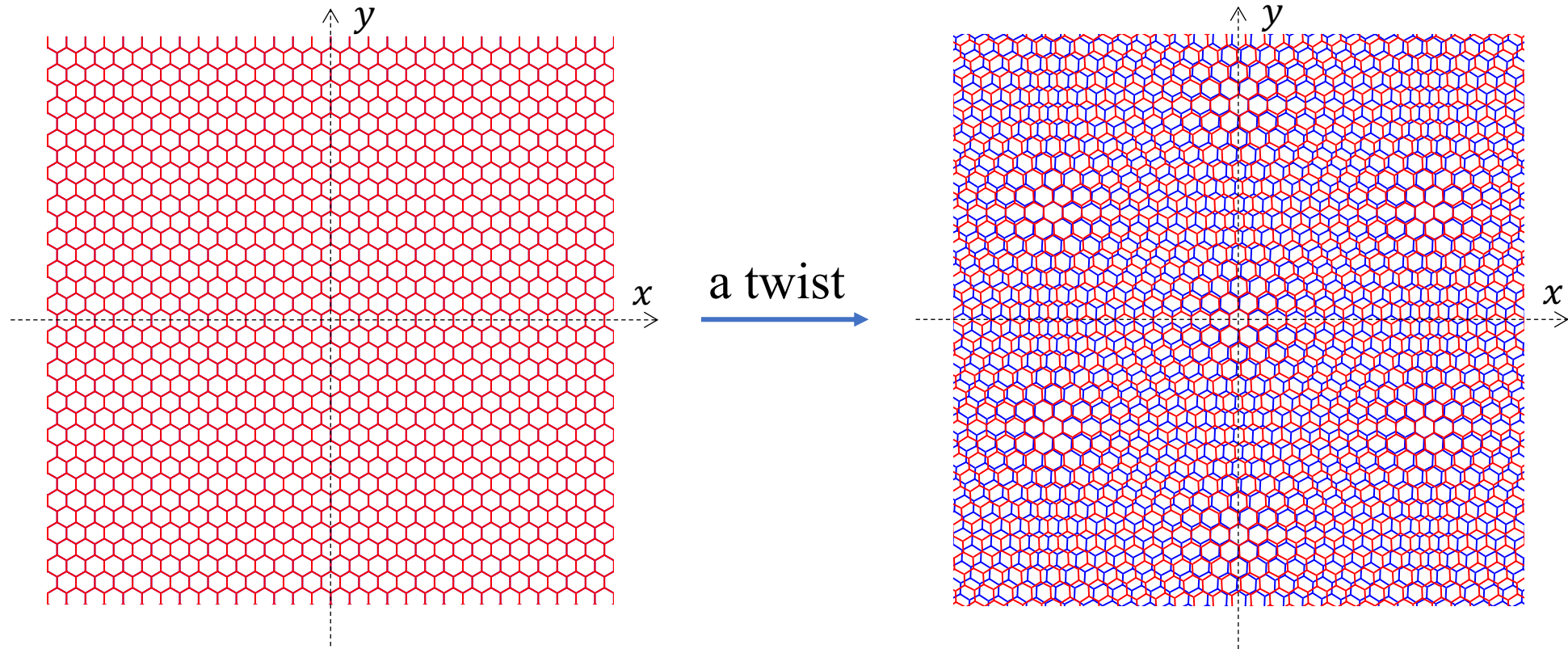
Generalized BCS (ok up to intermediate coupling)

$$T_c \sim \min(\epsilon_F, \omega_0) e^{-1/\lambda}$$

Outline

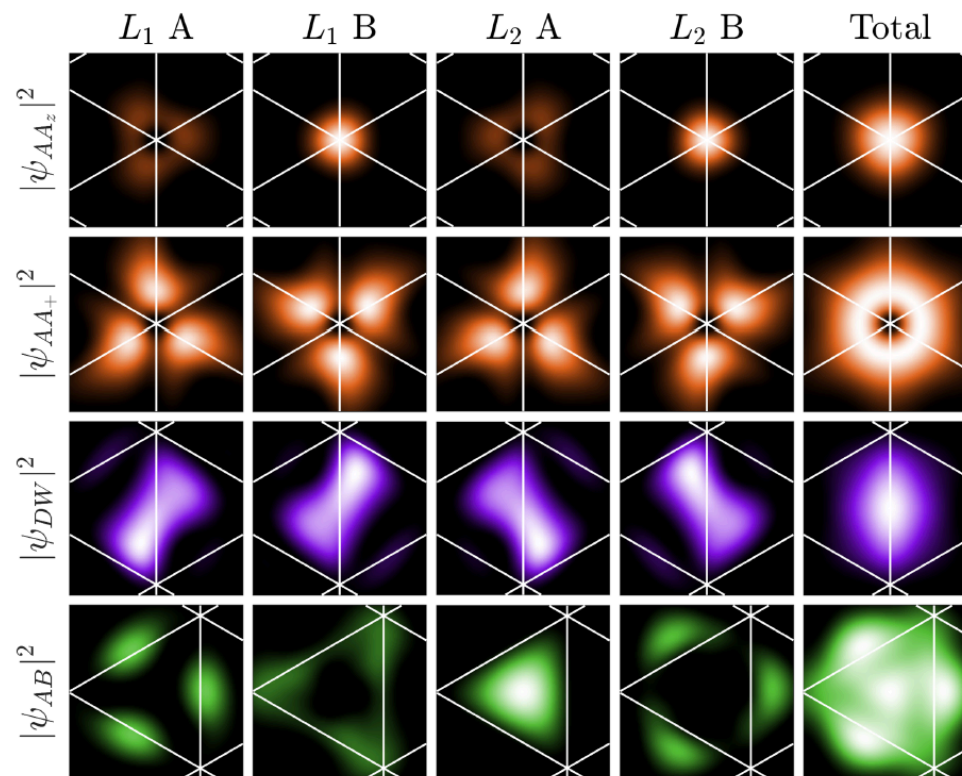
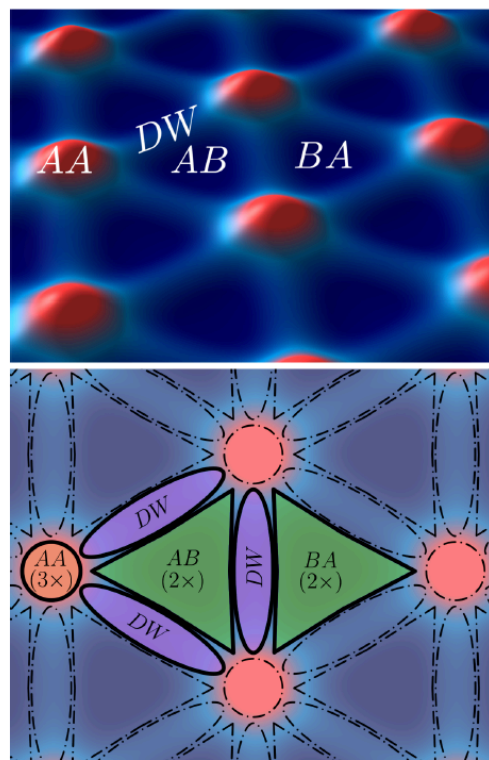
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Twisted bilayer graphene



