Weighted Birkhoff Averages: an Efficient Test for Chaos

J.D. Meiss
E. Sander† & N. Duignan*

† George Mason University
* Sydney University

SIMONS FOUNDATION

March 21, 2024

Maps and Flows

- Flow: $\varphi : \mathbb{R} \times M \to M$, so that

$$x(t) = \varphi_t(x(0))$$

  - Group property: $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$
  - Orbit is a curve $\{x(t) \,|\, t \in \mathbb{R}\}$
  - Typically obtained from ODEs:

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0$$

- Map: discrete time: $f : M \to M$ for phase space M:

- Flow: $\varphi : \mathbb{R} \times M \to M$, so that

$$x(t) = \varphi_t(x(0))$$

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  - Typically obtained from ODEs:

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0$$

- Map: discrete time: $f : M \to M$ for phase space M:

$$x_{t+1} = f(x_t)$$

  - Orbit is a sequence: $\{x_t \mid t \in \mathbb{Z}\}$
  - Can arise as a Poincaré map: $x_t = \varphi_1(x_{t-1}) = \varphi_t(x_0)$

Hamiltonian Flows

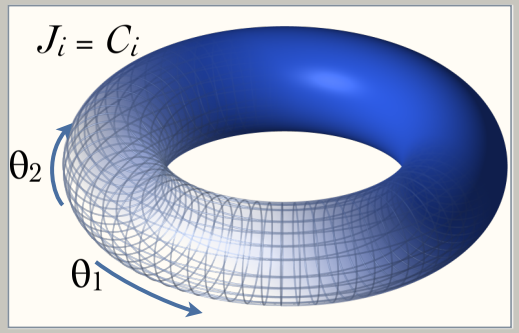
$$\frac{dq}{dt} = \frac{\partial}{\partial p} H_0(q,p) \quad \frac{dp}{dt} = -\frac{\partial}{\partial q} H_0(q,p)$$

- Integrability by Liouville-Arnold:
  An n degree-of-freedom Hamiltonian system is integrable if it has n integrals J_i in involution, $\{J_i, J_k\} = 0$.
- Each compact invariant surface is an n-torus, with n conjugate angle-action (θ_i, J_i) pairs.

$$\frac{d\theta}{dt} = \frac{\partial}{\partial J} H_0(J) \quad \frac{dJ}{dt} = 0$$

Frequency Map: $\nu = \Omega(J)$

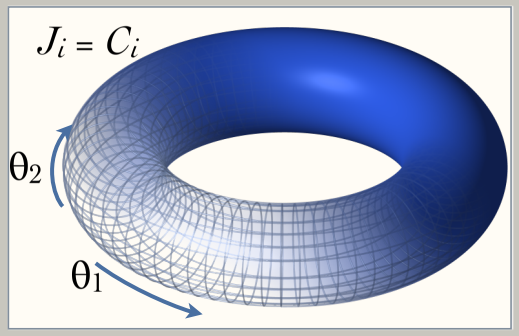
$$\theta(t) = \theta(0) + \nu t$$



$$\frac{d\theta}{dt} = \frac{\partial}{\partial J} H_0(J) \qquad \frac{dJ}{dt} = 0$$

Frequency Map: $\nu = \Omega(J)$

$$\theta(t) = \theta(0) + \nu t$$



KAM Theorem

$$H(\theta, J, \varepsilon) = H_0(J) + \varepsilon H_1(\theta, J)$$

- If H is sufficiently smooth and ν is sufficiently incommensurate, and $D\Omega = D^2H$ nonsingular, then there is an $\varepsilon_0 > 0$ such that H has an invariant torus with frequency ν for all $|\varepsilon| < \varepsilon_0$.

  - Smooth: $C^{3+\delta}$ is sufficient (Herman)
  - Incommensurate: ω satisfies a Diophantine condition

$$|m \cdot \omega| \geq \frac{c}{\|m\|^{\tau}}$$

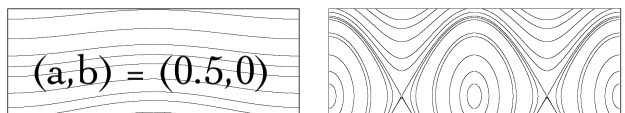
for some $\tau > n$ and $c > 0$ and any $m \in \mathbb{Z}^n \backslash \{0\}$, e.g., there are no integer resonance relations.

Two-Wave Model

- Particle in electrostatic potential of two waves

$$H(q,p,t) = \tfrac{1}{2}p^2 - \tfrac{1}{4\pi^2}\left(a\cos(2\pi q) - b\cos(2\pi(2q-t))\right)$$

- set mass to one and scale wave speeds & lengths
- Integrable cases: $a = 0$ or $b = 0$.

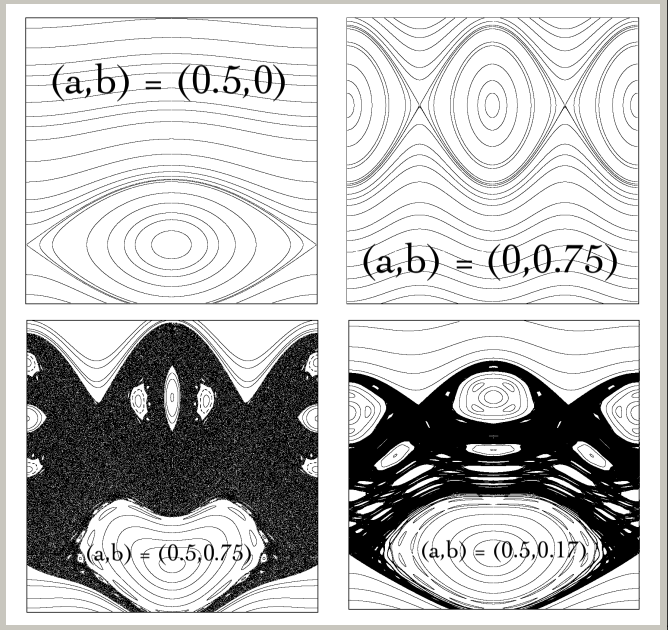


Two-Wave Model

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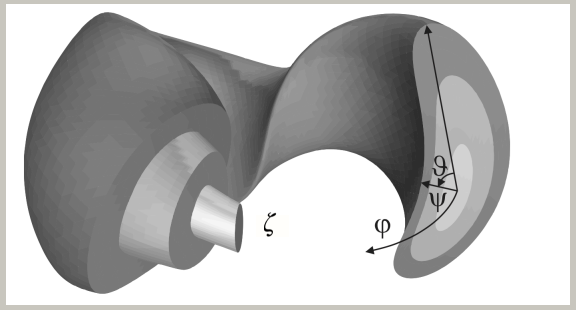
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- set mass to one and scale wave speeds & lengths
- Integrable cases: $a = 0$ or $b = 0$.
- Poincaré sections at $t = k$
- "Resonance overlap" gives chaotic motion



Magnetic Fields

- First problem: Design field coils to minimize chaos
- Coordinates (ψ, θ) + "time" ζ



$$A = \psi\nabla\theta - \chi\nabla\zeta$$

$$B = \nabla\times A = \nabla\psi\times\nabla\theta - \nabla\chi(\psi,\theta,\zeta)\times\nabla\zeta$$

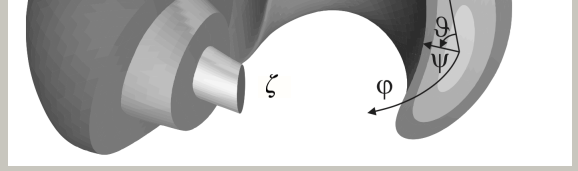
- Field line Hamiltonian $\chi(\psi,\theta,\zeta)$

$$\chi(\psi,\theta,\zeta) = \frac{1}{2}\psi^2 - \sum_{m,n\in\mathbb{Z}} \varepsilon_{m,n}\psi(\psi-1)\cos\left(2\pi(m\theta - n\zeta)\right)$$

- Coordinates (ψ, θ) + "time" ζ

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- Goal: Invariant tori = flux surfaces

†Paul, E. J., S. R. Hudson and P. Helander (2022). "Heat conduction in an irregular magnetic field." J. Plasma Phys. **88**(1): 905880107.

Maps

Arnold's Circle Map

- Two-parameter family of maps on $\mathbb{S}^1$

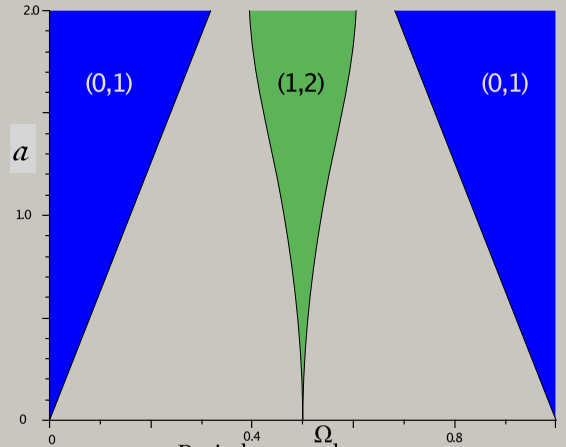
Arnold's Circle Map

- Two-parameter family of maps on $\mathbb{S}^1$

$$f(x) = x + \Omega + \frac{a}{2\pi}\sin(2\pi x)$$

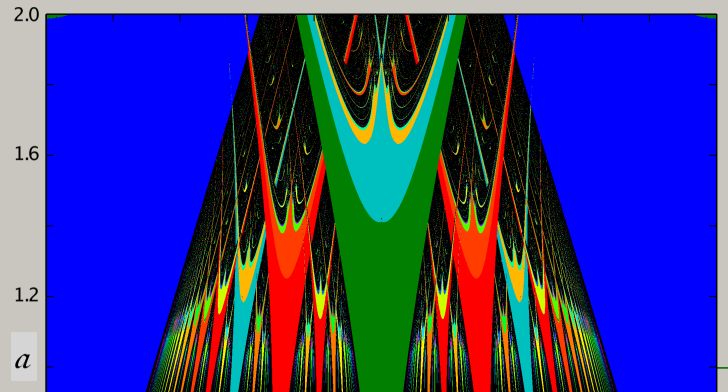
- $a = 0$: Rigid rotation, with rotation number $\omega = \Omega$
- $|a| \leq 1$: homeomorphism $\Rightarrow$ (Poincaré) every orbit has a rotation number
- Arnold Tongues: Regions in (Ω, a) with an attracting periodic orbit.

$$\omega = 0 \text{ when } |\Omega| \leq \frac{a}{2\pi}$$



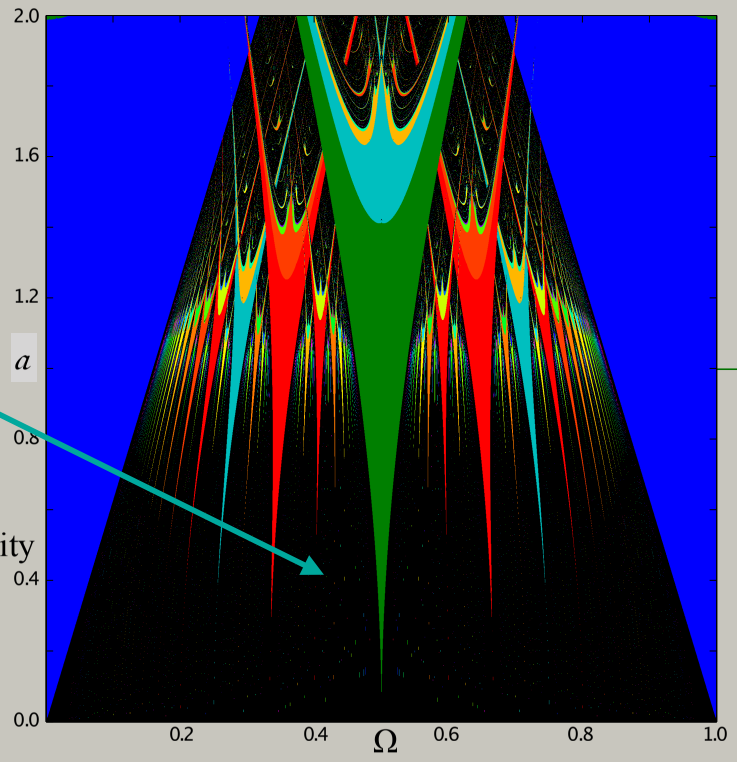
Period-one and two tongues

Arnold Tongues



Invariant

Arnold Tongues



Invariant Circles?

