

# Twistors and AdS deformations of celestial symmetries

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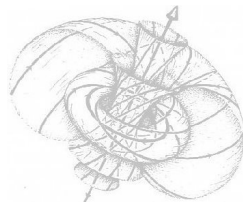
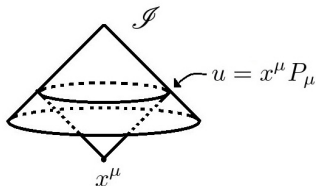
Celestial Simons, New York, 12/4/2024

On: 2403.18011 [Bittleston, Bogna, Kmec, Heuveline, & Skinner]  
also work in progress with Kmec, Ruzziconi & Yellespur.  
& older work with: Adamo & Sharma 2103.16984, 2110.06066,  
2203.02238, Dunajski, 00s.

**Slogan:** Taylor-Zhu's  $\Lambda$ -deformation of celestial  $Lw_{1+\infty}$  via  
Ward's  $\Lambda$ -extension of Penrose's nonlinear graviton.

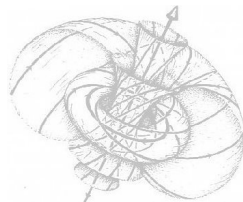
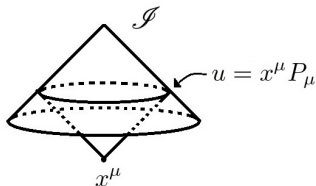
# Holography from null infinity, via light-cones

- Newman '70's: tries to rebuild space-time from 'cuts' of  $\mathcal{I}$ .
- Yields instead ' $\mathcal{H}$ -space' a complex self-dual space-time.
- Penrose:  $\leadsto$  asymptotic Twistor space  $PT$  & *nonlinear graviton*.
- Applies beyond asymptotically flat sector.
- Embodies integrability of SD sector.
- $\leadsto$  hierarchies of hidden symmetries.
- gives Noether charges via twistor actions.
- $\leftrightarrow$  Celestial Holography construction from soft OPEs of S-matrix?



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# Celestial symmetries and their deformations

[Strominger, Guevara, Himwich, Pate, ]

- Postulate amplitudes are correlators in a *celestial CFT*.

$$\mathcal{M}(k_1, k_2, \dots) = \langle \mathcal{O}(k_1) \mathcal{O}(k_2) \dots \rangle$$

- Set  $k_i = \omega_i(1 + |z_i|^2, \Re z_i, \Im z_i, 1 - |z_i|^2)$  soft expansion

$$\mathcal{O}(k) = \sum_p \omega^p H^p(z, \tilde{z})$$

- Collinear limits in amplitudes lead to ‘splitting functions’

$$\mathcal{M}(k_1, k_2, k_3, \dots) \xrightarrow{k_1 \parallel k_2} \text{Split}(k_1, k_2, k_1 + k_2) \mathcal{M}(k_1 + k_2, k_3, \dots)$$

- Splitting functions define OPE coefficients

$$H^{p_1}(z, \tilde{z}) H^{p_2}(0, \tilde{z}) \xrightarrow{z \rightarrow 0} \frac{C_{p_3}^{p_1 p_2}}{z} H^{p_3}(0, \tilde{z}) + \dots$$

- Mode expansion  $\leadsto Lw_{1+\infty} = L\text{ham}(\mathbb{C}^2)$ :

$$[h_{m,a}^p, h_{n,b}^q] = (m(q-1) - n(p-1)) h_{m+n, a+b}^{p+q-2}.$$

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- For AdS amplitudes get  $\Lambda$ -deformation: [Taylor-Zhu, cf Alday, et.al.]

$$\begin{aligned} [h_{m,a}^p, h_{n,b}^q]_\Lambda &= (m(q-1) - n(p-1)) h_{m+n,a+b}^{p+q-2} \\ &\quad - \Lambda(a(q-2) - b(p-2)) h_{m+n,a+b}^{p+q-1}, \end{aligned}$$

- 1 AdS symmetries in twistor space and their hierarchy.
- 2 Higher symmetries  $\leftrightarrow$  gravitons via Ward's extension of Penrose's nonlinear graviton.
- 3 Twistor action for Einstein gravity.
- 4 Noether charges, and Hamiltonians.
- 5 Recursion operators and hierarchies.