(Band) structure near magic angle

[Carr, Fang, Zhu, Kaxiras, [arXiv:1901.03420v1](#)]



$$\theta = 0.9^\circ$$

$$\theta \gtrsim 1^\circ$$

No lattice relaxation ~ BM 2011

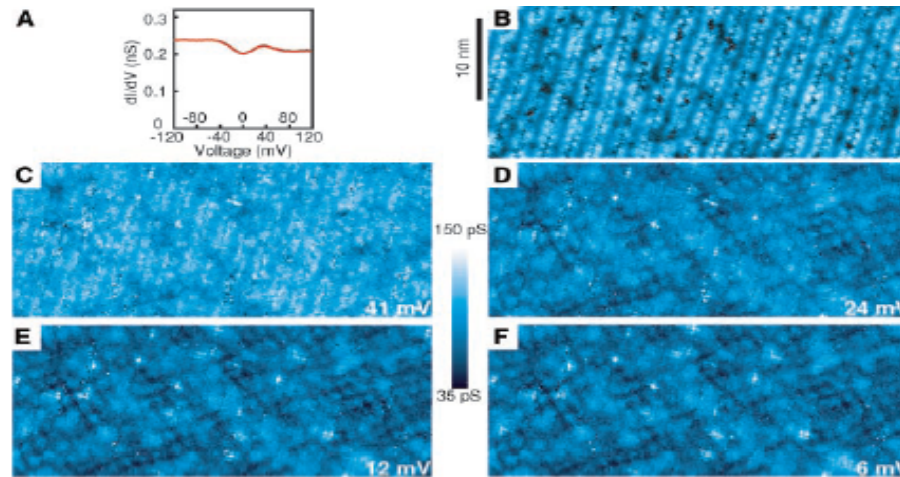
$$\theta \lesssim 1^\circ$$

Atoms relax to form

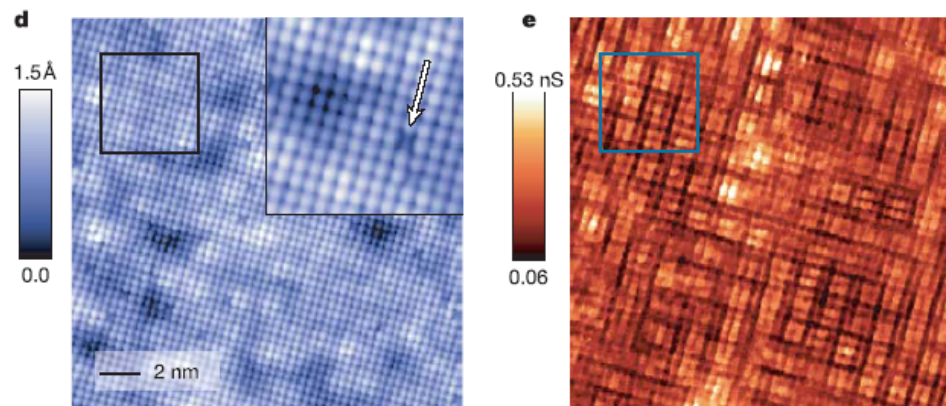
AA, **AB/BA**, and **DW** regions

- AA size stays indep of θ
- Domain walls carry low-E states)
- Smaller magic angles gone!

Looks familiar...