detecting periodicity by recurrence

no chaos

2D: Chirikov's Map

Area-Preserving Map on $\mathbb{T} \times \mathbb{R}$

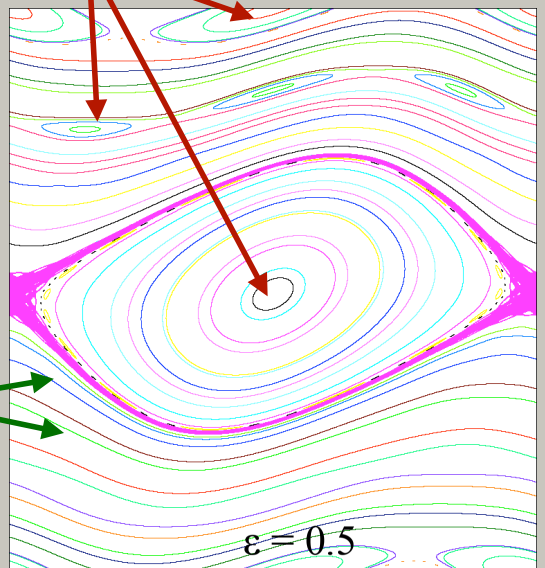
$$x' = x + \Omega(y')$$
$$y' = y + \varepsilon F(x)$$

Frequency map: $\Omega(y) = y$

Force $F(x) = -\frac{1}{2\pi}\sin(2\pi x)$

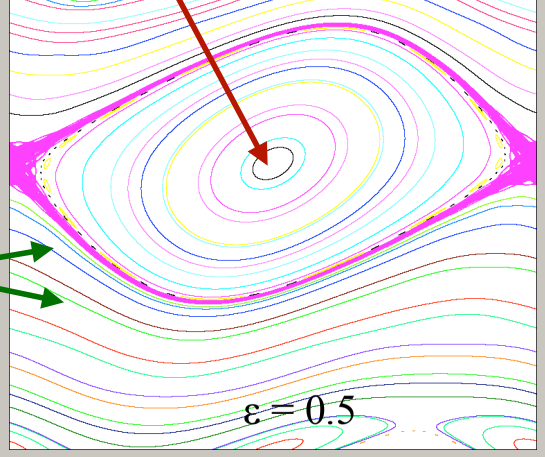
Resonant Island chains

Invariant Circles



Frequency map: $\Omega(y) = y$

Force $F(x) = -\frac{1}{2\pi}\sin(2\pi x)$



Invariant Circles

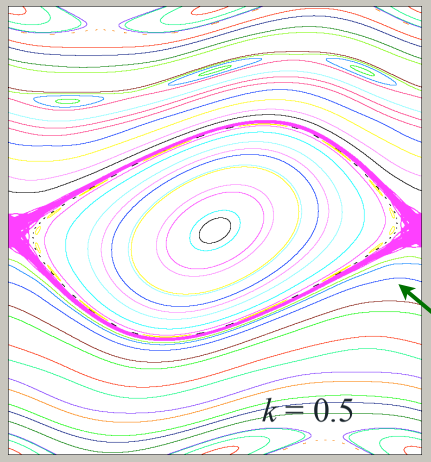
- Lift x from $\mathbb{S}$ to $\mathbb{R}$
- Orbit $(x_{t+1}, y_{t+1}) = f(x_t, y_t)$
- Rotation number $\omega = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \Omega(y_t)$

Chirikov's Map

$x' = x + y'$

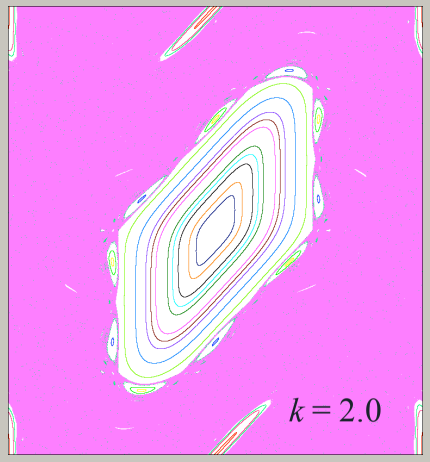
$y' = y - \frac{k}{2\pi}\sin(2\pi x)$

Many rotational circles



Diophantine circles?

No rotational circles



Higher Dimensions

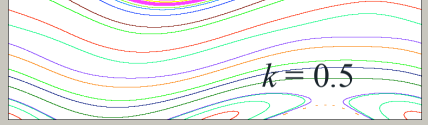
- 3D Volume Preserving map with 2 angles and 1 action $(x, y) \in \mathbb{T}^2 \times \mathbb{R}$

$x_1' = x_1 + y' + \gamma$

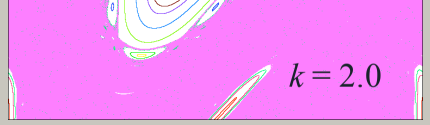
$x_2' = x_2 + 2y'^2 - \delta$

$\gamma = \frac{1}{2}(\sqrt{5} - 1) \qquad \delta = 0.1$

$\Omega = (y + \gamma, 2y^2 - \delta)$



Diophantine circles?



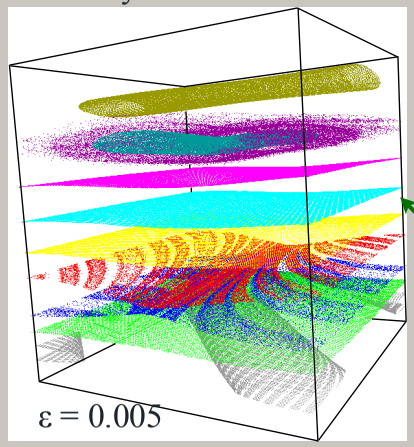
Higher Dimensions

◆ 3D Volume Preserving map with 2 angles and 1 action $(x, y) \in \mathbb{T}^2 \times \mathbb{R}$

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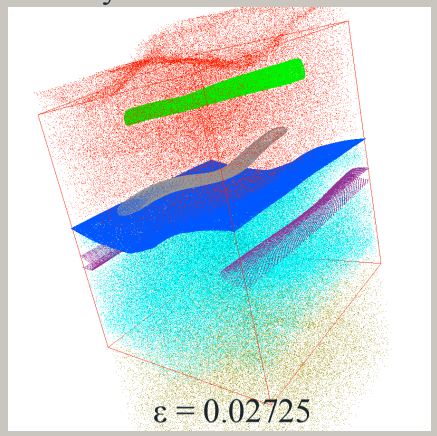
$$\gamma = \tfrac{1}{2}(\sqrt{5} - 1) \qquad \delta = 0.1$$
$$\Omega = (y + \gamma, 2y^2 - \delta)$$
$$g(x) = -\sin(2\pi x_1) - \sin(2\pi x_2) - \sin(2\pi(x_1 - x_2))$$

Many rotational tori



Diophantine tori?

Only one rotational torus

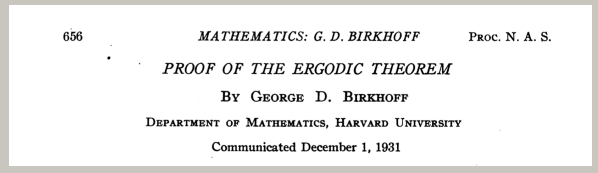


Birkhoff Averages

Birkhoff Averages

Birkhoff Averages

- Birkhoff says: *Time Average = Space Average*
  - a map $f : M \to M$, with ergodic invariant measure μ
  - an L^1 function $h : M \to \mathbb{R}$
  - for μ-almost all x


656 MATHEMATICS: G. D. BIRKHOFF PROC. N. A. S.

PROOF OF THE ERGODIC THEOREM

By George D. Birkhoff

Department of Mathematics, Harvard University

Communicated December 1, 1931

$$B_T(h)(x) = \frac{1}{T}\sum_0^T h(f^t(x)) \underset{T\to\infty}{\longrightarrow} \langle h \rangle \equiv \int_M h d\mu$$

- Flow case: Proof rarely given! (Pugh & Shub†)

$$B_T(h)(x) = \frac{1}{T}\int_0^T h(\varphi_t(x))dt \underset{T\to\infty}{\longrightarrow} \langle h \rangle \equiv \int_M h d\mu$$

- *Nota Bene*: convergence can be arbitrarily slow (Krengel*) and for almost all h is at most as $1/T$ (Kachurovskii•).

*Krengel, U. (1978) "On the speed of convergence in the ergodic theorem." Monatshefte für Mathematik 86: 3-6.
•Kachurovskii, A. G., I. V. Podvigin and A. A. Svishchev (2021). "The Maximum Pointwise Rate of Convergence in Birkhoff's Ergodic Theorem." Journal of Mathematical Sciences 255(2): 119-123.
†C.C. Pugh, M. Shub, (1971) "Ergodic elements of ergodic actions." Compos. Math. 23(1): 115–122.

Weighted Birkhoff Averages

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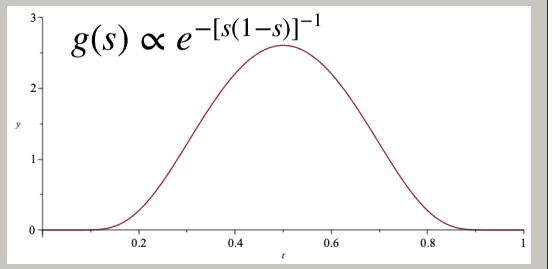
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Weighted Birkhoff Averages

- C^m Bump Functions: $$\mathcal{G}_m = \left\{ g \in C^m([0,1],\mathbb{R}^+) \ : \ \|g\|_1 = 1, \left.\frac{d^i g}{ds^i}\right|_{s=0,1} = 0, \ i = 0,1,\ldots,m-1 \right\}$$
- Add bump-function weighting

$$WB_T(h)(z) = \sum_{t=0}^{T-1} g\left(\frac{t}{T}\right) h(f^t(x))$$

C^∞ case



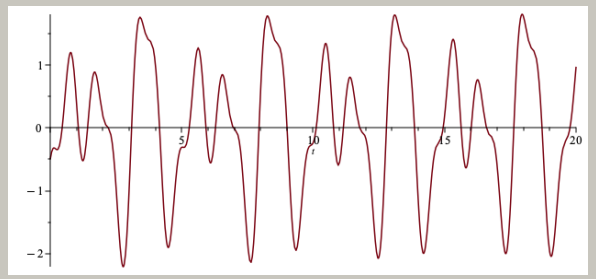
Das, S., C. B. Dock, et al. (2016). "Measuring Quasiperiodicity." Euro. Phys. Lett. 114: 40005.
Das, S., Y. Saiki, E. Sander and J. A. Yorke (2017). "Quantitative Quasiperiodicity." Nonlinearity 30(11): 4111.

- WBA Corollary to Birkhoff:
  - Under the Birkhoff hypotheses if $g \in \mathcal{G}_1$, then $\lim_{T\to\infty} WB_T(h) = \langle h\rangle$, for μ almost every $x \in M$.
  - Proof uses a summation criterion of Silverman.†

†Silverman, L. L. (1916). "On the Notion of Summability for the Limit of a Function of a Continuous Variable." Trans. Am. Math. Soc. 17(3): 284–294.

