

Mathematical modeling for stellarator design: Symmetries and optimization

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CONTEXT AND MOTIVATION

Context and motivation

- ▶ *Promise of carbon-free, safe, dispatchable baseload energy production with widely available fuel*
- ▶ Towards sustained fusion energy

Hidden symmetries and Fusion energy collaboration

- ▶ Website <https://hiddensymmetries.princeton.edu/>
- ▶ Focus on stellarator design
 - ▶ Magnetic field (and more) to confine particles
 - ▶ Breaking the toroidal symmetry
- ▶ Leveraging other symmetries
 - ▶ Optimization theory, plasma physics, dynamical systems, analysis of partial differential equations
- ▶ Goals
 - ▶ Stellarator optimization code for large scale computation
 - ▶ A few optimal designs

Need for a common vocabulary

- ▶ Magnetic nuclear fusion
 - ▶ Non-linear coupling between plasma and electromagnetic fields
- ▶ Toroidal geometry : toroidal and poloidal angle

Stellarator design

- ▶ Equilibrium magnetic field + Coils
- ▶ Two-stage approach vs single-stage approach

Toroidal field modeling

- ▶ Ideal MHD equilibrium
- ▶ Flux surfaces
 - ▶ *Assumption of surfaces* vs Multiregion approach
 - ▶ Flux/magnetic/Boozer coordinates

Coil modeling

- ▶ Winding surface model vs filamentary model

SYMMETRIES IN STELLARATORS

Symmetries

- ▶ Structure-preserving transformations of a physical system
 - ▶ Discrete transformations
 - ▶ Continuous transformations

Symmetries in physics

- ▶ Strong connection
continuous symmetries $\leftrightarrow$ **conserved quantities**
 - ▶ Noether's theorem

Symmetries in stellarators

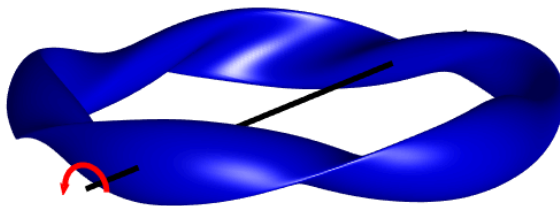
- ▶ Magnetic field symmetry
 - ▶ Discrete symmetry : global simplification of the geometry
 - ▶ Continuous symmetry : improved confinement properties
- ▶ Theory versus practice
 - ▶ Abstract definition
 - ▶ Mathematical modeling
 - ▶ Approximation in an actual stellarator

Discrete symmetries

Stellarator symmetry : discrete reflection symmetry

- In cylindrical coordinates :

$$\begin{cases} -B_R(R, \phi, Z) &= B_R(R, -\phi, -Z) \\ B_\phi(R, \phi, Z) &= B_\phi(R, -\phi, -Z) \\ B_Z(R, \phi, Z) &= B_Z(R, -\phi, -Z) \end{cases}$$

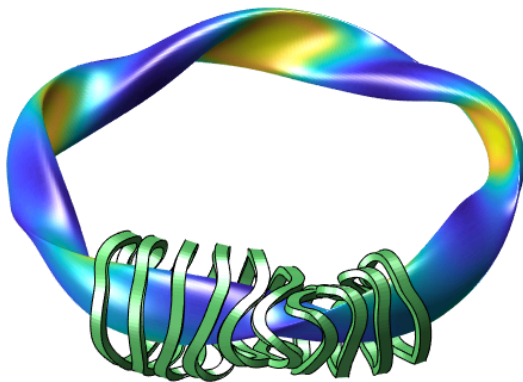


W7-X configuration (stellarator symmetric)
Last closed magnetic surface

- Leverage parity in Fourier representations w. flux coordinates

N_P symmetry or periodicity in the number of field periods

- ▶ In cylindrical coordinates : $\mathbf{B}(R, \phi + 2\pi/N_P, Z) = \mathbf{B}(R, \phi, Z)$
- ▶ In flux coordinates : $\mathbf{B}(\psi, \theta, \zeta + 2\pi/N_P) = \mathbf{B}(\psi, \theta, \zeta)$



W7-X configuration (5 toroidal field periods)

Last closed magnetic surface with field strength and coils over one period

- ▶ Choice for the value of N_P ?