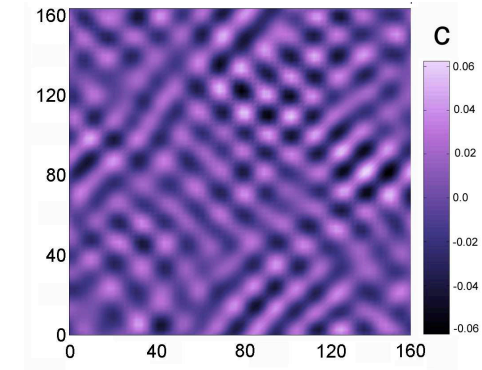


Vershinin et al, Science 2004

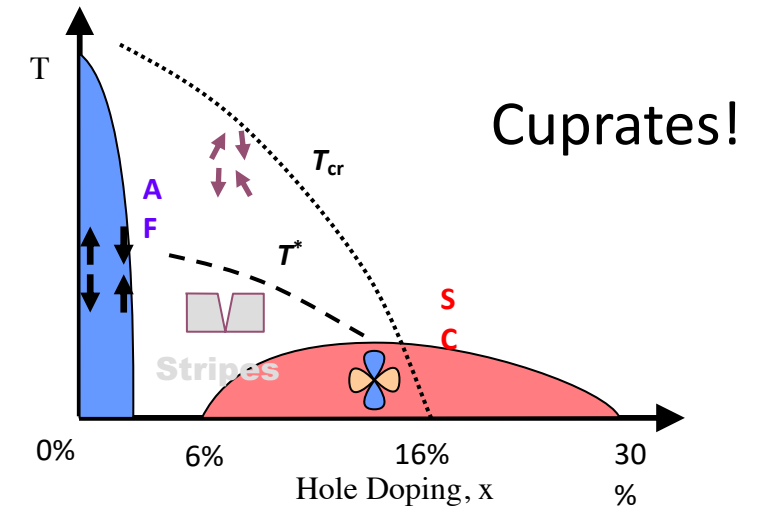


Hanaguri et al, Nature 2004

Flatiron CCQ: Jan 2020



Howald et al PNAS (2003)



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How can spatial modulation help?

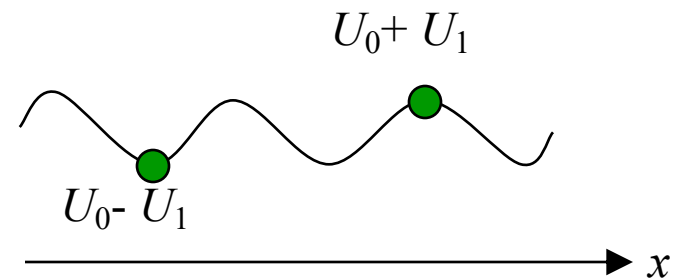
Inhomogeneous **negative** U Hubbard model:

$$\mathcal{H} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_i U(x_i) n_\uparrow(x_i) n_\downarrow(x_i)$$

For constant $U \rightarrow$ s-wave superconductivity.

Consider effects of **weak inhomogeneity**

$$U(\mathbf{x}) = U_0 + U_1 \cos \mathbf{Q} \cdot \mathbf{x}$$



Condensation at non-zero momentum

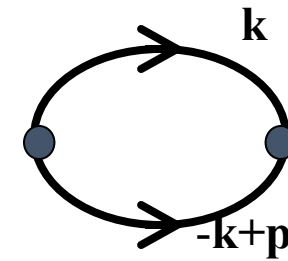
Hierarchy of condensates:

where... $\Delta_0, \Delta_{\pm Q}, \Delta_{\pm 2Q} \dots$

$$\Delta_{\mathbf{q}} = \sum_{\mathbf{k} \mathbf{p}} U(\mathbf{k}) \langle c_{\mathbf{q}/2 - \mathbf{k}/2 + \mathbf{p} \uparrow} c_{\mathbf{q}/2 - \mathbf{k}/2 - \mathbf{p} \downarrow} \rangle$$

BCS **self-consistency** for T_c

$$\Delta_{\mathbf{q}} = \int \frac{d^d p}{(2\pi)^d} U(\mathbf{q} - \mathbf{p}) K(\mathbf{p}) \Delta_{\mathbf{p}},$$



Kernel:
$$K(\mathbf{p}) \approx N_f \ln \left[\frac{2\gamma\omega_D}{\pi\sqrt{T^2 + (v_f p)^2}} \right] \Theta(\omega_D - |v_f p|),$$

$$\begin{pmatrix} \vdots \\ \Delta_{-2Q} \\ \Delta_{-Q} \\ \Delta_0 \\ \Delta_Q \\ \Delta_{2Q} \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & & & & & & \\ & U_0 K(-2Q) & U_1 K(-Q) & 0 & 0 & 0 & \\ & U_1 K(-2Q) & U_0 K(-Q) & U_1 K(0) & 0 & 0 & \\ & 0 & U_1 K(-Q) & U_0 K(0) & U_1 K(Q) & 0 & \\ & 0 & 0 & U_1 K(0) & U_0 K(Q) & U_1 K(2Q) & \\ & 0 & 0 & 0 & U_1 K(Q) & U_0 K(2Q) & \\ & & & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \Delta_{-2Q} \\ \Delta_{-Q} \\ \Delta_0 \\ \Delta_Q \\ \Delta_{2Q} \\ \vdots \end{pmatrix}$$

BCS equation as **particle hopping on a lattice**:
 “Energy” can always be lowered by delocalizing
 wave function $\rightarrow$ **higher T_c**

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Enhanced local Interaction – Holstein supersized

