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Restriction theorems for curves

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X : a smooth Fano variety over $\mathbb{C}$.

$\mathcal{E}$: a vector bundle on X .

Problem

Given a family of curves C on X , how does $\mathcal{E}|_C$ change as we vary C ?

More precisely, suppose M denotes a parameter space of curves on X .

Question

What is the codimension of the sublocus of M parametrizing curves C such that $\mathcal{E}|_C$ “jumps” in some way?

Classical example: jumping lines in $\mathbb{P}^n$ (Barth, Hulek,...)

Application: understanding singular locus of

$\text{Mor}(\mathbb{P}^1, X)_d =$ parameter space of rational
curves of anticanonical degree d .

Example (Browning-Sawin)

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree s such that

$$n > 3(s-1)2^{s-1}.$$

There is a linear function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ such that the codimension of the singular locus of $\text{Mor}(\mathbb{P}^1, X)_d$ is bounded below by $\lfloor \ell(d) \rfloor$.

Example (Starr)

Let X be a smooth cubic threefold. Then for any $d \geq 3$

$$\mathrm{Mor}(\mathbb{P}^1, X)_d = M_d \cup \underbrace{R_d}_{d\text{-fold covers of lines}}$$

The intersection $M_d \cap R_d$ consists of degree d covers of non-free lines.

In particular, the singular locus of $\mathrm{Mor}(\mathbb{P}^1, X)_d$ has codimension 1 for every d .

Theorem

Let X be a smooth Fano variety. There is a linear function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ with positive leading coefficient satisfying the following.

Suppose $V \subset \text{Sing}(\text{Mor}(\mathbb{P}^1, X)_d)$. Then either

- 1** *$\text{codim}(V) \geq \lfloor \ell(d) \rfloor$, or*
- 2** *the evaluation map over V factors through an “accumulating map” $f : Y \rightarrow X$.*

Remark

Recently Peyre formulated a version of Manin's Conjecture using the notion of an “arithmetic slope” of a rational point. His conjectures suggests certain structural results about the set of rational points with given slopes.

Our work addresses similar issues for curves, clarifying the structure in the geometric case.

A key tool: slope stability. We will use a notion developed by Campana and Peternell.

Definition

Let X be a smooth projective variety, $\mathcal{T}$ a torsion-free coherent sheaf on X . Suppose α is a nef curve class on X . The α -slope of $\mathcal{T}$ is

$$\mu_{\alpha}(\mathcal{T}) = \frac{c_1(\mathcal{T}) \cdot \alpha}{\mathrm{rk}(\mathcal{T})}.$$

We say that $\mathcal{T}$ is α -semistable if every non-zero coherent subsheaf $\mathcal{F} \subset \mathcal{T}$ satisfies $\mu_{\alpha}(\mathcal{F}) \leq \mu_{\alpha}(\mathcal{T})$. This leads to a theory of α -Harder-Narasimhan filtrations.

The following definition identifies the discrete data in the Harder-Narasimhan filtration.

Definition

Let X be a smooth projective variety, $\mathcal{T}$ a torsion-free coherent sheaf on X . Suppose α is a nef curve class on X . Write the α -HN-filtration of $\mathcal{T}$ as

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_s = \mathcal{T}.$$

The slope panel $\mathrm{SP}_{X,\alpha}(\mathcal{T})$ is the non-increasing ordered $\mathrm{rk}(\mathcal{T})$ -tuple consisting of

- $\mathrm{rk}(\mathcal{F}_1/\mathcal{F}_0)$ copies of $\mu_\alpha(\mathcal{F}_1/\mathcal{F}_0)$,
- $\mathrm{rk}(\mathcal{F}_2/\mathcal{F}_1)$ copies of $\mu_\alpha(\mathcal{F}_2/\mathcal{F}_1)$,
- etc.

Notation

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Notation: X a smooth projective variety, $\mathcal{E}$ a vector bundle on X , C a smooth projective curve

We denote a family of curves using:

$$\begin{array}{ccc} & \mathcal{U} & \longrightarrow X \\ & \downarrow & \\ \mathrm{Mor}(C, X) & \supset & W \end{array}$$

ev will denote the composition of the normalization of $\mathcal{U}$ with the map $\mathcal{U} \rightarrow X$.