Analog: Smoothing Fourier Transforms

- Using filters to compute Fourier Series
  - quasiperiodic function
  - ω_k set of frequencies (periodic if $\omega_k = k\omega_1$) $$x(t) = \sum_k a_k e^{i\omega_k t}$$
  - Compute finite, weighted inner product



signal with golden mean and $1/\sqrt{2}$ frequencies

Analog: Smoothing Fourier Transforms

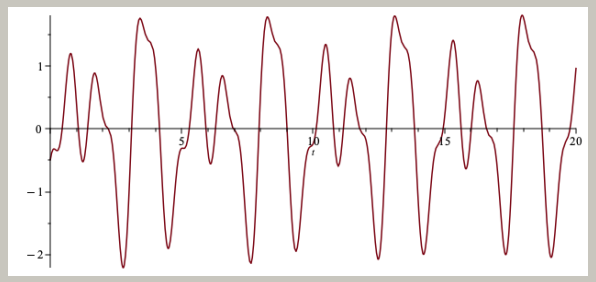
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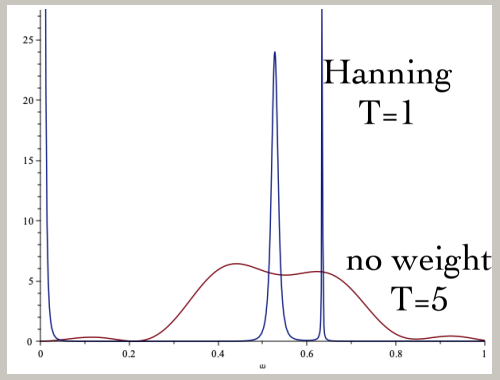
  - Compute finite, weighted inner product

$$a(\omega) \approx \frac{1}{T}\int_0^T g(t)x(t)e^{-i\omega t}$$

  - Ex: Hanning window: $g(t) = 1 - \cos(2\pi t/T)$
    - C^2 weight function
  - ω_k given by peaks in amplitude $a(\omega)$



signal with golden mean and $1/\sqrt{2}$ frequencies



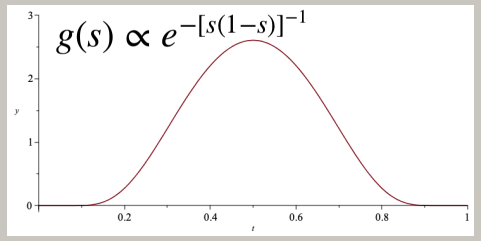
a(ω) vs ω

Laskar, J. (1993). "Frequency Analysis of a Dynamical System." Celestial Mech. and Dyn. Astron. 56: 191-196.

Super Convergence!

$$WB_T(h)(z) = \sum_{t=0}^{T-1} g\left(\frac{t}{T}\right) h(f^t(x))$$

- $g : [0,1] \to \mathbb{R}^+$ is a C^∞ bump function if $\frac{d^k}{dt^k} g(t)\big|_{t=0,1} = 0,\ \forall k \in \mathbb{N}$

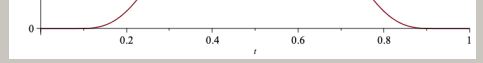


- Theorem: Suppose $f : M \to M$ is C^∞ and has a d-dimensional invariant torus $\mathcal{T}$ on which the dynamics is C^∞ conjugate to a rigid rotation with Diophantine frequency $\rho \in \mathbb{R}^d$. Then for any $h \in C^\infty(\mathcal{T}, \mathbb{R})$, $x_0 \in \mathcal{T}$, $\exists c_n \in \mathbb{R}^+$, s.t. for all $T \in \mathbb{N}$ any $n \in \mathbb{N}$

$$\|WB_T(h)(x_0) - \langle h \rangle\|_\infty \leq \frac{c_n}{T^n}$$

Maps:
Das, S. and J. A. Yorke (2018). "Super convergence of ergodic averages for quasiperiodic orbits." Nonlinearity **31**(2): 491-501.

Flows:



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Maps:
Das, S. and J. A. Yorke (2018). "Super convergence of ergodic averages for quasiperiodic orbits." Nonlinearity **31**(2): 491-501.
Flows:
Duignan, N. and J. D. Meiss (2023). "Distinguishing between Regular and Chaotic orbits of Flows by the Weighted Birkhoff Average." Physica D **449**: 133749.

Super-Convergence

$$WB_T(h)(z) = \sum_{t=0}^{T-1} g\left(\frac{t}{T}\right) h(f^t(x))$$

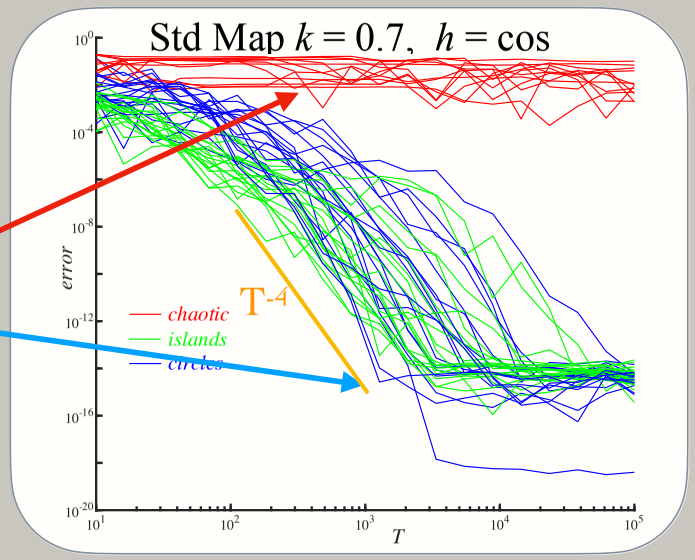
- Chirikov Standard Map

$$x' = x + y'$$
$$y' = y - \frac{k}{2\pi}\sin(2\pi x)$$

- $h(x) = \cos(2\pi x)$, $k = 0.7$
- Expect:
  - Chaotic orbits: error $\sim T^{-\frac{1}{2}}$ by central limit theorem
  - Invariant circles: error $< T^{-n}$

No convergence for Chaos

machine precision after 10^3 steps!

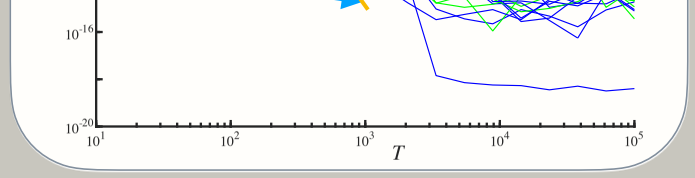


Super-convergence observed only for multi-precision computations

Sander, E. and J. D. Meiss (2020). "Birkhoff Averages and Rotational Invariant Circles for Area-Preserving Maps." Physica D 411(October): 132569.

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Sander, E. and J. D. Meiss (2020). "Birkhoff Averages and Rotational Invariant Circles for Area-Preserving Maps." Physica D 411(October): 132569.



Detecting Chaos

Example:Circle Map

$$F(x) = x + \Omega + \frac{a}{2\pi}\sin(2\pi x)$$

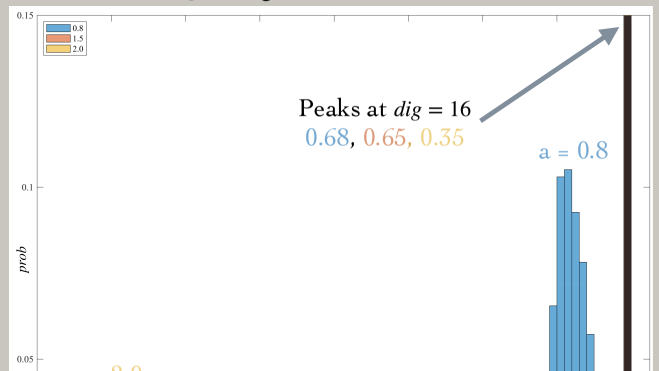
$$\omega_T = WB_T(F(x) - x) = \Omega + \frac{a}{2\pi}\sum_{t=0}^{T-1} g(\tfrac{t}{T})\sin(2\pi x_t)$$

- Digits of Accuracy

$$dig_T = -\log_{10}\left|WB_T(h)(z) - WB_T(h)(f^T(z))\right|$$

- If $|a| < 1$ every orbit has a rotation number

dig_T Histograms for a = 0.8,1.5,2.0



$$F(x) = x + \Omega + \frac{a}{2\pi}\sin(2\pi x)$$

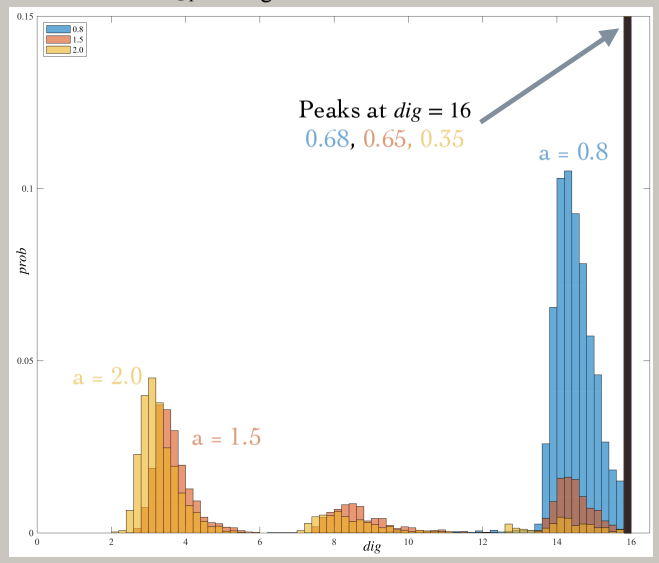
$$\omega_T = WB_T(F(x) - x) = \Omega + \frac{a}{2\pi}\sum_{t=0}^{T-1} g(\tfrac{t}{T})\sin(2\pi x_t)$$

- Digits of Accuracy

$$dig_T = -\log_{10}\left|WB_T(h)(z) - WB_T(h)(f^T(z))\right|$$

- If $|a| \leq 1$ every orbit has a rotation number
- If $|a| > 1$, no invariant circles
- For $a = 0.8, 0.11\,\%$ have $dig_T < 9$

dig_T Histograms for $a = 0.8, 1.5, 2.0$

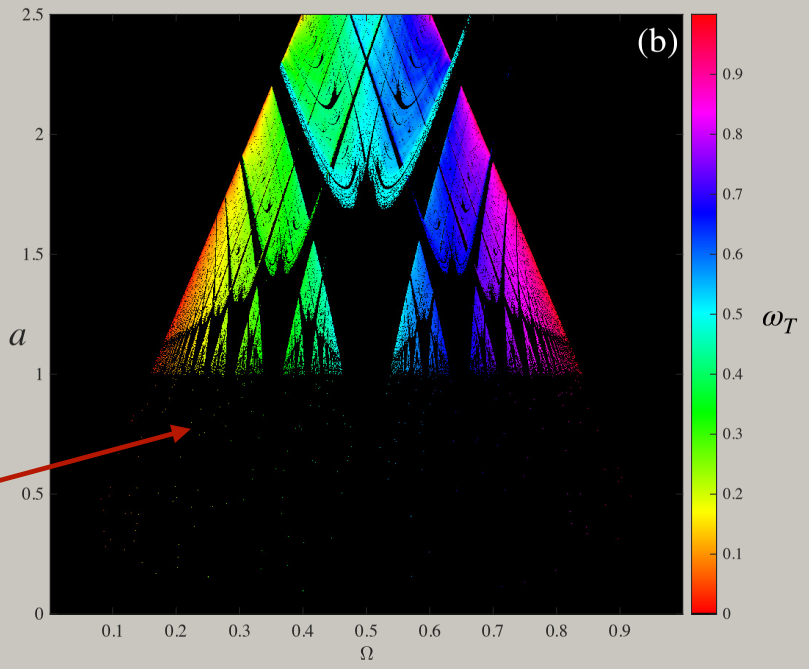


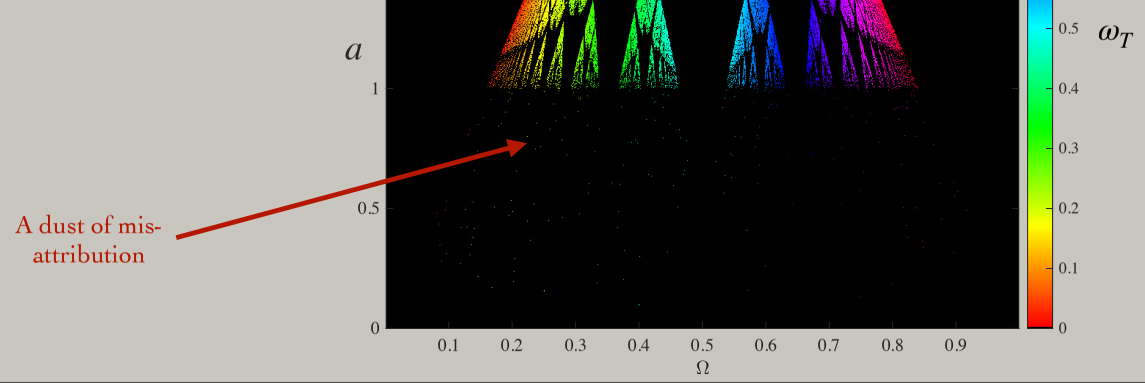
$T = 10^5$, grid of 10,000 Ω

Example: Circle Maps

$$dig_T = -\log_{10}\left|WB_T(h)(z) - WB_T(h)(f^T(z))\right|$$

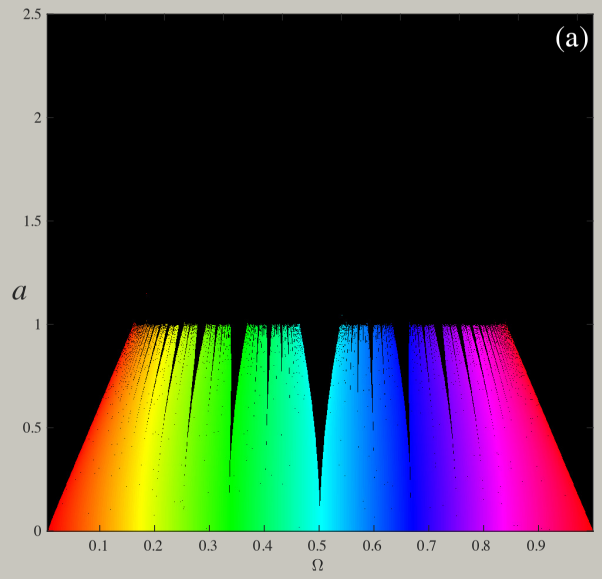
- $T = 10^5$
- Chaotic Orbits ($dig_T < 9$)