- Modified Interaction

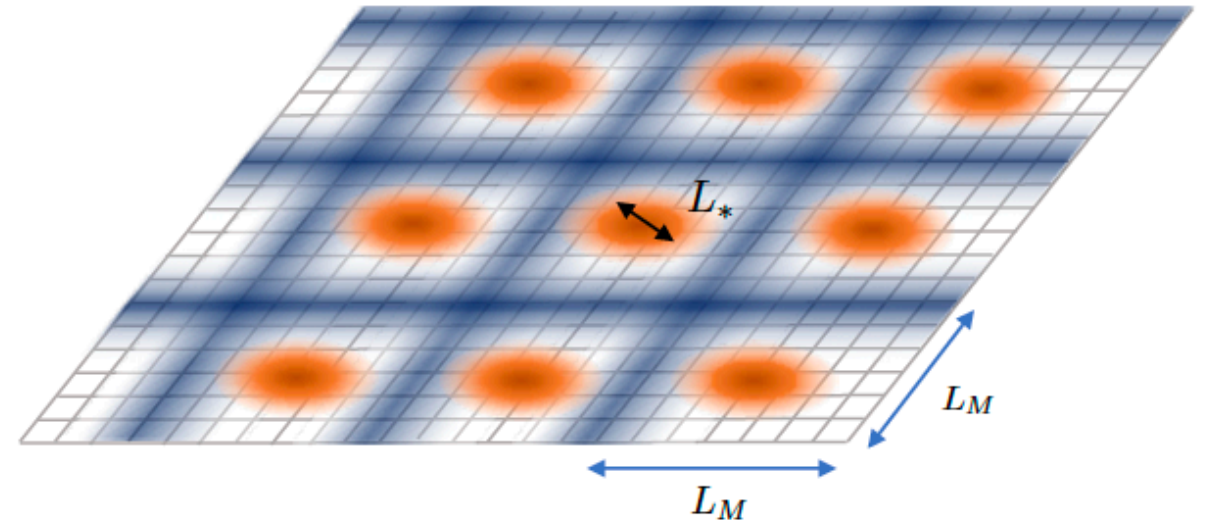
$$\frac{\alpha^2}{2k} \sum_{j \in J} n_j^2 \approx \frac{\alpha^2}{2k N_*} n_J^2$$

- Rescaled DOS

$$\nu_{\epsilon_F}^M \sim \nu_{\epsilon_F} \bar{N}_M$$

- SC coupling enhancement

$$\lambda \rightarrow \lambda \times (N_M / N_*)$$



$$N_M = L_M^2$$

$$N_* = L_*^2$$

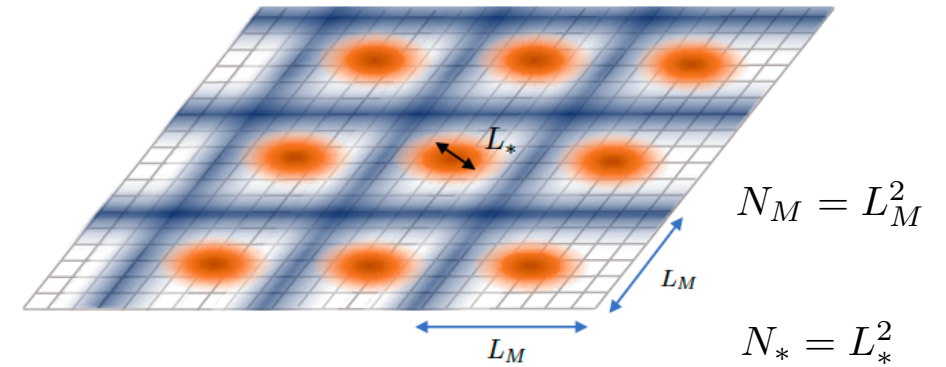
General range interactions in Moire

$$H_{int} = \sum_{r,r'} v(r - r') n_r n_{r'}$$

$$\psi_{\sigma}^{\dagger}(r) = \sum_{q,m} e^{iqr} u_{q,m}(r) c_{q,m}^{\dagger}$$

$$|u_{k=0,m}(r)|^2 \equiv \rho^m(r)$$

$$\rho(r) = \frac{1}{N} \sum_G e^{iGr} \rho_G, \quad G = mG_1 + nG_2 \quad \text{Reciprocal moire lattice vectors}$$



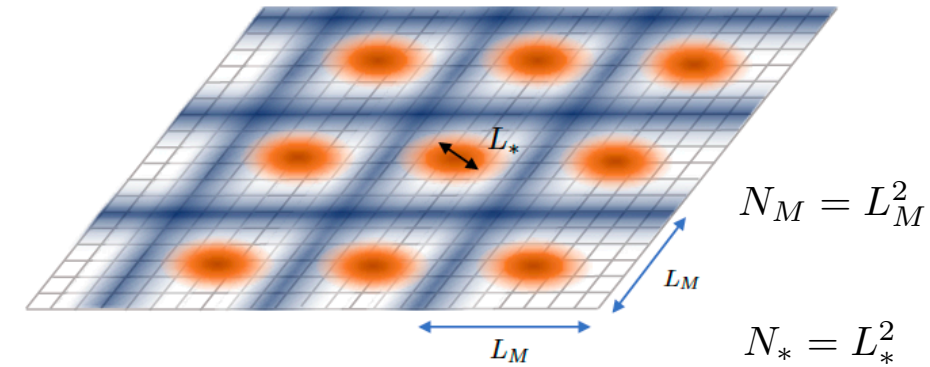
For well localized bands (deep states)

$$H_{Cooper} = \frac{1}{N} \sum_{k,p,G,\sigma,\sigma'} \tilde{v}_{k-p+G} c_{k\sigma}^{\dagger} c_{-k\sigma'}^{\dagger} c_{-p\sigma'} c_{p\sigma} \rho_G^2$$