Observation

Suppose that either:

- 1 ev is not dominant, or
- 2 ev is dominant with disconnected fibers.

Then there is a generically finite non-birational morphism $f : Y \rightarrow X$ from a smooth projective Y such that ev factors rationally through f . We can furthermore ensure that ev defines a family of curves on Y whose evaluation map is dominant with connected fibers.

Thus every family of curves falls into one of the following two categories:

- Case (1) ev is dominant with connected fibers.
- Case (2) ev factors rationally through a generically finite $f : Y \rightarrow X$ as above.

We will use different techniques to address these two cases.

Case (1): $\mathcal{E}$ matters, Fano irrelevant

Case (2): Fano matters, $\mathcal{E}$ irrelevant

The prototypical restriction theorem is:

Theorem (Mehta-Ramanathan)

Let X be a smooth projective variety, $\mathcal{E}$ a vector bundle on X . Fix a “sufficiently ample” divisor H on X .

Suppose C is a curve which is a general complete intersection of elements of $|H|$. Then

$$\text{HN-filt of } \mathcal{E}|_C = \text{restriction of } [C]\text{-HN-filt of } \mathcal{E}$$

More generally, suppose $s : C \rightarrow X$ is a general member of a dominant family of curves on X . Then we expect

$$\mathrm{SP}_C(s^*\mathcal{E}) \approx \mathrm{SP}_{X,s_*[C]}(\mathcal{E}).$$

Under certain natural assumptions, this is verified by a Grauert-Mulich type theorem of L-Riedl-Tanimoto: the difference between the two is bounded by a function of $\dim(X), g(C), \mathrm{rk}(\mathcal{E})$.

Theorem

Let X be a smooth projective variety, $\mathcal{E}$ a torsion-free coherent sheaf on X .

Suppose $M \subset \text{Mor}(C, X)$ is an irreducible component satisfying:

- *the map ev is dominant with connected fibers,*
- *the general map $s : C \rightarrow X$ parametrized by M is birational onto its image,*
- *the flat locus of ev contains the general fiber of $\mathcal{U} \rightarrow M$.*

Then for $s : C \rightarrow X$ general in M we have

$$\|\text{SP}_C(s^* \mathcal{E}) - \text{SP}_{X, s_*[C]}(\mathcal{E})\|_{\text{sup}} \leq Q(\dim(X), g(C)) \cdot \text{rk}(\mathcal{E})$$

for an (explicit) constant $Q(\dim(X), g(C))$.

Remark

We can only expect a result of this type when the evaluation map is dominant with connected fibers.

Otherwise ev factors rationally through a morphism $f : Y \rightarrow X$ as explained earlier. Then we should expect (and the theorem proves) that $s^*\mathcal{E}$ is controlled by the stability of $f^*\mathcal{E}$ on Y and not by $\mathcal{E}$ on X .

Remark

If we impose additional assumptions on the positivity of s^*T_X then the constant Q becomes smaller.

Idea of proof: Suppose for simplicity that $\mathcal{E}$ is α -semistable. Consider the map $ev : \mathcal{U} \rightarrow X$ and suppose that $\mathcal{F}$ is a term in the relative HN-filtration of $ev^*\mathcal{E}$ over M . We would like to show that if $\mathcal{F}$ is “too positive”, then it descends to a sheaf on X , violating semistability.

It suffices to prove that the map induced by $\mathcal{F}$

$$\mathcal{U} \rightarrow \mathrm{Gr}(\mathcal{E}, \mathrm{rk}(\mathcal{F}))$$

factors through X , or equivalently, that the map $T_{\mathcal{U}/X} \rightarrow ev^*T_{\mathrm{Gr}/X}$ vanishes. Fix a general curve C in our family. If $\mathcal{F}$ is “too positive” then $T_{\mathrm{Gr}/X}|_C = \mathrm{Hom}(\mathcal{F}, \mathcal{E}/\mathcal{F})|_C$ is very negative. On the other hand $T_{\mathcal{U}/X}|_C$ is (more-or-less) a syzygy bundle for s^*T_X , which is not very negative by a result of Butler. Thus there are no non-zero maps between them.