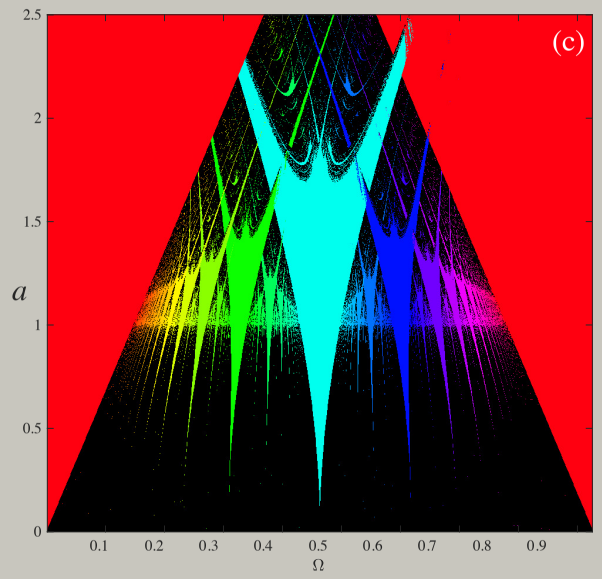




Example: Circle Map

- Nonresonant and Resonant Orbits ($dig_T \geq 9$)

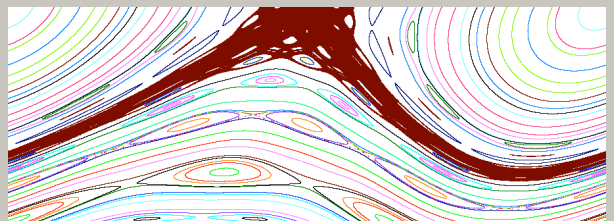




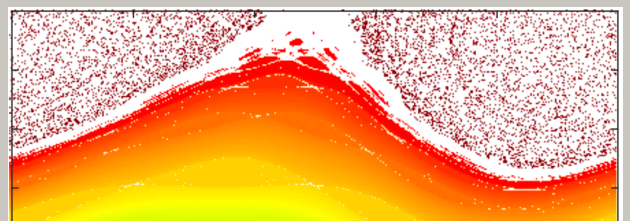
Sander, E. and J.D. Meiss (2023) "Weighted Birkhoff Averages for Torus Maps." ArXiv: <https://arxiv.org/abs/2310.11600>

Standard Map, $k = 0.7$

Chaotic



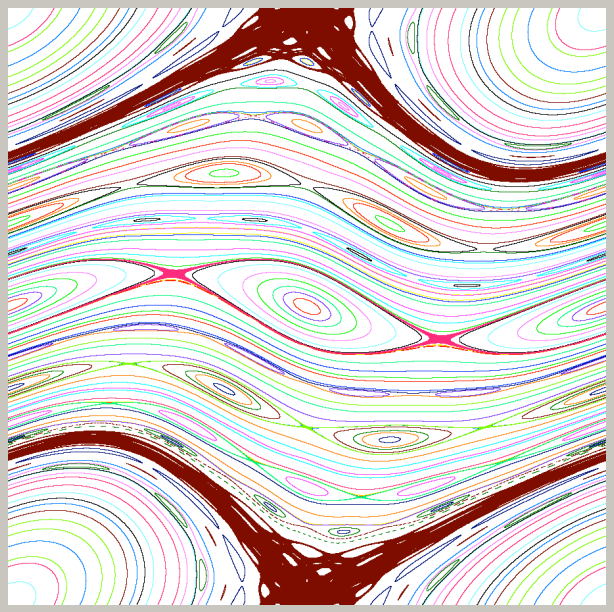
Regular



Sander, E. and J.D. Meiss (2023) "Weighted Birkhoff Averages for Torus Maps." ArXiv: <https://arxiv.org/abs/2310.11600>

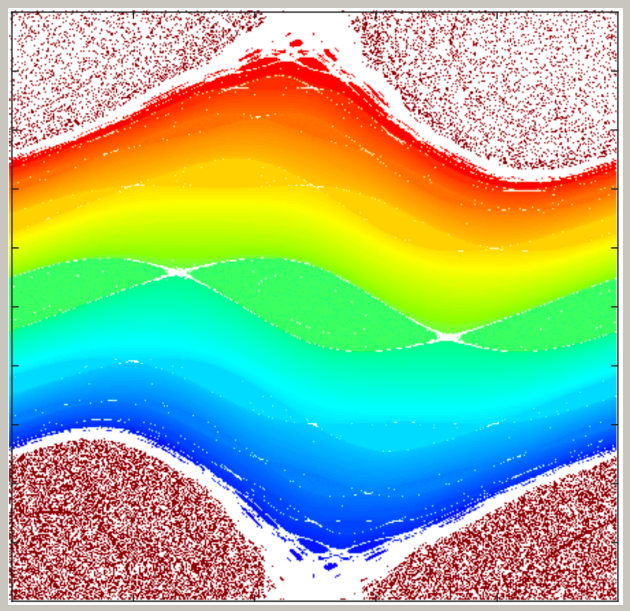
Standard Map, $k = 0.7$

Chaotic



$dig_T < 5$

Regular



$dig_T > 5$

$T = 10^4$

Birkhoff vs Lyapunov

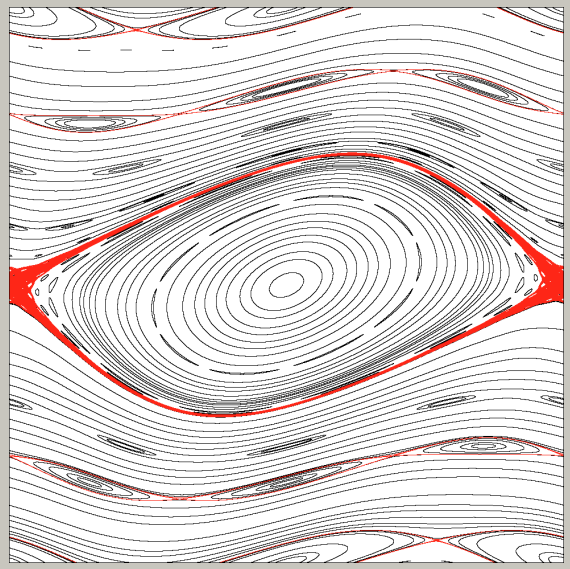
Place your bets now!

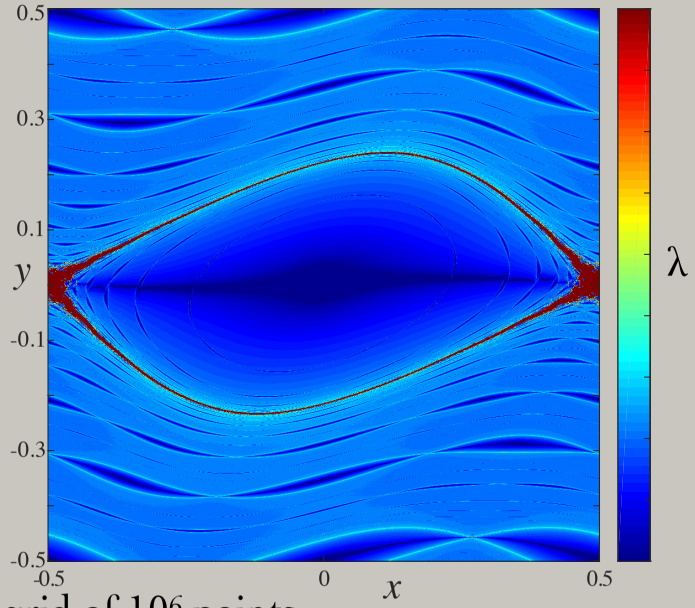
Birkhoff vs Lyapunov

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Distinguishing Chaos

$$\lambda_T(v) = \frac{1}{T}\log\left(\left|Df^T(x_0, y_0)v\right|\right)$$



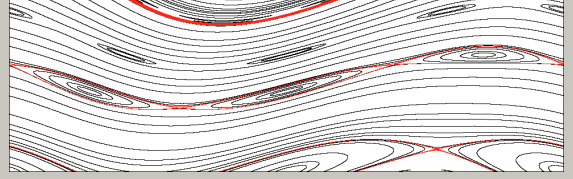


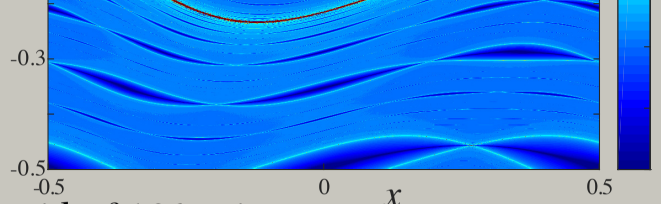
k = 0.52, T = 10^3, grid of 10^6 points

Distinguishing Chaos

Regular?

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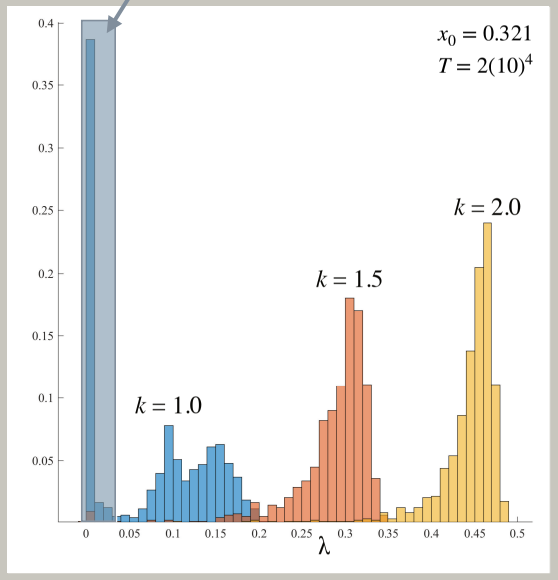


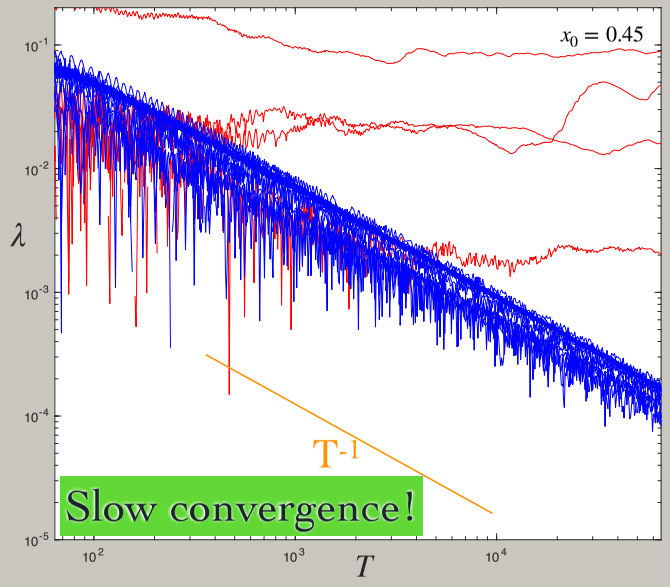
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Distinguishing Chaos

Regular?