Short vs long range interaction

$$H_{Cooper} = \frac{1}{N} \sum_{k,p,G,\sigma,\sigma'} \tilde{v}_{k-p+G} c_{k\sigma}^\dagger c_{-k\sigma'}^\dagger c_{-p\sigma'} c_{p\sigma} \rho_G^2$$

$$= \frac{1}{N} (\sum_G \rho_G^2) \sum_{k,p,\sigma,\sigma'} \tilde{v}_{k-p} c_{k\sigma}^\dagger c_{-k\sigma'}^\dagger c_{-p\sigma'} c_{p\sigma}$$



- Short range, $L_{int} < L_*$: $\tilde{v}_q = const$

$$\approx \frac{1}{N} \frac{N_M}{N_*} \sum_{k,p,\sigma,\sigma'} \tilde{v}_{k-p} c_{k\sigma}^\dagger c_{-k\sigma'}^\dagger c_{-p\sigma'} c_{p\sigma}$$

- Finite range interaction, $L_{int} > L_*$

$$\approx \frac{1}{N} \frac{N_M}{L_{int}^2} \sum_{k,p,\sigma,\sigma'} \tilde{v}_{k-p} c_{k\sigma}^\dagger c_{-k\sigma'}^\dagger c_{-p\sigma'} c_{p\sigma}$$

Weaker enhancement!

Phonons vs Coulomb in carbon SC

- Doped graphite

$$\left| \frac{V_q^{ph}}{V_q^C} \right| \sim \epsilon \left(\frac{t_\pi}{\hbar\omega_0} \right)^2 \frac{m_e}{M_C} q_{sc} a_0$$

10 x 1000 x 10⁻⁴ x 1

- Moire graphene

$$\left| \frac{V_q^{ph}}{V_q^C} \right|_{moire} \sim \epsilon \left(\frac{t_\pi}{\hbar\omega_0} \right)^2 \frac{m_e}{M_C} \frac{a_0 L_M}{L_*^2}$$