# Symmetries in twistor space

Recall 4d spinors:  $SO(4, \mathbb{C}) = SL(2\mathbb{C}) \times SL(2, \mathbb{C})/\mathbb{Z}_2$ ,

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}, \quad \alpha = 0, 1, \dot{\alpha} = \dot{0}, \dot{1}.$$

Nonprojective twistor space  $\mathbb{T} = \mathbb{C}^4 \ni Z^A = (\lambda_\alpha, \mu^{\dot{\alpha}})$ ,

- Conformal group

$$SO(6) = SL(4)/\mathbb{Z}_2 \ni C_A^B = \begin{pmatrix} R_\beta^{-\alpha} & T^{\alpha\dot{\beta}} \\ K_{\dot{\alpha}\beta} & R_{\dot{\alpha}}^{+\dot{\beta}} \end{pmatrix}.$$

- (Anti-) de Sitter groups preserve skew forms  $I_\Lambda^{AB}, I_{AB}^\Lambda$ :

$$I_{AB}^\Lambda = \frac{1}{2} \varepsilon_{ABCD} I_\Lambda^{CD}, \quad I_\Lambda^{AC} I_{BC}^\Lambda = \Lambda \delta_B^A.$$

- $\leadsto$  spin group of (A)dS group  $SO(5) = SP(2)$ .
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# Infinity twistors embedding formalism

- Points  $X$  in space-time  $\leftrightarrow$  lines  $L_X \subset \mathbb{PT}$  joining  $Z_1^A, Z_2^A$

$$L_X = \{\sigma_1 Z_1^A + \sigma_2 Z_2^A\} \longleftrightarrow X^{AB} = Z_1^{[A} Z_2^{B]},$$

- Complex compactified Minkowski space  $\mathbb{CM}^4 \subset \mathbb{CP}^5$

$$\mathbb{CM} = \{X^{AB} \in \mathbb{CP}^5 \mid X \cdot X := \varepsilon_{ABCD} X^{AB} X^{CD} = 0.\}$$

- Choice of infinity twistor  $I_{AB}$  gives

$$\mathcal{I} := \{X \in \mathbb{CM} \mid X \cdot I = 0\}$$

- In Lorentz signature  $I \cdot I = 4\Lambda \leftrightarrow \text{AdS/dS}$ .
- $\mathcal{I}$  is then timelike/spacelike.
- If  $I \cdot I = 0$ ,  $I^{AB} \in \mathbb{CM}$ ,  $\mathcal{I}$  is null and  $\leftrightarrow$  reduce to Poincaré.

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# Twistor Poisson structure for Poincaré ball

- Lines  $L_X \subset \mathbb{PT}$  with  $Z^A = (\lambda_\alpha, \mu^{\dot{\alpha}})$  given by:

$$\mu^{\dot{\alpha}} = x^{\dot{\alpha}\alpha} \lambda_\alpha.$$

- $I^{AB}, I_{AB}$  define Poisson and contact structures

$$\{f, g\}^\Lambda = I_\Lambda^{AB} \frac{\partial f}{\partial Z^A} \frac{\partial g}{\partial Z^B}, \quad \tau = I_{AB}^\Lambda Z^A dZ^B.$$

- Defines infinity by

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- Ball representation  $\mathbb{H}^4 = \{x \in \mathbb{R}^4 \mid x^2 < -2/\Lambda\}$ :

$$\tau = \mu_{\dot{\alpha}} d\mu^{\dot{\alpha}} + \Lambda \lambda^\alpha d\lambda_\alpha \quad \rightsquigarrow \quad \mathcal{I} = \left\{ x \mid x^2 = \frac{-2}{\Lambda} \right\}$$

- Associated Poisson bracket

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Note: could also do upper-half-space.

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- The Lie algebra of the (anti-) de Sitter group

$$sp(2) = \{h_{AB} = h_{(AB)}\}.$$

- Acts via Hamiltonians vector fields for  $\{, \}_\Lambda$  with

$$h(Z) = h_{AB}Z^AZ^B \in \Gamma(\mathbb{PT}, \mathcal{O}(2)).$$

- Locally extends to  $\infty$ -diml Poisson symmetries i.e.

$$h_{m,a}^p = \frac{(\mu^{\dot{0}})^{p+m-1}(\mu^{\dot{1}})^{p-m-1}}{2\lambda_0^{p-2-a}\lambda_1^{p+a-2}} \in \Gamma(U, \mathcal{O}(2)).$$

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- Reduces to celestial  $Lw_{1+\infty}$  for  $\Lambda = 0$ .
- What role do these play?