Quasisymmetry

- ▶ *Quasi* : symmetry of the field strength B
 - ▶ Unlike symmetry of the vector field $\mathbf{B}$ (as in axisymmetry)
- ▶ Benefit : confinement of guiding center trajectories

A formal definition

- ▶ Assumption of continuously nested toroidal flux surfaces
- ▶ Relies on notion of Boozer coordinates
 - ▶ $(\psi, \vartheta_B, \varphi_B)$
- ▶ B independent of a linear combination of the two angles
 - ▶ $\chi = M\vartheta_B - N\varphi_B$

Types of quasisymmetry

- ▶ Quasi-axisymmetry $N = 0$
Designs : Aries-CS, CFQS, ESTELL, QuASDEX, MUSE, NCSX
- ▶ Quasi-helical symmetry $MN \neq 0$
Exp : Helically Symmetric Experiment (HSX) at U Wisconsin

Guiding center motion in quasisymmetry

Gyroaveraged Lagrangian

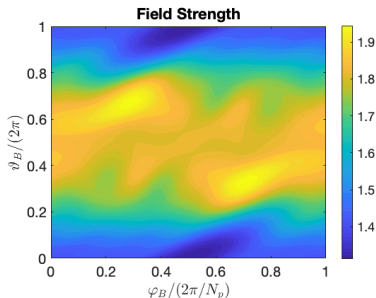
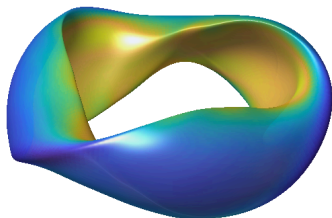
- ▶ Assumption of strong given stationary magnetic field
 - ▶ Single particle motion : neglect effect of particle on fields
 - ▶ Separation of time and length scale :
Motion **along** vs fast motion **across** field lines
(non-periodic) (periodic)
guiding center gyromotion
- ▶ Derivation of the Lagrangian for the guiding center motion $\mathcal{L}$
 - ▶ Changes of coordinates and **asymptotic expansion**

Properties of the Lagrangian and the gyromotion

- ▶ Conservation of the canonical momentum for $\mathcal{L}$
 - ▶ Similar to angular momentum conservation in axisymmetry
 - ▶ Integrability of the guiding center motion in phase space
- ▶ Consequence on physical space trajectories
 - ▶ Further **asymptotic expansion**
 - ▶ **Approximate confinement** of the motion to flux surfaces

In practice : approximation of quasisymmetry

Field strength of NCSX's quasi-axisymmetry magnetic field

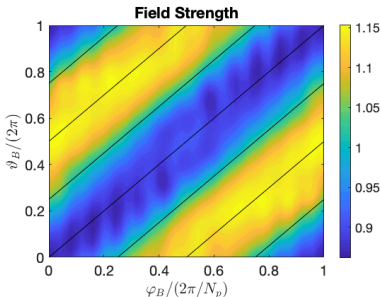
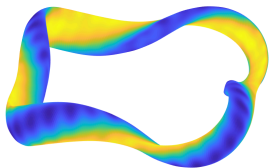