10 x 1000 x 10⁻⁴ x 100/10²

$$\text{DOS} \sim 1/(t_\pi a_0^2)$$

About the same

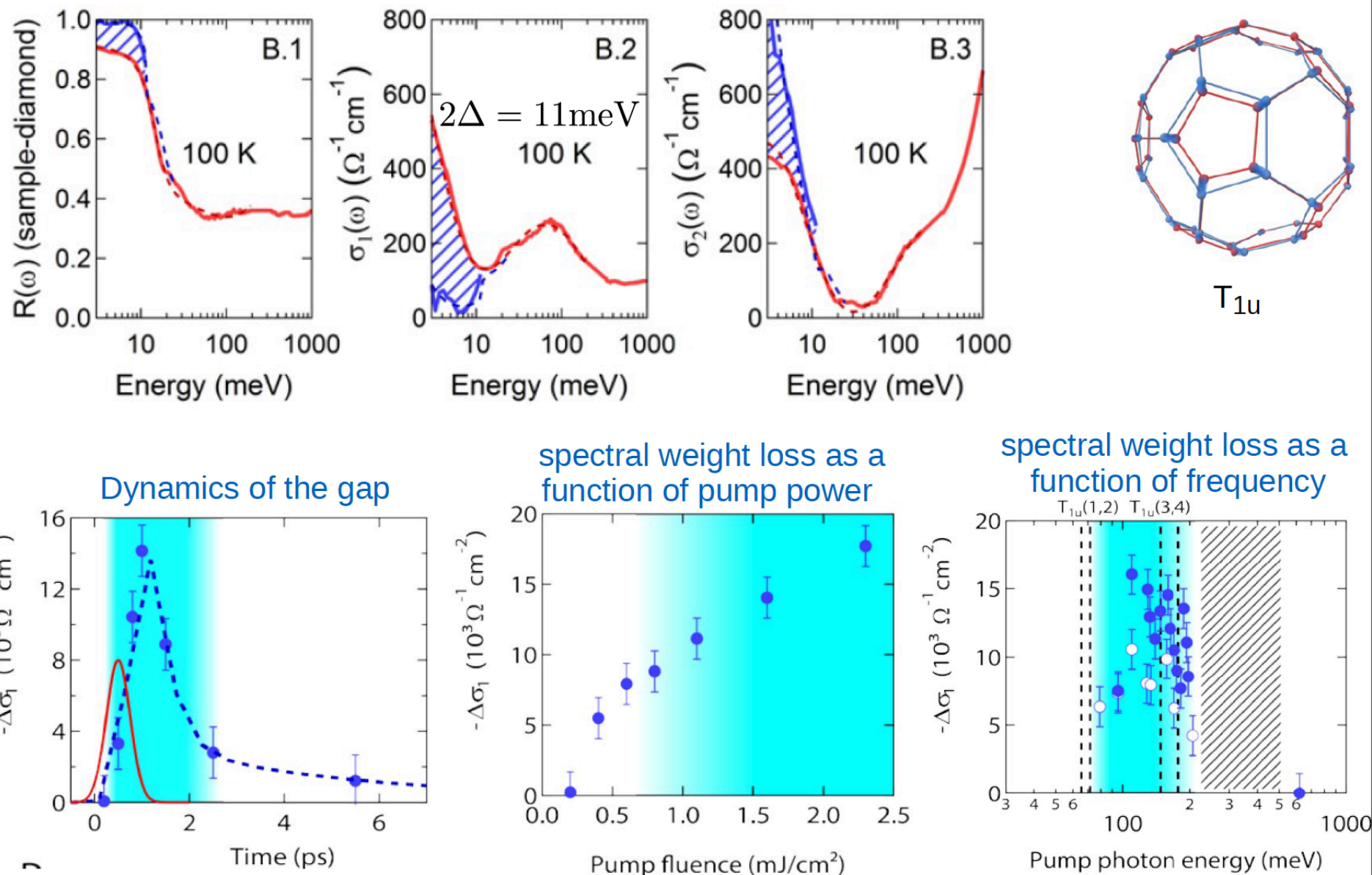
$$\text{DOS} \sim 1/(w_M L_M^2)$$

$$T_c \sim \omega_0 e^{-\frac{1}{\lambda}}$$

$$T_c \sim E_F e^{-\frac{N_*}{N_M} \frac{1}{\lambda}}$$

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IR Driven superconductivity

- renormalization of the parameters: effective static Hamiltonian
- emergent squeezing correlations
- periodic modulation of the parameters: Floquet state

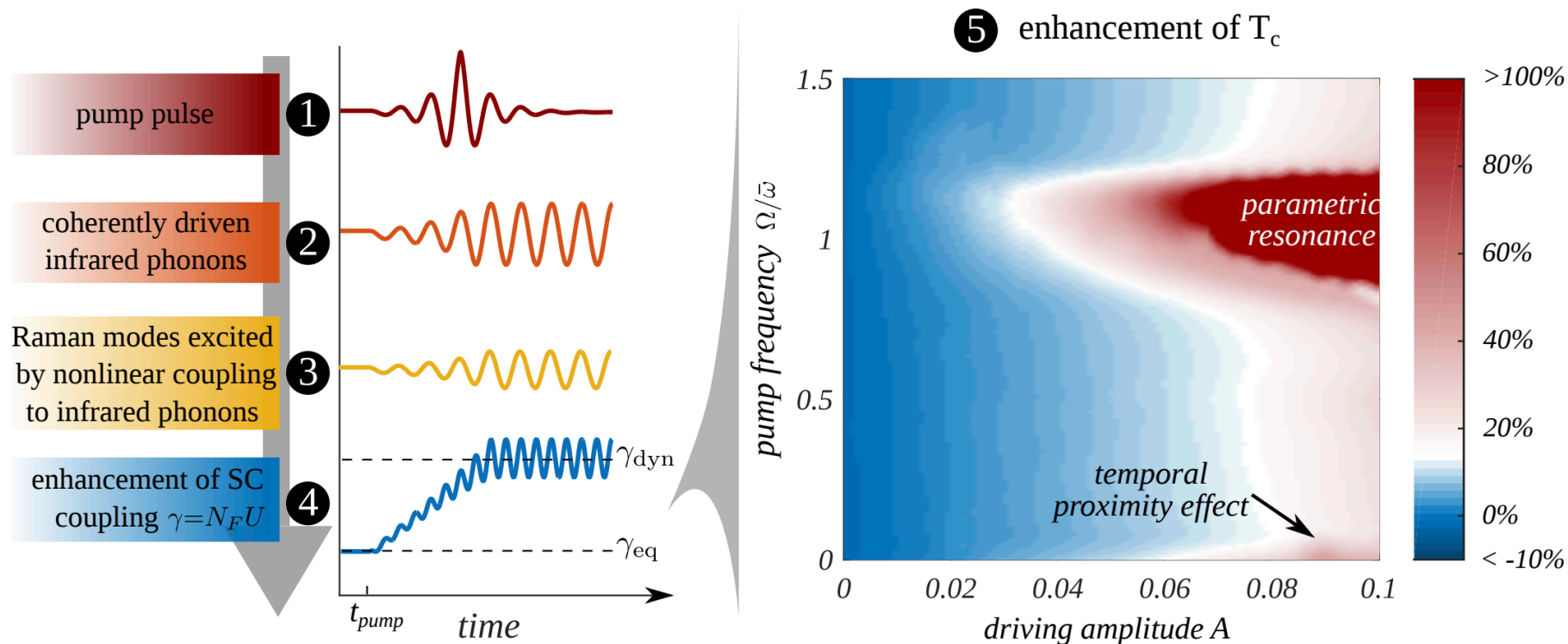
	static renorm. of Hamiltonian	dynamical squeezing	periodic Floquet
I: $(Q_0^{\text{IR}})^2 Q_0^{\text{R}}$	✓	×	✓
II: $(Q_0^{\text{IR}})^2 Q_k^{\text{R}} Q_{-k}^{\text{R}}$	✓	✓	✓
III: $Q_0^{\text{IR}} Q_k^{\text{R}} Q_{-k}^{\text{R}}$	×	✓	✓

static

dynamical

Knap equations

2



$$\tilde{H}(t) = J_{\text{eq}} e^{-\zeta} \sum_{ij} c_{i\sigma}^\dagger c_{j\sigma} - U(1 + \mathcal{A} \cos 2\Omega t) \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$0 = (U^{-1} + F_n) \Delta_n + \frac{\mathcal{A}}{2} F_n (\Delta_{n-1} + \Delta_{n+1})$$

$$F_n = \frac{1}{V} \sum_k \frac{1 - 2n_k}{E + 2n\Omega - 2(\epsilon_k + i/\tau - \mu - U\rho)}$$

Same equations -> same result!