Next suppose that W is a family of curves in Case (1) and that the general curve $s : C \rightarrow X$ parametrized by W satisfies

$$\|\mathrm{SP}_C(s^*\mathcal{E}) - \mathrm{SP}_{X,s_*[C]}(\mathcal{E})\|_{\mathrm{sup}} = \mu.$$

Deformation theory suggests that the codimension of W is bounded by a linear function in μ .

This expectation is verified by the following theorem.

Theorem

Let X be a smooth projective variety, $\mathcal{E}$ a torsion-free coherent sheaf on X . Fix an ample divisor H on X , a positive constant μ , and a closed cone $\mathcal{C} \subset \text{Nef}_1(X)$ such that $\mathcal{C} \setminus \{0\}$ is contained in $\text{Nef}_1(X)^\circ$.

There are linear functions $\ell, B : \mathbb{R} \rightarrow \mathbb{R}$ with positive leading coefficients satisfying the following: Suppose $W \subset \text{Mor}(C, X)$ satisfies:

- the map ev is dominant with connected fibers,
- the general map $s : C \rightarrow X$ parametrized by W is birational onto its image,
- the curves parametrized by W have class contained in $\mathcal{C}$ and have H -degree $\geq B(\mu)$,

Suppose that for $s : C \rightarrow X$ general we have

$$\|\text{SP}_C(s^* \mathcal{E}) - \text{SP}_{X, s_*[C]}(\mathcal{E})\|_{\text{sup}} \geq \mu.$$

Then $\text{codim}(W) \geq \lfloor \ell(\mu) \rfloor$.

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Idea of proof: First suppose that ev is flat along a general fiber of $\mathcal{U} \rightarrow M$. Then the proof is similar to the earlier result: we try to descend sheaves over $\mathcal{U} \rightarrow X$.

Next suppose ev is not flat. Then we pass to a birational model $\phi : X' \rightarrow X$ that flattens ev . The earlier result implies that it suffices to understand the stability of $\phi^* \mathcal{E}$ with respect to the strict transforms C' of our curves to X' .

Problem

How does stability behave with respect to strict transforms of curves under birational maps?

(Stability is preserved by pulling back curves, but not by strict transforms.)

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Suppose $\alpha \in \text{Nef}_1(X)$ is the class of C and $\alpha' \in \text{Nef}_1(X')$ is the class of C' .

Consider a quotient $\mathcal{E} \rightarrow \mathcal{Q}$ and set $\mathcal{Q}' = (\phi^* \mathcal{Q})_{\text{tf}}$. The difference in first Chern classes between $\mathcal{Q}$ and $\mathcal{Q}'$ is controlled by a Fitting ideal of $\mathcal{Q}$.

We estimate the log canonical thresholds of the Fitting ideal of $\mathcal{Q}$. This allows us to compare the difference $\mu_\alpha(\mathcal{Q}) - \mu_{\alpha'}(\mathcal{Q}')$ against the intersection $K_{X'/X} \cdot \alpha'$. In turn, the latter quantity controls the difference in dimension between our original family on X and our strict transform family on X' .

Example

Consider the tangent bundle $T_{\mathbb{P}^2}$. This is stable, but can be destabilized on birational models. The easiest example is the blow-up of a point $\phi : X' \rightarrow X$. If we set $\alpha' = \phi^*H - tE$, then the relative tangent bundle $T_{X'/\mathbb{P}^1}$ destabilizes $\phi^*T_{\mathbb{P}^2}$ once $t > 1/2$.

We show that this map ϕ is the “most efficient” way to destabilize $T_{\mathbb{P}^2}$ birationally. Thus either:

- 1 $T_{\mathbb{P}^2}$ remains semistable under pulling back to the flattening model $\phi : X' \rightarrow \mathbb{P}^2$ (and we get linear bound from earlier result), or
- 2 $\frac{\dim(W)}{\dim(M)} \lesssim \frac{5}{6}$.