(Left) On the last magnetic surface of NCSX

(Right) As a function of the two Boozer angles

In practice : approximation of quasisymmetry

Field strength of HSX's quasi-helical magnetic field



(Left) On the last magnetic surface of HSX

(Right) As a function of the two Boozer angles

Omnigeneity

- ▶ Generalization of quasisymmetry

Background

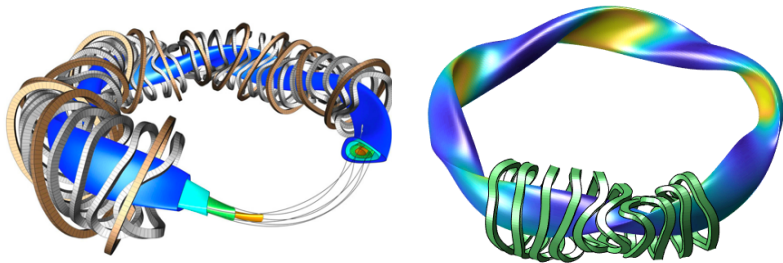
- ▶ Assumption of continuously nested toroidal flux surfaces
- ▶ Based on asymptotic expansion of guiding center motion
 - ▶ Order 0 : motion along field lines
 - ▶ Order 1 : drift perpendicular to field lines
- ▶ Average along field lines
 - ▶ Introduced in terms of field-line following coordinates
- ▶ Condition to cancel the averaged radial drift
 - ⇔ Symmetry of *parallel adiabatic invariant* $J_{||}$ across field lines
- ▶ Drift orbits approximately confined to surfaces of constant $J_{||}$

Definition

- ▶ Surfaces of constant $J_{||}$ aligned with flux surfaces

In practice : approximation of omnigenity

Approximately omnigenous configuration of Wendelstein7X



(Left) Modular and toroidal field coils

(Right) Field strength on the last magnetic surface of W7X

OPTIMIZATION FOR STELLARATOR DESIGN

Three-dimensional magnetic field geometry

- ▶ Extensive configuration space
 - ▶ confinement primarily provided by external fields
- ▶ Explored via numerical optimization
- ▶ As opposed to two-dimensional tokamak geometry
 - ▶ more sensitive to plasma profile changes vs external fields

Different types of objectives to optimize

- ▶ Physics objectives
 - Favorable confinement capabilities
- ▶ Engineering and manufacturing objectives
 - Limiting challenges in coil construction

Reminder : **Two-stage** vs single-stage approach

- ▶ Reduced computational cost
- ▶ Additional flexibility in balancing both aspects

FIRST STAGE

Differential eq. model for the equilibrium magnetic field

Standard model : Ideal MHD equilibrium

- ▶ Force balance $\begin{cases} (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$
- ▶ Prescribed pressure profile
- ▶ Assumption of continuously nested flux surfaces

Two approaches

- ▶ Fixed-boundary problem
 - ▶ Domain boundary = Plasma boundary
 - ▶ Implemented in NSTAB, VMEC, DESC
- ▶ Free-boundary problem
 - ▶ Domain includes plasma, a vacuum region, location of coils
 - ▶ Implemented in free-boundary VMEC

Fixed-boundary problem for MHD equilibrium

Boundary value problem

$$\left. \begin{aligned} (\nabla \times \mathbf{B}) \times \mathbf{B} &= \mu_0 \nabla p, \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned} \right\} \quad \begin{array}{l} \text{in } \Omega \\ \mathbf{B} \cdot \hat{\mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{array}$$

Together with prescribed

- ▶ value of the toroidal flux
- ▶ profile for the pressure
- ▶ profiles for the rotational transform or the toroidal current

Free-boundary problem for MHD equilibrium (for comparison)

Seek plasma boundary $\partial\Omega$ within the domain

- ▶ Given coil currents and positions, the total toroidal flux, pressure profile, rotational transform profile
- ▶ Start from initial guess for $\partial\Omega$

Approximation via an iterative process

Repeat the following steps

- ▶ Determine $\mathbf{B}$ in Ω using a fixed-boundary approach
Evaluate the total pressure along $\partial\Omega$ from the inside
- ▶ Determine $\mathbf{B}$ from a vacuum field model outside of Ω
Evaluate the total pressure along $\partial\Omega$ from the outside
- ▶ While the total pressures do not match,
update the boundary $\partial\Omega$

Fixed-boundary MHD optimization

Framework

- ▶ Given boundary $\partial\Omega$:
equilibrium field inside the plasma $\mathbf{B}(\partial\Omega)$
defined as solution of the fixed-boundary problem
- ⇒ Optimize the location of the plasma boundary
With respect to various physics objectives - with target values

Parameterization of the boundary

- ▶ Toroidal surface : poloidal angle θ and toroidal angle ϕ
- ▶ Via Fourier series with coefficients $\mathcal{C}$
- ▶ Define $\partial\Omega(\mathcal{C})$ in terms of R, Z the cylindrical coordinates