Temporal modulation

$$\tilde{H}(t) = J_{\text{eq}} e^{-\zeta} \sum_{ij} c_{i\sigma}^\dagger c_{j\sigma} - U(1 + \mathcal{A} \cos 2\Omega t) \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$0 = (U^{-1} + F_n) \Delta_n + \frac{\mathcal{A}}{2} F_n (\Delta_{n-1} + \Delta_{n+1})$$

$$F_n = \frac{1}{V} \sum_k \frac{1 - 2n_k}{E + 2n\Omega - 2(\epsilon_k + i/\tau - \mu - U\rho)}$$

Spatial modulation

$$\mathcal{H} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_i U(x_i) n_{\uparrow}(x_i) n_{\downarrow}(x_i)$$

$$U(\mathbf{x}) = U_0 + U_1 \cos \mathbf{Q} \cdot \mathbf{x}$$

$$\begin{pmatrix} \vdots \\ \Delta_{-2Q} \\ \Delta_{-Q} \\ \Delta_0 \\ \Delta_Q \\ \Delta_{2Q} \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & & & & & & \\ & U_0 K(-2Q) & U_1 K(-Q) & 0 & 0 & 0 & \\ & U_1 K(-2Q) & U_0 K(-Q) & U_1 K(0) & 0 & 0 & \\ & 0 & U_1 K(-Q) & U_0 K(0) & U_1 K(Q) & 0 & \\ & 0 & 0 & U_1 K(0) & U_0 K(Q) & U_1 K(2Q) & \\ & 0 & 0 & 0 & U_1 K(Q) & U_0 K(2Q) & \\ & & & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \Delta_{-2Q} \\ \Delta_{-Q} \\ \Delta_0 \\ \Delta_Q \\ \Delta_{2Q} \\ \vdots \end{pmatrix}$$

$$K(\mathbf{p}) \approx N_f \ln \left[\frac{2\gamma\omega_D}{\pi \sqrt{T^2 + (v_f p)^2}} \right] \Theta(\omega_D - |v_f p|),$$

T_c enhanced in both cases!

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Parametrically softened phonons

Holstein Model

$$H = H_e + H_{ph} + \alpha \sum_j n_{j\sigma} x_j,$$

$$H_e = \sum_k \epsilon_k c_{k\sigma}^\dagger c_{k\sigma},$$

$$H_{ph} = \sum_j \frac{kx_j^2}{2} + \frac{p_j^2}{2M}.$$

Phonon nonlinearity

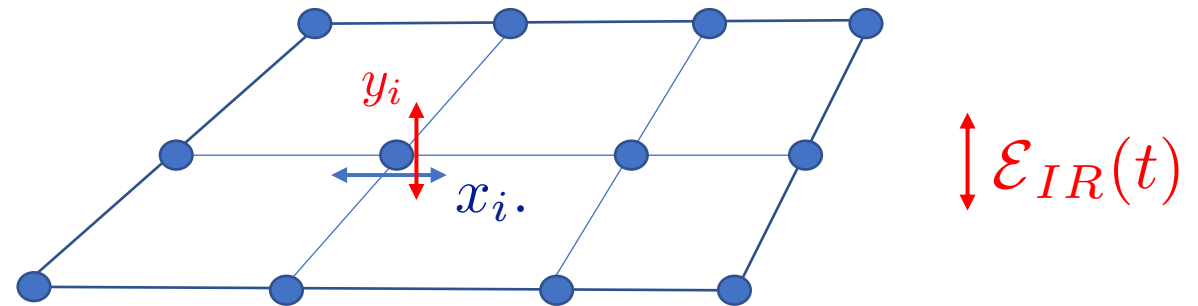
$$H_{nl} = \eta \sum_i y_i x_i^2.$$

Drive

$$H_{dr} = \mathcal{E}(t) \sum_i y_i.$$

Hypothesis:

Driving a local phonons near parametric instability makes them softer (without exciting them) – enhances e-ph coupling. **Thus can boost Tc**
(Buzzi talk from Thu)



Effective Hamiltonian for parametrically softened phonons – 1

$$H = H_e + H_{ph} + \alpha \sum_{j,\sigma} n_{j\sigma} x_j, \quad \longrightarrow \quad H_{e-ph} = \frac{\alpha \ell_0}{\sqrt{2}} \sum_{j,\sigma} n_{j\sigma} \left[(\cosh \xi a^\dagger - \sinh \xi a) e^{i\frac{\Omega}{2}t} + h.c. \right]$$

$$H_e = \sum_k \epsilon_k c_{k\sigma}^\dagger c_{k\sigma},$$

$$H_{ph} = \sum_j \frac{(k + \eta y_0 \cos \Omega t) x_j^2}{2} + \frac{p_j^2}{2M}.$$

Enhanced e-ph coupling!