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Now suppose we are in Case (2): W parametrizes a family of curves on X that come from a generically finite non-birational morphism $f : Y \rightarrow X$. In this case we will always assume that X is Fano.

Since we do not have good control on how the stability of $f^*\mathcal{E}$ behaves with respect to curve classes on Y , we can no longer use restriction theorems. Instead, we will ignore $\mathcal{E}$ and use invariants from the MMP to analyze how f interacts with families of curves.

Fujita invariant

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Definition

Let Y be a smooth projective variety, L a big and nef divisor on Y . The Fujita invariant is

$$a(Y, L) = \inf\{t \in \mathbb{R} \mid K_Y + tL \text{ is pseudo-effective}\}.$$

(If Y is singular, take a resolution $\phi : Y' \rightarrow Y$ and set $a(Y, L) = a(Y', \phi^*L)$.)

The Fujita invariant $a(Y, -f^*K_X)$ measures the “amount of negativity” of K_X when compared against K_Y . Thus we can expect it to control the “amount of curves” on X that come from Y .

Theorem

Let X be a smooth Fano variety. Fix a positive rational number a . There is a linear function $B : \mathbb{R} \rightarrow \mathbb{R}$ such that the following holds:

Suppose $W \subset \text{Mor}(C, X)$ satisfies:

- *ev factors rationally through a generically finite morphism $f : Y \rightarrow X$ and defines a dominant family of curves on Y ,*
- *W parametrizes curves of anticanonical degree $\geq B(a)$,*
- *letting $M \subset \text{Mor}(C, X)$ denote the irreducible component containing W , we have $\dim(W) \geq a \dim(M)$*

Then

$$a(Y, -f^* K_X) \geq a.$$

Example

Consider finite dominant morphisms $f : Y \rightarrow \mathbb{P}^2$ with Y smooth. The largest possible values of $a(Y, -f^*K_{\mathbb{P}^2})$ are:

- 1 $a = \frac{2}{3}$. Then Y is a quadric surface and f is a projection to $\mathbb{P}^2$.
- 2 $a = \frac{1}{2}$. Then $Y = \mathbb{P}^2$ and f is a degree 2 map.
- 3 etc.

These can contribute families of curves on $\mathbb{P}^2$ whose dimension is roughly $\frac{2}{3}, \frac{1}{2}$, etc., of the dimension of the ambient space.

Maps with Fujita invariant at least 1 are particularly useful.

Definition

Let X be a smooth Fano variety and let $f : Y \rightarrow X$ be a generically finite non-birational morphism. We say that f is an accumulating map if $a(Y, -f^*K_X) \geq 1$.

Following Birkar's work, we know that accumulating maps satisfy a certain type of “boundedness”.

Theorem

Let X be a smooth Fano variety. There is a linear function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ with positive leading coefficient such that the following holds:

Suppose $W \subset \text{Mor}(C, X)$ parametrizes curves of anticanonical degree d such that ev factors rationally through a generically finite morphism $f : Y \rightarrow X$ and defines a dominant family of curves on Y . Then either:

- 1** $\text{codim}(W) \geq \lfloor \ell(d) \rfloor$, or
- 2** f is an accumulating map.

Summary: let X be a smooth Fano variety, $\mathcal{E}$ a vector bundle on X .

Suppose $W \subset \text{Mor}(C, X)$ parametrizes a family of curves on X of anticanonical degree d and such that

$$\|\text{SP}_C(s^*\mathcal{E}) - \text{SP}_{X, s_*[C]}(\mathcal{E})\|_{\text{sup}} = \mu.$$

- Case 1: assuming the class of C is “sufficiently interior” to $\text{Nef}_1(X)$, we get $\text{codim}(W) \geq \lfloor \ell_1(\mu) \rfloor$.
- Case 2: either $\text{codim}(W) \geq \lfloor \ell_2(d) \rfloor$ or ev factors rationally through an accumulating map.