Optimization problem

$$\min_{\mathcal{C}} \chi^2 \left(\mathbf{B}(\partial\Omega(\mathcal{C})) \right).$$

Complete optimization process

Given an optimization algorithm and a fixed-boundary pb solver

1. Find Fourier coefficients $\mathcal{C}^{init}$ and $\partial\Omega^{init} = \partial\Omega(\mathcal{C}^{init})$
 - 1.1 Compute $\mathbf{B}^{init} = \mathbf{B}(\partial\Omega^{init})$
 - 1.2 Evaluate objective $\chi^2(\mathbf{B}^{init})$
2. Until χ^2 satisfies a stopping criterion, repeat the following :
 - 2.1 Adjust $\partial\Omega$ according to an optimization algorithm
 - 2.2 Compute $\mathbf{B}^{new} = \mathbf{B}(\partial\Omega)$
 - 2.3 Evaluate the objective $\chi^2(\mathbf{B}^{new})$

In practice

- ▶ Finite Fourier series for the boundary representation
- ▶ Code implementing the fixed-boundary problem solver
- ▶ Code implementing the optimization algorithm

Some common stellarator equilibrium objectives

Physics parameter	Equilibrium properties	Stability	Collisionless part. confinement	Neoclassical part. confinement
Island width	✓			
Shafranov shift	✓			
Magnetic shear	✓	✓		
Bootstrap current	✓	✓		
Geodesic curvature	✓	✓	✓	✓
Mercier criterion		✓		
Magnetic well		✓		
Quasisymmetry			✓	✓
Omnigeneity			✓	✓
Effective ripple				✓

SECOND STAGE

Stellarator coil modeling

Coils are modeled as support of a current

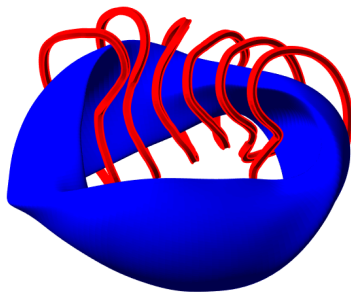
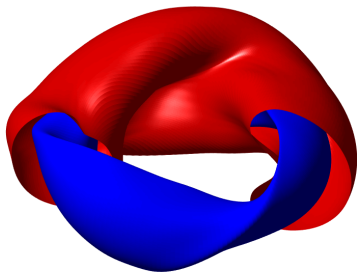
Winding surface model

Restricted to a toroidal surface

For example uniformly offset from the plasma boundary

Filamentary model

Restricted to a finite number of filamentary lines



Coil design

Problem setting

- ▶ Given plasma boundary $\partial\Omega$ and $\mathbf{B}^{plasma}$ generated by plasma
- ▶ Find currents outside Ω to produce the field inside Ω

Magnetic field generated by currents

- ▶ Magnetostatics assumptions

$$\partial\mathbf{E}/\partial t = 0 \quad \text{and} \quad \partial\mathbf{B}/\partial t = 0$$

- ▶ Vacuum region Ω_{vac} without currents

$$\begin{cases} \mathbf{B} = \nabla\Phi_B \\ \Delta\Phi_B = 0 \end{cases} \quad \text{in } \Omega_{vac}$$

- ▶ Electromagnetic coils outside of Ω_{vac} supporting current $\mathbf{J}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \times \int_{\mathbb{R}^3 \setminus \Omega_{vac}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'$$

Coil design

- ▶ Assume a volume Ω_c includes the support of the current $\mathbf{J}$ and is bounded away from Ω

Total magnetic field

$\mathbf{B} = \mathbf{B}^{curr} + \mathbf{B}^{plasma}$ must satisfy $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$

Seek $\mathbf{J}$ such that

$$\forall \mathbf{r} \in \partial\Omega, \mathbf{B}^{plasma}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int_{\Omega_c} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}') \cdot \hat{\mathbf{n}}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}'$$