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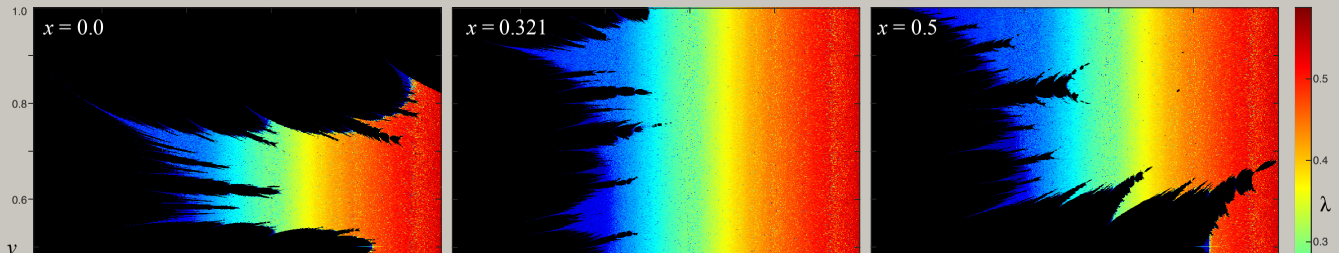




Lyapunov Exponents

$$\lambda_T(v) = \frac{1}{T}\log\left(|Df^T(x_0, y_0)v|\right)$$

$T = 2^{16}$
$v = (0,1)^T$

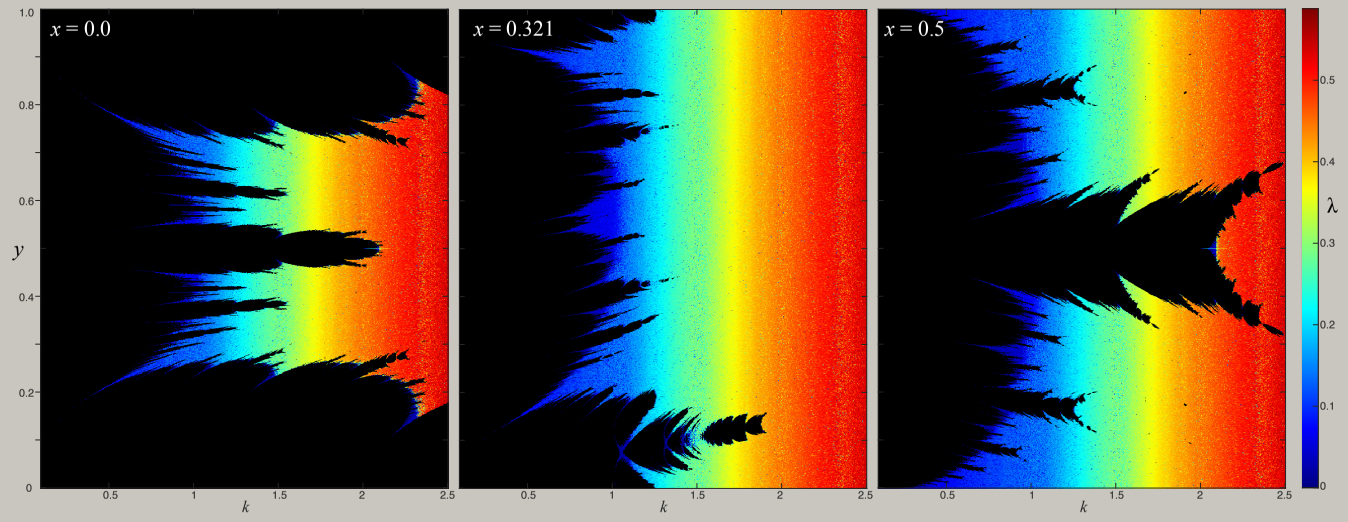


Lyapunov Exponents

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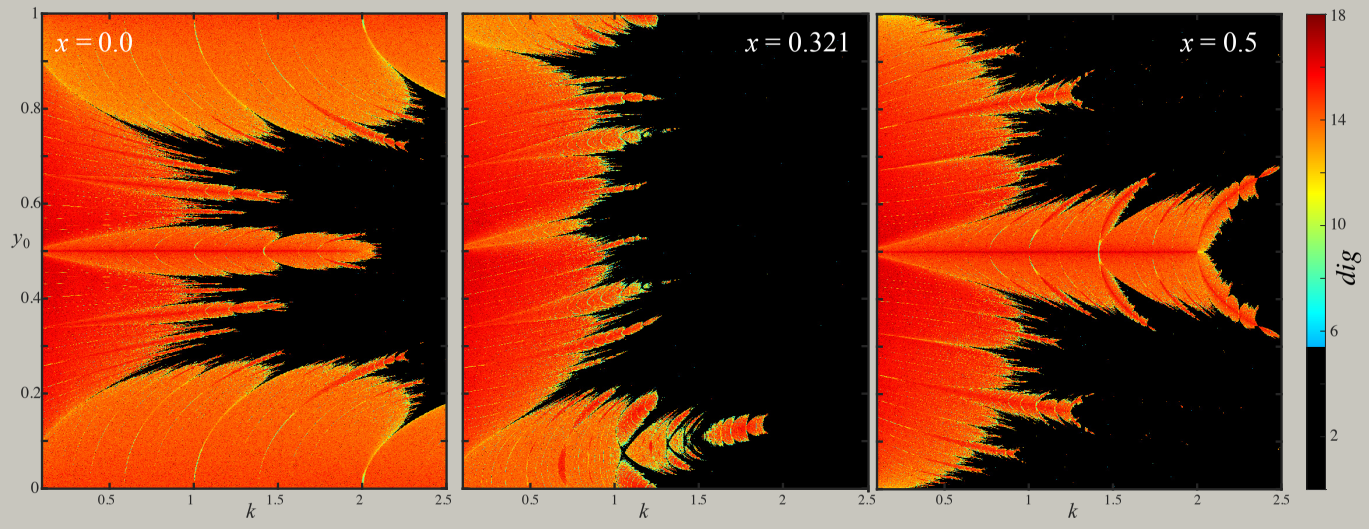
Black corresponds to $\lambda < 0.0022$

Birkhoff Averages

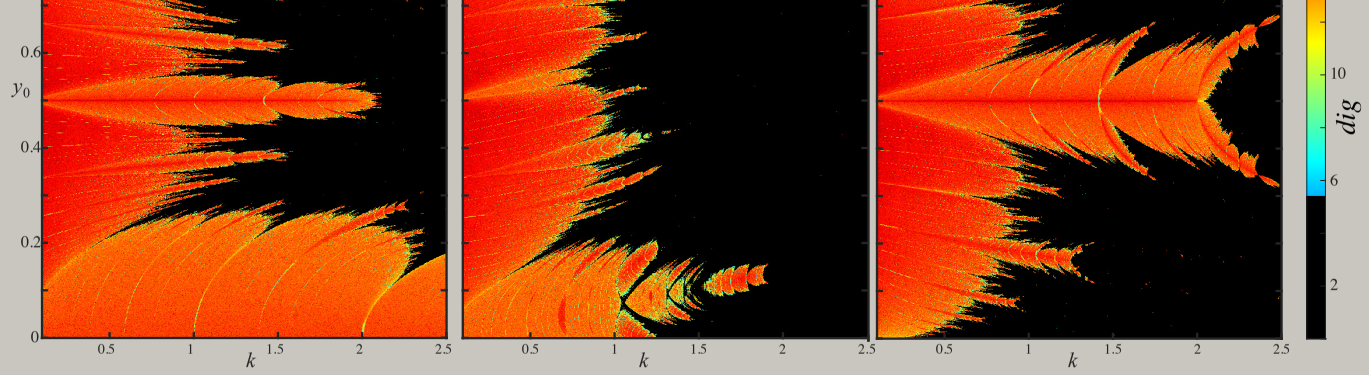
$$dig_T = -\log_{10}\left|WB_T(h)(z) - WB_T(h)(f^T(z))\right|$$

$h(x,y) = \cos(2\pi x)$

$T = 10^4$



Initial conditions with $dig_T < 5.5$ are colored black.

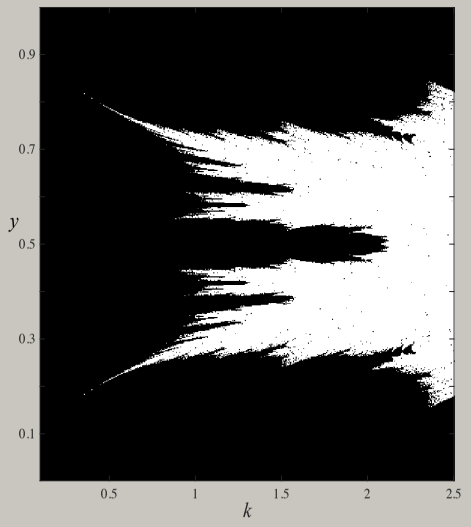


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WBA vs 0-1 Test

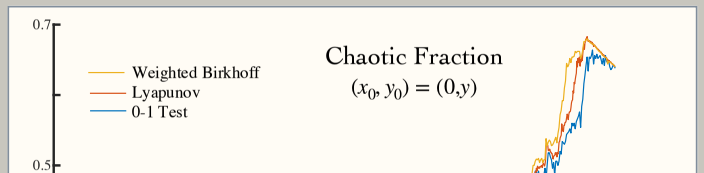
- 0-1Test for Chaos of Gottwald and Melbourne
  - Compute a time series, say $h(x_t)$
  - An auxiliary series: $z_n(\nu) = \sum_{t=0}^{n-1} h(x_t)\, e^{2\pi i \nu t}, \quad \nu \in \mathbb{R}\backslash\mathbb{Q}$
  - Look for diffusive behavior $K_T(\nu) = \dfrac{\log\left(\frac{1}{n}\sum_{j=0}^{n} |\, z_{T+j} - z_j|^2\right)}{\log T}$
  - $\langle K_T \rangle$ = median over 100 ν values:
  - Regular: $\langle K_T \rangle < 0.5$
  - Chaotic: $\langle K_T \rangle > 0.5$

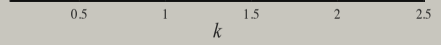
$T = 1000$, $h(x) = \sin(2\pi x)$



Gottwald, G. A. and I. Melbourne (2009). "On the Implementation of the 0–1 Test for Chaos." SIAM J. Appl. Dyn. Sys., **8**(1): 129-145.

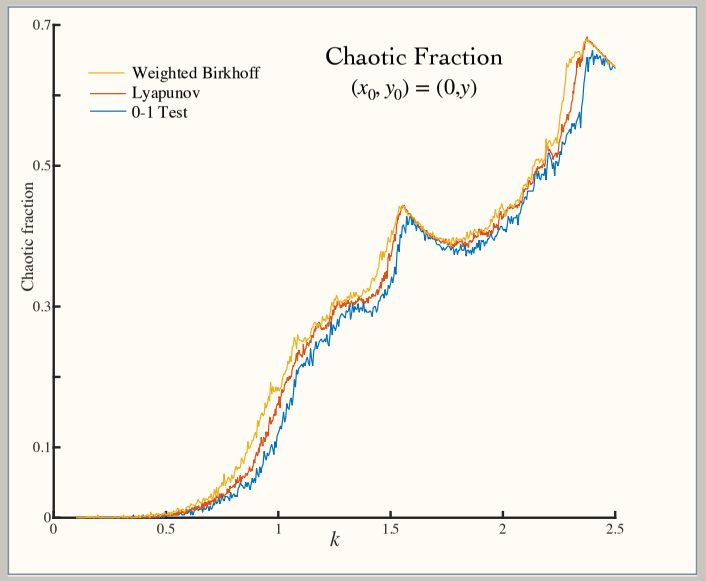
Comparing the Methods



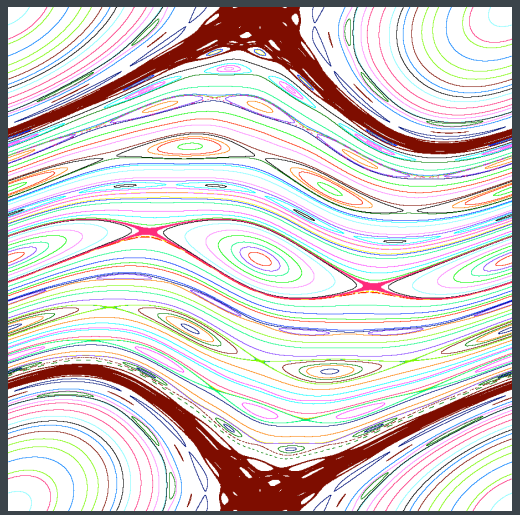


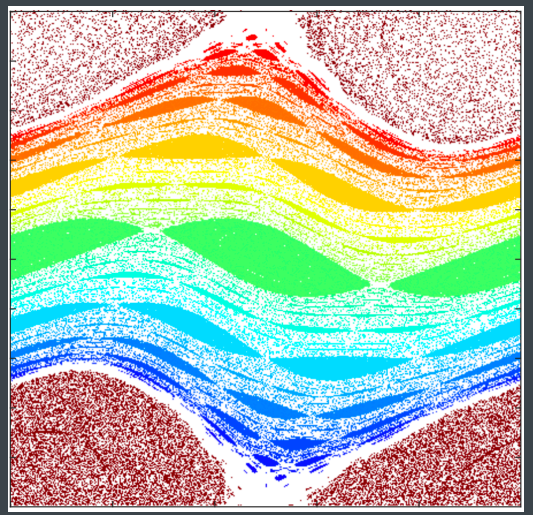
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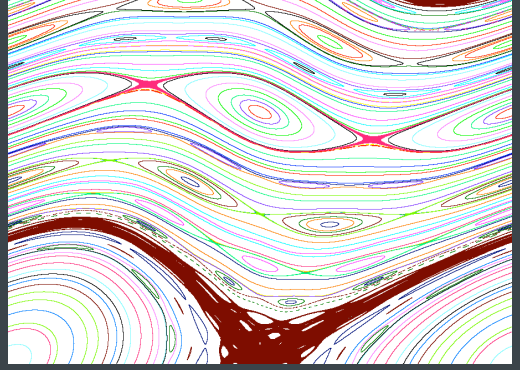
Comparing the Methods