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# Twistor construction for self-dual space-times with $\Lambda$

Deform flat twistor space:  $\mathbb{PT} = \mathbb{CP}^3$ , projectively in hgs coords:

$$Z = (\lambda_\alpha, \mu^{\dot{\alpha}}) \in \mathbb{T}, \quad Z \sim aZ, a \neq 0.$$

with Poisson bracket:

$$\{f, g\}_\Lambda = I^{AB} \frac{\partial f}{\partial Z^A} \frac{\partial g}{\partial Z^B} = \varepsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial f}{\partial \mu^{\dot{\alpha}}} \frac{\partial g}{\partial \mu^{\dot{\beta}}} + \Lambda \varepsilon_{\alpha\beta} \frac{\partial f}{\partial \lambda_\alpha} \frac{\partial g}{\partial \lambda_\beta}.$$

## Theorem (Ward 1980 after Penrose 1976)

*There is a 1 : 1 correspondence between:*

- *SD Einstein metrics with  $\Lambda$  on regions in  $\mathbb{C}^4$ , and*
- *deformations  $PT$  of region  $\lambda_\alpha \neq 0$  in twistor space  $\mathbb{PT}$  preserving Poisson structure  $\{, \}_\Lambda$  on  $\mathcal{O}(2)$ .*

# Patching functions for deformations

SD metrics  $\leftrightarrow$  deformations  $\mathbb{PT} \rightarrow \mathcal{PT} \leftrightarrow$  Patching functions

- Cover  $\mathbb{PT} - \mathbb{CP}^1$  by coord patches  $Z$  on  $U$  and  $\tilde{Z}$  on  $\tilde{U}$ :

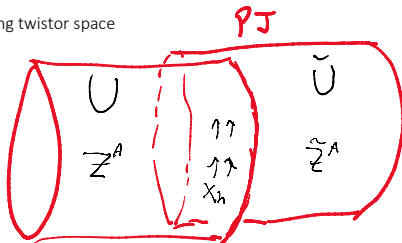
$$U = \{\lambda_0 \neq 0\}, \quad \tilde{U} = \{\lambda_1 \neq 0\}$$

- to preserve  $\{, \}_\Lambda$ , exponentiate Hamiltonian vector field

$$X_h = I^{AB} \frac{\partial h}{\partial Z^A} \frac{\partial}{\partial Z^B}, \quad h \in \Gamma(U \cap \tilde{U}, \mathcal{O}(2)).$$

- on  $U \cap \tilde{U}$  'Patch:'  $\tilde{Z} = \tilde{Z}(Z) = \exp(X_h)Z$
- infinitesimally, gravitons  $\leftrightarrow$  such  $h(Z)$ .

Patching twistor space



# Patching functions and symmetries

## Proposition

*Space of (linearized) SD Einstein metrics  $\simeq \mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$ , coords  $(\mu^{\dot{\alpha}}.\lambda_1)/\lambda_0$ , group of Poisson diffeos.*

- Gravitons  $h$  can be expanded in

$$h_{m,a}^p = \frac{(\mu^{\dot{0}})^{p-m-1}(\mu^{\dot{1}})^{p+m-1}}{\lambda_0^{p-2-a}\lambda_1^{p-2+a}}, \quad p \pm m \in \mathbb{N}, \quad p \pm a \in \mathbb{Z}.$$

- When  $p - 2 - a < 0 \rightsquigarrow$  a coord transformations on  $U$  or, when  $p - 2 + a < 0$ , on  $\tilde{U}$ .
- $\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$  acts on SD Einstein metrics + coord choices on  $U, \tilde{U}$ .
- Reduces to  $L\mathfrak{ham}(\mathbb{C}^2)$  at  $\Lambda = 0$