- ▶ First kind integral equation
- ▶ Ill-posedness and regularization

Parameterization of the support of the current and optimization problem

- ▶ Coefficients $\mathcal{C}$

$$\min_{\mathcal{C}} \chi^2(\mathbf{J}(\mathcal{C}))$$

Some common engineering metrics for stellarator optimization

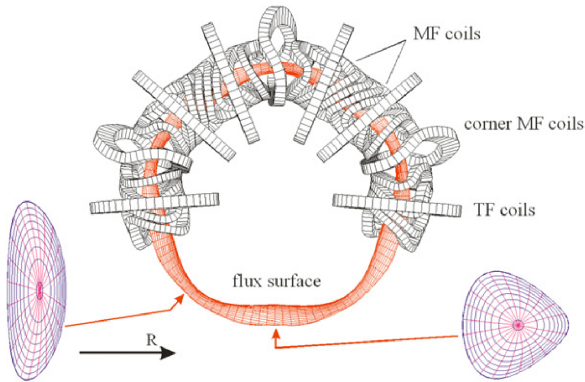
- ▶ *Coil length* : costly conducting material
- ▶ *Coil torsion* : strains on the conducting material
- ▶ *Coil curvature* : bends may reduce the integrity of material
min. radius of curvature due to the cross-section of material
- ▶ *Coil-coil spacing* :
space needed for diagnostic and maintenance access
- ▶ *Coil-plasma distance* :
room needed for other components of the experiment

Some common engineering metrics for stellarator optimization

Maximum current density	$J_{\max} = \max_{\theta, \zeta} \mathbf{J}(\theta, \zeta) $
Curvature	$f_{\kappa} = \int_{S_{\text{coil}}} \kappa^2 \mathbf{J} ^2 d^2 r$
Coil-plasma spacing	$\min_{\theta, \zeta, \theta_p, \zeta_p} \text{dist}(\mathbf{R}_P(\theta, \zeta), \mathbf{R}_W(\theta_p, \zeta_p))$
Length	$L_k = \oint_{C_k} d\ell_k$
Curvature	$\kappa(\ell) = \mathbf{R}_k''(\ell) $
Torsion	$\tau_k(\ell) = \frac{ \mathbf{R}_k'(\ell) \cdot (\mathbf{R}_k''(\ell) \times \mathbf{R}_k'''(\ell)) }{ \mathbf{R}_k''(\ell) ^2}$
Coil-coil separation	$\min_{i \neq j, \ell} \text{dist}(\mathbf{R}_i(\ell), \mathbf{R}_j(\ell))$
Coil-plasma separation	$\min_{k, \theta_p, \zeta_p, \ell} \text{dist}(\mathbf{R}_k(\ell), \mathbf{R}(\theta_p, \zeta_p))$

EXAMPLES OF (OPTIMIZED) STELLARATOR EXPERIMENTS

Wendelstein 7-Advanced Stellarator (W7-AS)



- ▶ Operating from 1988 to 2002
- ▶ Experimental verification of improved properties compared to previous version

Summary of experimental devices

Device	Location	Type	N_P	R [m]	a [m]	B [T]	Status
Heliotron E	Kyoto	hel	19	2.2	0.2	1.9	1980-1997
ATF	Oak Ridge	hel	12	2.1	0.27	2	1988-1994
W7-AS	Garching	O	5	2.0	0.18	3	1988-2002
CHS	Nagoya/Toki	hel	8	1.0	0.2	2	1989-2006
H1	Canberra	heliac	3	1.0	0.2	0.5	1992-2017
TJ-II	Madrid	heliac	4	1.5	0.2	1	1998-now
LHD	Toki	hel	10	3.9	0.6	3	1998-now
HSX	Madison	QH, O	4	1.2	0.15	1.4	1999-now
Heliotron J	Kyoto	hel	4	1.2	0.2	1.5	2000-now
CNT	Columbia	IL, O	2	0.30	0.15	0.2	2004-2017
CTH	Auburn	hel	2	0.75	0.29	0.7	2007-now
W7-X	Greifswald	O	5	5.5	0.53	3	2015-now

O for optimized

hel for heliotron