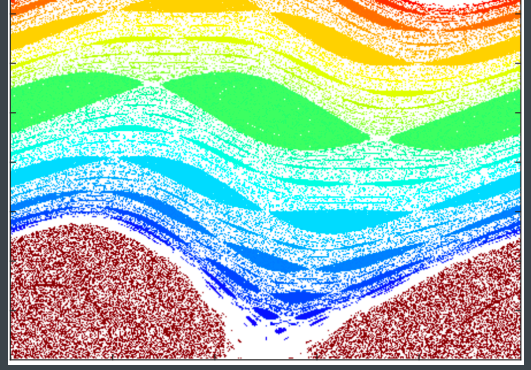


Computation Time (25,000 IC)
7 sec. WBA ($T = 2^{15}$)
10.5 sec. Lyapunov ($T = 2^{15}$)
45 min. 0-1 Test ($n = 10T = 10^4$)









Finding Resonances

Or: When is a Computed number Rational?

Nearest Rational

Rational?

Nearest Rational

- Given ω computed to precision δ, is it rational?
  - Nonsensical question since rational are dense!
- Better question: what is the smallest denominator of a rational in interval $I_\delta(\omega) \equiv (\omega - \delta, \omega + \delta)$?

$$q_{min}(I) \equiv \min\{q \in \mathbb{N} : \frac{p}{q} \in I, p \in \mathbb{Z}\}$$

- Standard software:
  Matlab's *rat*(*x*, δ)—incorrect!
  Mathematica's *Rationalize*[*x*,δ]— recently corrected!
- algorithms based on Continued Fractions do not give q_{min}!
- For random $\omega \in [0,1]$ we observe

RATIONALITY

- Goal: Find rational with smallest denominator within δ of *x:*

$$q_{min}(I) \equiv \min\{q \in \mathbb{N} : \frac{p}{q} \in I, p \in \mathbb{Z}\}$$

- Claim: Stern-Brocot or Farey Tree gives the smallest denominator
- Lemma: The smallest denominator rational in an interval I ∈ ℝ is the first rational on the Stern-Brocot tree that falls in I.

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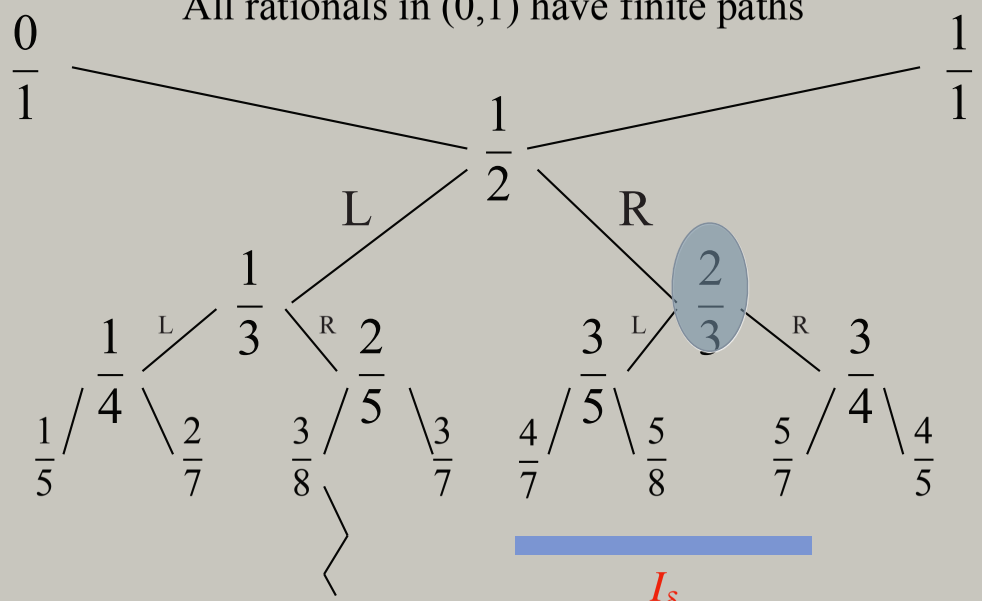
$$\langle \log_{10} q_{min} \rangle = -\tfrac{1}{2} \log_{10} \delta - 0.05 \pm 0.001,$$
$$\sigma = 0.2935 \pm 0.0006$$

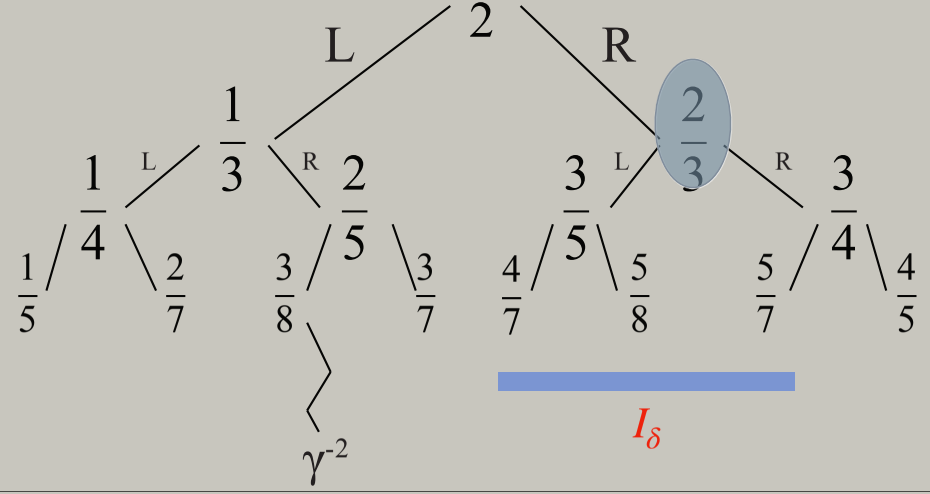
Forisek, M. (2007). Approximating Rational Numbers by Fractions. Fun with Algorithms. **LNCS 4475:** 156-165.
Citterio, M. and R. Pavani (2016). "A Fast Computation of the Best k-Digit Rational Approximation to a Real Number." Mediterranean J. Math. **13**(6): 4321-4331.

STERN-BROCOT TREE

$$\frac{p}{q} = \frac{p_1 + p_2}{q_1 + q_2}$$

All rationals in (0,1) have finite paths





Farey & Cont. Fractions

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0; a_1, a_2, \ldots]$$

$$\begin{aligned}
\pi &= R^3L^7R^{15}LR^{292}LRLR^2LR^3L\ldots \\
&= [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots] \\
e &= RRLR^2LRL^4RLR^6LRL^8RLR^{10}\ldots \\
&= [2; 1, 2, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10\ldots]
\end{aligned}$$

$$q_{min}(\pi, 10^{-8}) = 32085, \quad \frac{p_m}{q_m} = [3; 7, 15, 1, 283],$$
$$q_{min}(e, 10^{-10}) = 154257, \quad \frac{p_m}{q_m} = [2; 1, 2, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 8]$$

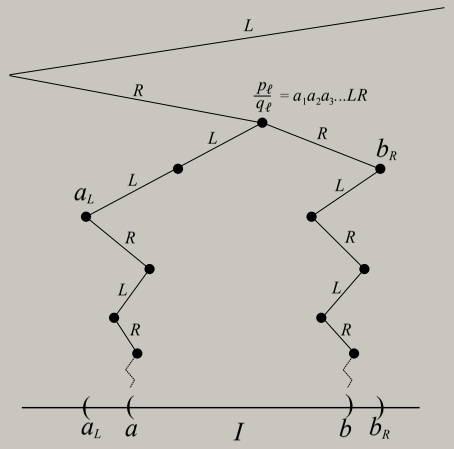
Smallest Denominator

- Lemma: The smallest denominator rational in an interval $I \in \mathbb{R}$ is the first rational on the Stern-Brocot tree that falls in I.
- Lemma: For an irrational interval $I = (a,b)$ q_{min} is the denominator of the common Farey

$$q_{min}(e, 10^{-10}) = 154257, \quad \frac{p_m}{q_m} = [2; 1, 2, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 8]$$

Smallest Denominator

- Lemma: The smallest denominator rational in an interval $I \in \mathbb{R}$ is the first rational on the Stern-Brocot tree that falls in I.
- Lemma: For an irrational interval $I = (a,b)$ q_{min} is the denominator of the common Farey ancestor of a and b.




```
procedure SmallDenom(x, δ)
    (n, d) = (p_l, q_l) = (0, 1)
    (p_r, q_r) = (1, 0)
    while |x − n/d| ≥ δ do
        (n, d) = (p_l + p_r, q_l + q_r)
        if x < n/d then
            (p_r, q_r) = (n, d)
        else
            (p_l, q_l) = (n, d)
        end if
    end while
    return (n, d)
end procedure
```

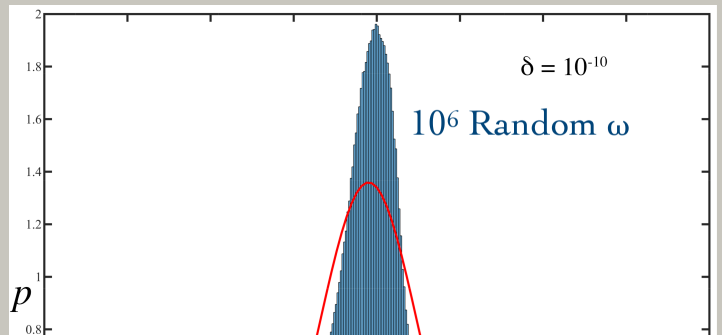

Forisek, M. (2007). Approximating Rational Numbers by Fractions. **LNCS 4475:** 156-165.
Sivignon, I. (2016). "A note on the computation of the fraction of smallest denominator in between two irreducible fractions." Discrete Applied Mathematics **202**: 197-201.

Statistics of Q_{MIN}

- For *i.i.d.* $\omega \in [0,1]$

$$\langle \log_{10} q_{min} \rangle = -\tfrac{1}{2} \log_{10} \delta - 0.05 \pm 0.001,$$
$$\sigma = 0.2935 \pm 0.0006$$

What is this distribution?