$$H_{ph} = \sum_j \delta b_j^\dagger b_j + \mu (b_j^{\dagger 2} + b_j^2)$$

RWA

Bogoliubov

$$H_{ph} = \sum_j \tilde{\delta} a_j^\dagger a_j.$$

$$\tilde{\delta} = \sqrt{\delta^2 - \mu^2}$$

$$b^\dagger = \cosh \xi a^\dagger - \sinh \xi a$$

$$\tanh 2\xi = 2\mu/\delta$$

Effective Hamiltonian for parametrically softened phonons – 2

$$H_{e-ph} = \frac{\alpha \ell_0}{\sqrt{2}} \sum_{j,\sigma} n_{j\sigma} \left[(\cosh \xi a^\dagger - \sinh \xi a) e^{i\frac{\Omega}{2}t} + h.c. \right]$$

$$\mathcal{U}_{eff}(t) = \langle 0_a | \mathcal{U}(t) | 0_a \rangle \equiv e^{-iH_{eff}t}$$

$$H_{eff} = H_e - \tilde{U} \sum_j n_j^2$$

$$\tilde{U} = \ell_0^2 \alpha^2 \left(\frac{\cosh^2 \xi}{\Omega + 2\tilde{\delta}} - \frac{\sinh^2 \xi}{\Omega - 2\tilde{\delta}} \right) \approx \frac{\ell_0^2 \alpha^2}{2\omega_0}$$

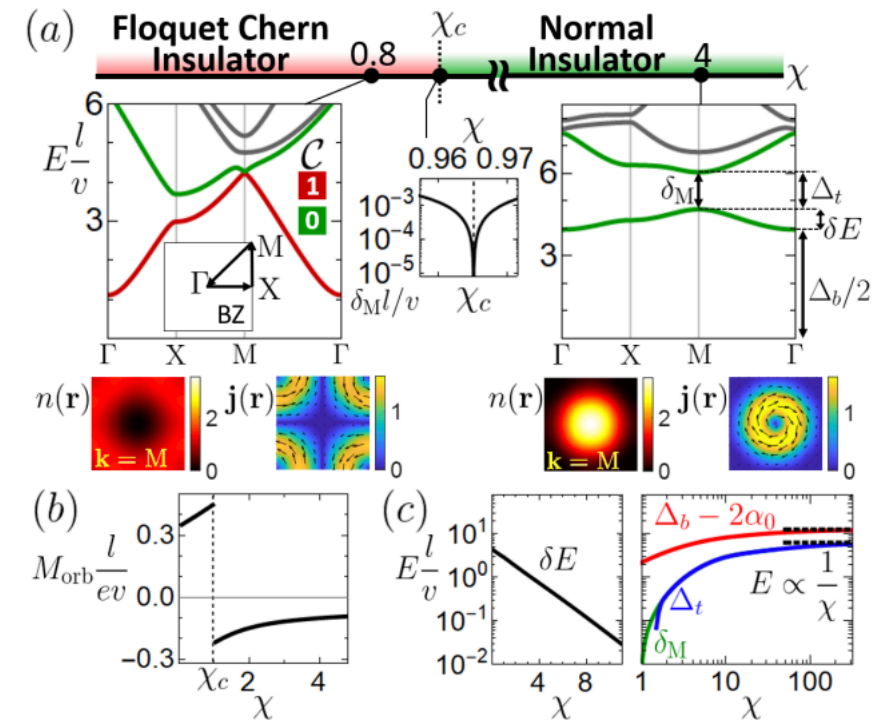
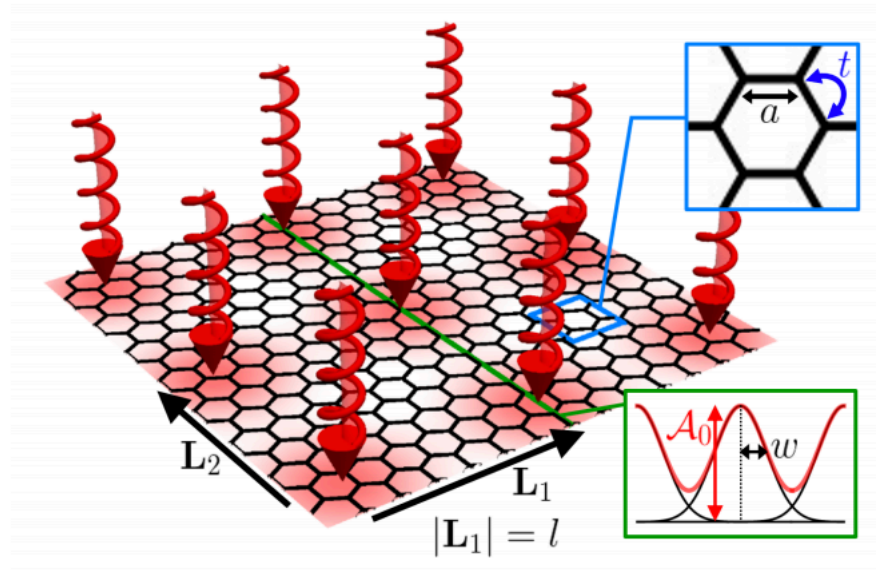
Attraction strength remains essentially unaffected by parametric drive!



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Combining spatial and temporal modulation



A route to moire Floquet superconductors?

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Summary

- Spatial modulation can enhance e-e attraction *more* than Coulomb repulsion – lead to intermediate coupling superconductivity
- Phonon excitation outside ground state does not affect pairing strength
- Parametric phonon drive (softening) *does not* increase pairing interaction in effective Hamiltonian
- *Optical imprinting* – combines benefits of Floquet and moire