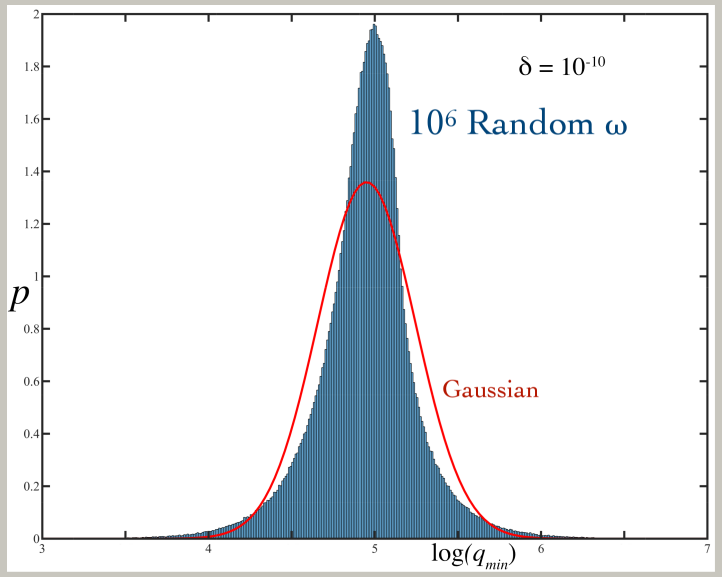


Chen, H. Y. and A. L. Haynes (2023). "Expected value of the smallest denominator in a random interval of fixed radius." Int. J. Number Theory **19**(06): 1405-1413
Marklof, J. "Smallest Denominators" (2023) arXiv:2310.11251v1

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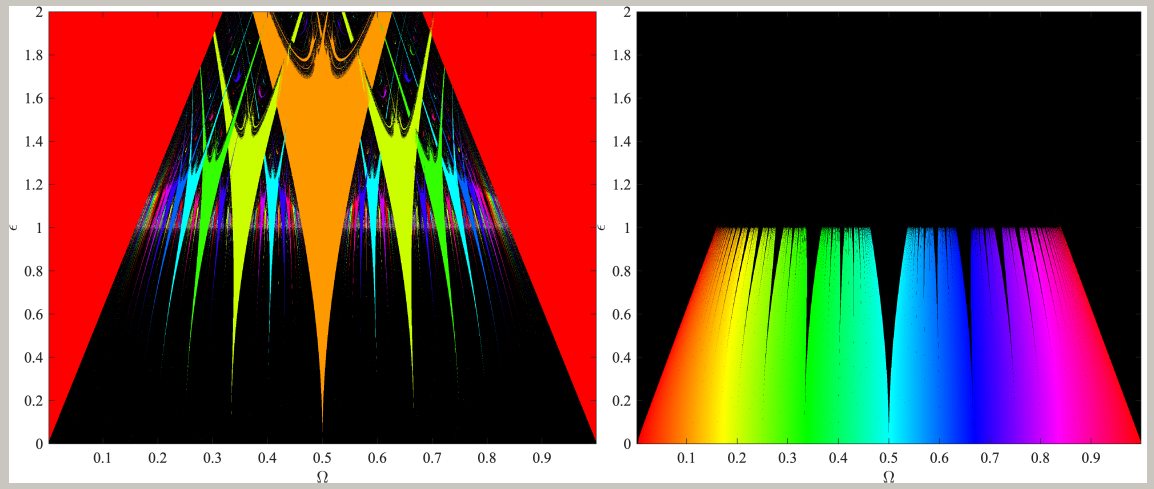
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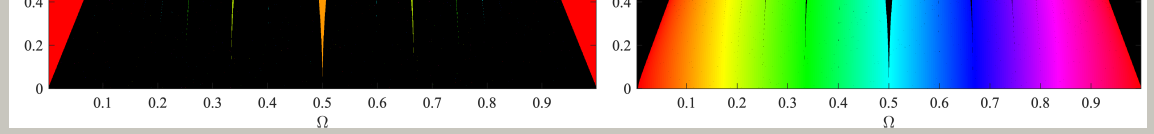
Resonant vs Non-Resonant orbits



colors = q_{min}

colors = ω

$\delta = 10^{-9}$ "Irrational" if $|\log_{10}(q_{min}) - 4.5| < 5.75\sigma$



colors = q_{min} colors = ω

$\delta = 10^{-9}$ "Irrational" if $|\log_{10}(q_{min}) - 4.5| < 5.75\sigma$

2D Case: Rotational Circles

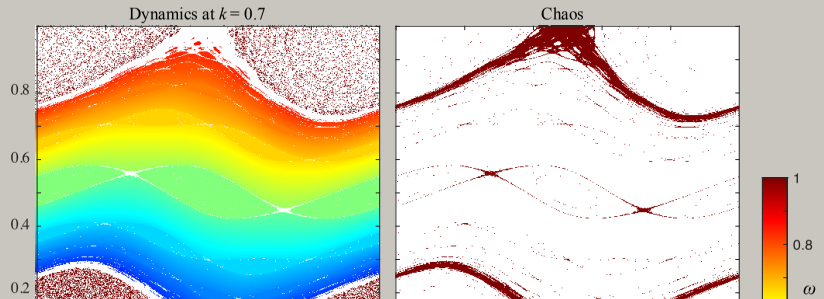
Chaos-Islands-Circles

$T = 10^4$

$\delta = 10^{-8}$

Chaotic:

$dig_T < 5.5$



Chaos-ISLANDS-CIRCLES

$T = 10^4$

$\delta = 10^{-8}$

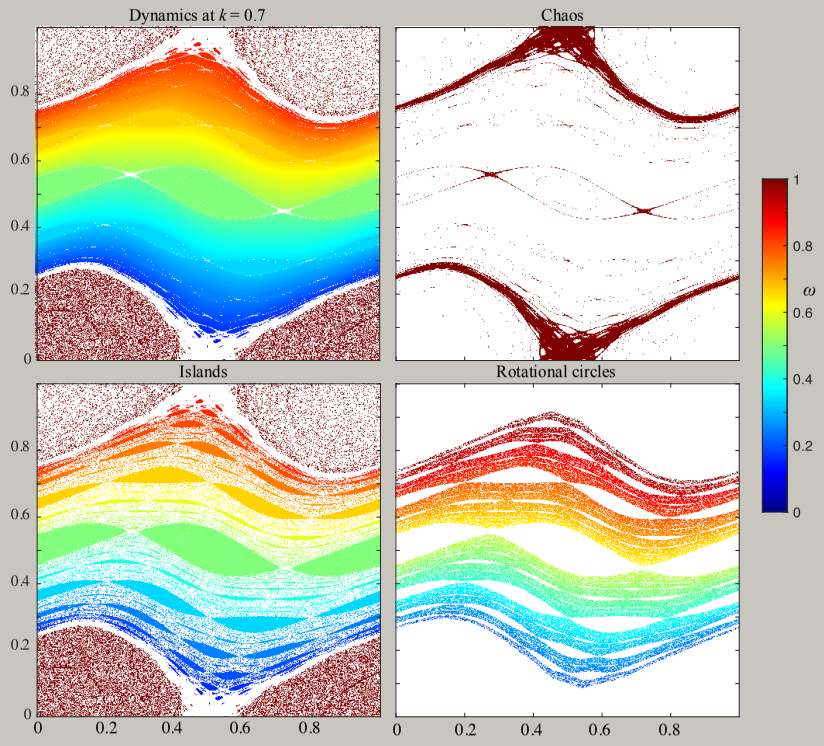
Chaotic:

$dig_T < 5.5$

Irrational:

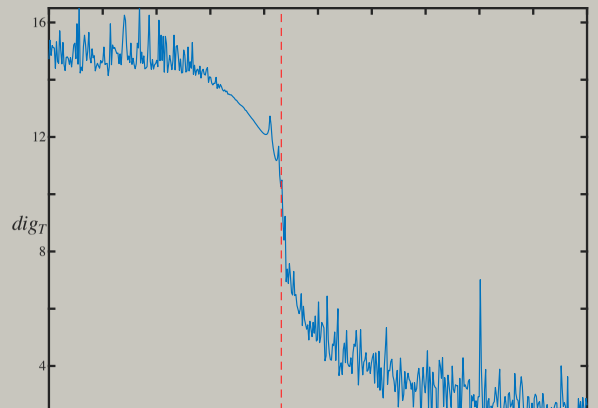
$prob(\omega \in \mathbb{Q}) < 0.2$

$|\log(q_{min}) - 4| < 0.34$



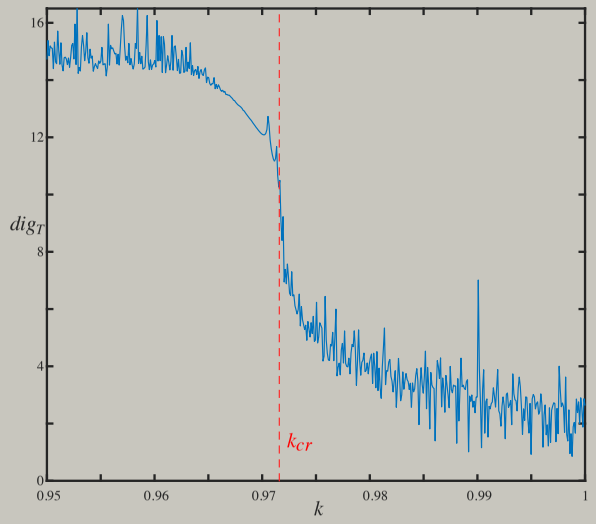
Break-up Of Circles

- Greene-MacKay: Renormalization Theory
  - No rotational circles if $k > 0.971635406$
  - Last circle has $\omega = \gamma$, the golden mean
- MacKay-Percival: Converse KAM proves no circles $k \geq 63/64$
  - Computations with interval arithmetic bounds
- *WB* for golden circle:
  - dig_T drops from 12 to 6 for $k \in [0.9706, 0.9731]$



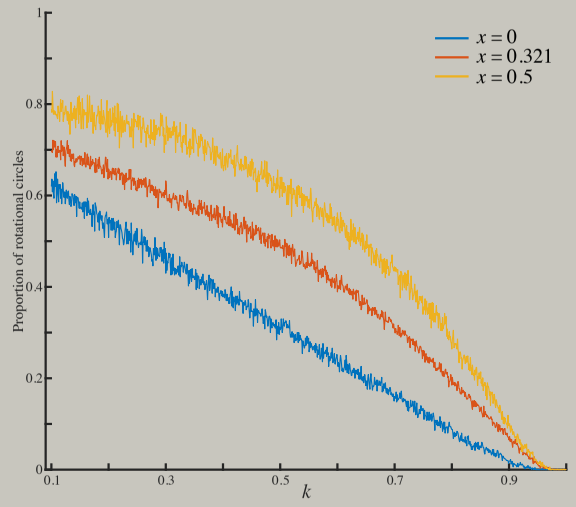
- MacKay-Percival: Converse KAM proves no circles $k \geq 63/64$
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- *WB* for golden circle:
  - dig_T drops from 12 to 6 for $k \in [0.9706, 0.9731]$

$T = 10^4$

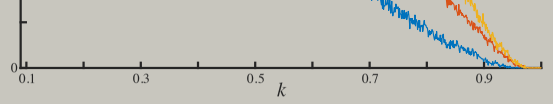


Break-up of Circles

Unlike Greene's method, no symmetry
(to find periodic orbits) is required!



Proportion of Circles for 1000 I.C's on x_0 = constant lines



Proportion of Circles for 1000 I.C's on x_0 = constant lines

3D Case: Invariant Tori

3D Volume-Preserving

$$x_1' = x_1 + y' + \gamma$$
$$x_2' = x_2 + 2y'^2 - \delta$$
$$y' = y + \varepsilon g(x)$$

$$(x, y) \in \mathbb{T}^2 \times \mathbb{R}$$

$$\gamma = \tfrac{1}{2}(\sqrt{5} - 1)$$

$$\Omega = (y + \gamma, 2y^2 - \delta)$$

$$g(x) = -\sin(2\pi x_1) - \sin(2\pi x_2) - \sin(2\pi(x_1 - x_2))$$

$$\sum_{}^{T-1}$$

$$x_1' = x_1 + y' + \gamma$$
$$x_2' = x_2 + 2y'^2 - \delta$$
$$y' = y + \varepsilon g(x)$$

$(x, y) \in \mathbb{T}^2 \times \mathbb{R}$

$\gamma = \frac{1}{2}(\sqrt{5} - 1)$

$\Omega = (y + \gamma, 2y^2 - \delta)$

$g(x) = -\sin(2\pi x_1) - \sin(2\pi x_2) - \sin(2\pi(x_1 - x_2))$

$$\omega(x, y) = \lim_{T \to \infty} \sum_{0}^{T-1} \Omega(y_t)$$

Dullin, H. R. and J. D. Meiss (2012). "Resonances and Twist in Volume-Preserving Mappings." SIAM J. Appl. Dyn. Sys. **11**: 319-349.

Fox, A. M. and J. D. Meiss (2013). "Greene's Residue Criterion for the Breakup of Invariant Tori of Volume-Preserving Maps." Physica D **243**(1): 45-63.

Higher Dimensions

- 3D Volume Preserving map with 2 angles and 1 action $(x, y) \in \mathbb{T}^2 \times \mathbb{R}$

$$x_1' = x_1 + y' + \gamma$$
$$x_2' = x_2 + 2y'^2 - \delta$$
$$y' = y + \varepsilon g(x)$$

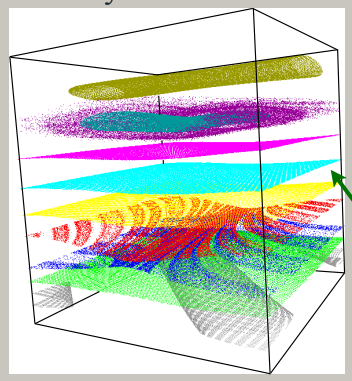
$\Omega = (y + \gamma, 2y^2 - \delta)$

$g(x) = -\sin(2\pi x_1) - \sin(2\pi x_2) - \sin(2\pi(x_1 - x_2))$

$\gamma = \frac{1}{2}(\sqrt{5} - 1)$

$\delta = 0.1$

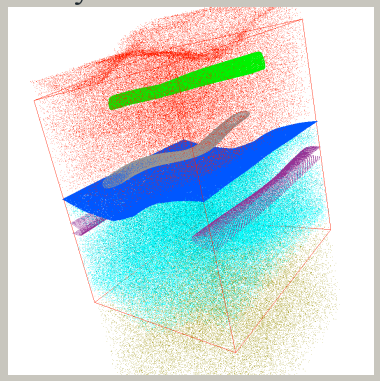
Many rotational tori



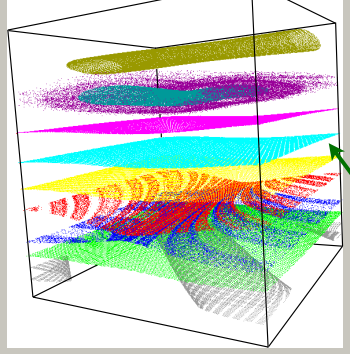
$\varepsilon = 0.005$

Diophantine tori?

Only one rotational torus

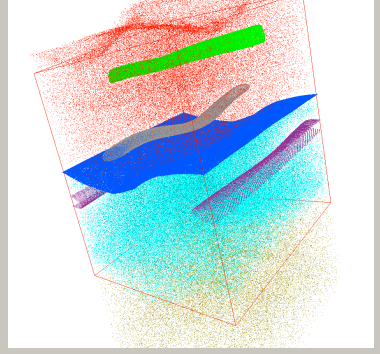


$\varepsilon = 0.02725$



ε = 0.005

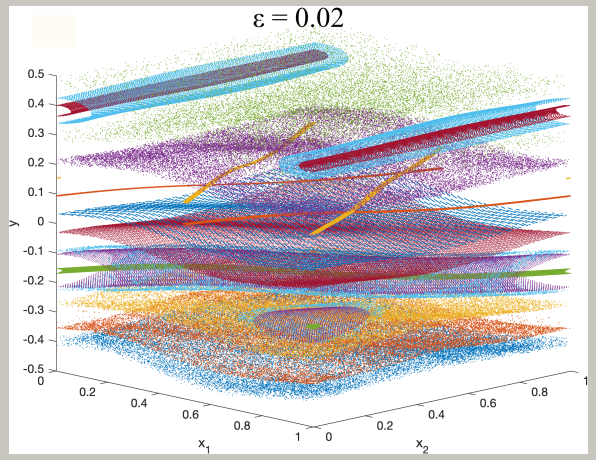
Diophantine tori?



ε = 0.02725

KAM Theory

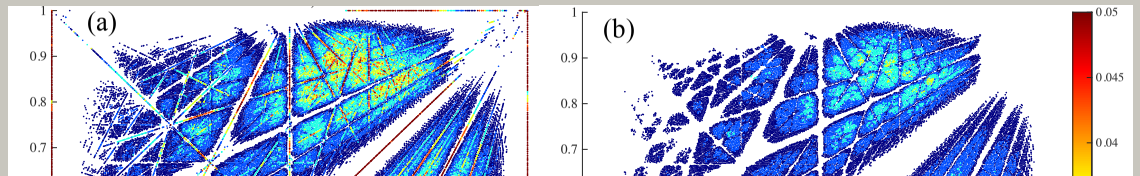
- Xia and Cheng-Sun show when $\varepsilon \ll 1$ there exist many rotational invariant two-tori
- ω Diophantine: $$|m \cdot \omega - n| \geq \frac{c}{|m|^2} \qquad m \in \mathbb{Z}^2 \setminus \{0\}, \quad n \in \mathbb{N}$$
- Key point:
  - Cannot specify frequency vectors.
  - Resonances $m \cdot \omega = n$



Xia, Z. (1992). "Existence of invariant tori in volume-preserving diffeomorphisms." Erg. Th. Dyn. Sys. **12**(3): 621-631.

Persistent Tori

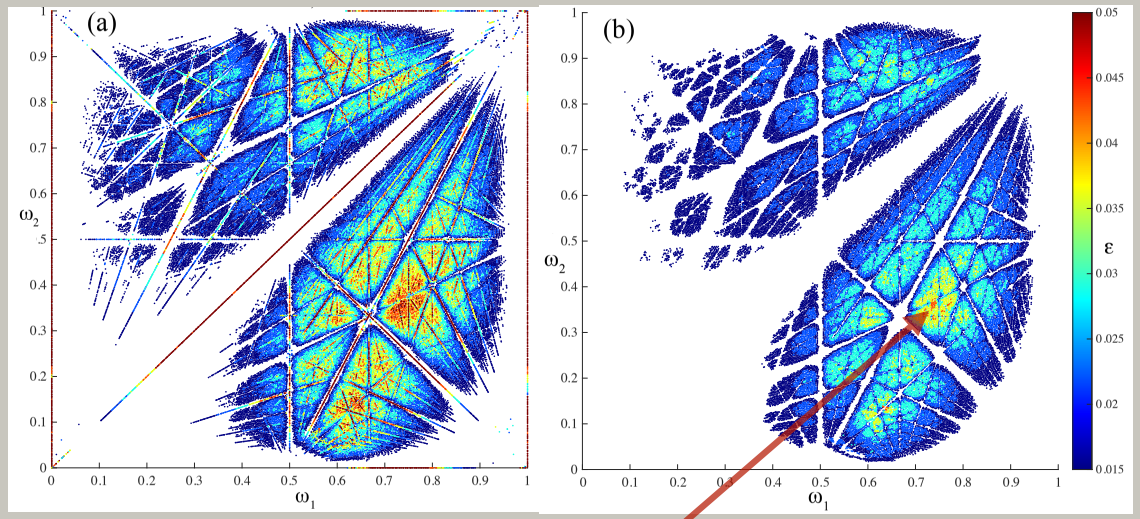
$T = 10^6$



Xia, Z. (1992). "Existence of invariant tori in volume-preserving diffeomorphisms." Erg. Th. Dyn. Sys. **12**(3): 621-631.

Persistent Tori

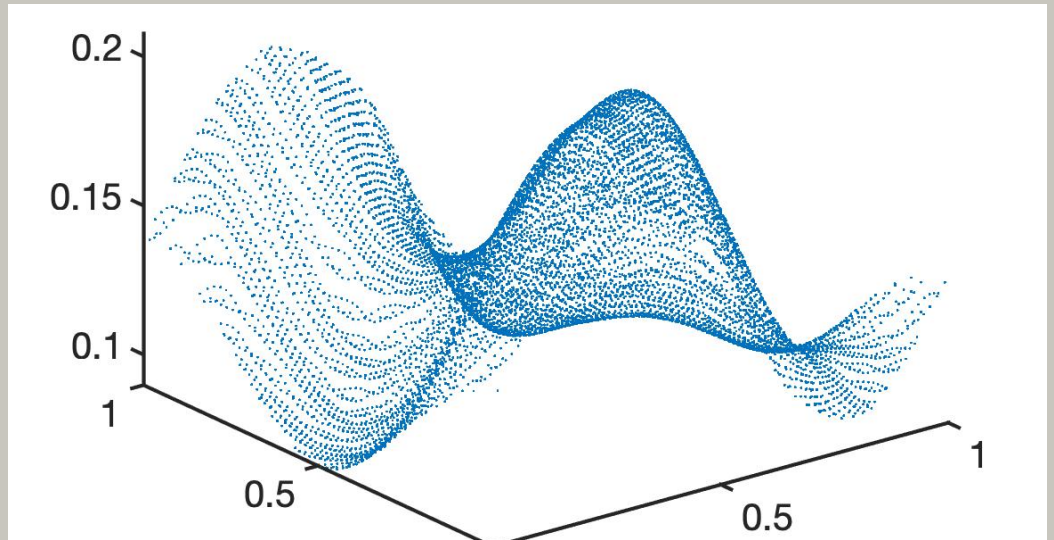
$T = 10^6$

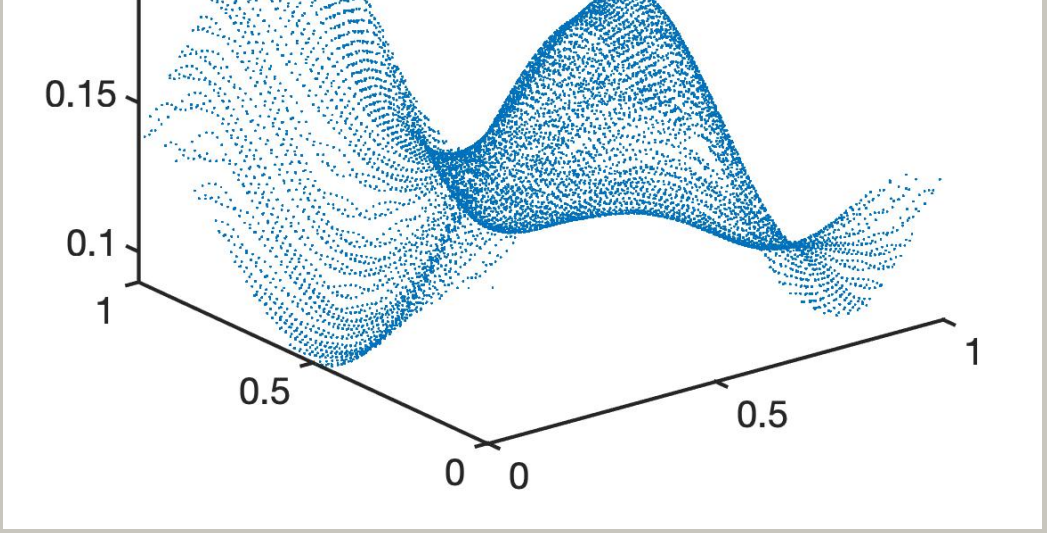


Resonances Tori

Most Robust Torus?

Critical Torus?





$\varepsilon = 0.045, \delta = -0.305$

- Meiss, J. D. (2012). "The Destruction of Tori in Volume-Preserving Maps." Comm. Nonlinear Sci. Numer. Simulat. 17: 2108-2121.
- Dullin, H. R. and J. D. Meiss (2012). "Resonances and Twist in Volume-Preserving Mappings." SIAM J. Appl. Dyn. Sys. 11: 319-349.
- Fox, A. M. and J. D. Meiss (2013). "Greene's Residue Criterion for the Breakup of Invariant Tori of Volume-Preserving Maps." Physica D 243(1): 45-63.
- Das, S., Y. Saiki, E. Sander and J. A. Yorke (2017). "Quantitative quasiperiodicity." Nonlinearity 30(11): 4111.
- Das, S. and J. A. Yorke (2018). "Super convergence of ergodic averages for quasiperiodic orbits." Nonlinearity 31(2): 491-501.
- Sander, E. and J. D. Meiss (2020). "Birkhoff Averages and Rotational Invariant Circles for Area-Preserving Maps." Physica D 411: 132569.
- Meiss, J. D. and E. Sander (2021). "Birkhoff Averages and the Breakdown of Invariant Tori in Volume-Preserving Maps." Physica D 428(15): 133048.
- Duignan, N. and J. D. Meiss (2023). "Distinguishing between Regular and Chaotic orbits of Flows by the Weighted Birkhoff Average." Physica D 449(July): 133749.
- Sander, E. and J.D. Meiss (2023) "Weighted Birkhoff Averages for Torus Maps." ArXiv: <https://arxiv.org/abs/2310.11600>

- Duignan, N. and J. D. Meiss (2023). “Distinguishing between Regular and Chaotic orbits of Flows by the Weighted Birkhoff Average.” Physica D 449(July): 133749.

- Sander, E. and J.D. Meiss (2023) “Weighted Birkhoff Averages for Torus Maps.” ArXiv: <https://arxiv.org/abs/2310.11600>