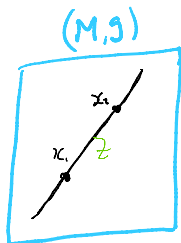
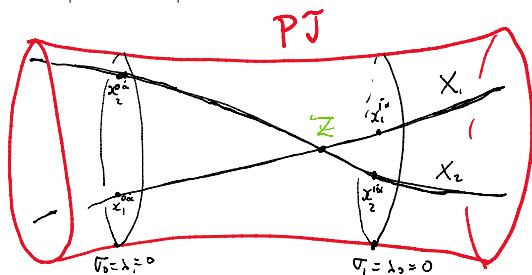
# The self-dual space-time from holomorphic curves

To reconstruct self-dual space-time

$$(M^4, g) = \{ \text{Holomorphic degree-1 } \mathbb{CP}^1 \text{ s in } \mathbb{PT} \}$$

- Parametrize the  $\mathbb{CP}^1_{x,\sigma} \subset \mathbb{PT}$ , with coords  $\sigma$ ,  $Z = Z(x, \sigma)$

Correspondence with space-time



Dolbeault version: deform  $\bar{\partial}$ -operator with Poisson compatible Beltrami differential  $\leadsto$  d-bar eq for  $\mathbb{C}$ -curves in deformed  $\mathbb{PT}$ :

$$\bar{\partial}_\sigma Z^A = \{H, Z^A\}_\Lambda, \quad H \in \Omega^{0,1}(2).$$

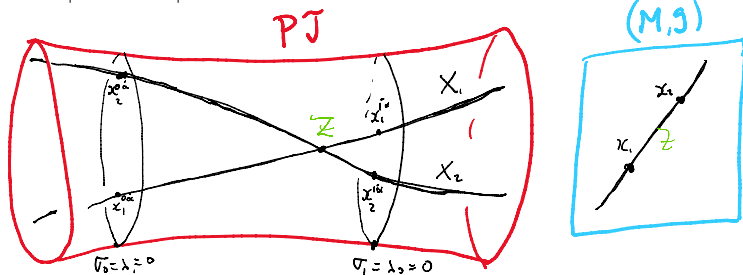
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To go off-shell, use Dolbeault framework:

- Gravitational data, SD:  $H \in \Omega^{0,1}(2)$ , ASD:  $G \in \Omega^{0,1}(-6)$ ,

$$H(aZ) = a^2 H(Z), \quad G(aZ) = a^{-6} G(Z), \quad H = H_{\bar{A}} d\bar{Z}^{\bar{A}} \text{ etc..}$$

- To guarantee compatibility with  $\{, \}$ , deform  $\bar{\partial}$ -operator by

$$\bar{\partial}_H f(Z, \bar{Z}) := \bar{\partial}_0 f + \{H, f\}$$

- Need integrability  $\bar{\partial}_H^2 = 0 \leadsto$  SD field equations

$$\bar{\partial}_0 H + \frac{1}{2} \{H, H\} = 0, \quad \bar{\partial}_H G = 0.$$

- Follow from action, with  $D^3 Z = \frac{1}{24} \varepsilon_{ABCD} Z^A dZ^B \wedge \dots \wedge dZ^D$ ,

$$S[G, H] := \int_{\text{PT}} G \wedge (\bar{\partial} H + \frac{1}{2} \{H, H\}) \wedge D^3 Z.$$

# Hamiltonians for symmetries

Noether analysis  $\leadsto$  local symmetries  $= (g, h) \in (\mathcal{O}(-6), \mathcal{O}(2))$ .

$$\delta(G, H) = (\bar{\partial}_H g + \{h, G\}, \bar{\partial}_H h)$$

So  $[G] \in H^1(\mathcal{PT}, \mathcal{O}(-6))$  and  $H$  a nonlinear  $H^1(\mathcal{PT}, \mathcal{O}(2))$ .

- $h \leftrightarrow$  generators of smooth local Poisson diffeos.
- $g \leftrightarrow$  gauge for  $G \in$  cohomology class.
- Action  $\leadsto$  symplectic form

$$\Omega(\delta_1, \delta_2) = \int_{\Sigma^5} \delta_1 G \wedge \delta_2 H \wedge D^3 Z + (1 \leftrightarrow 2).$$

- Gauge generated by Hamiltonians

$$\mathcal{H}_{g,h} = \int_{\Sigma^5} (\bar{\partial}_H h \wedge G + (\bar{\partial}_0 g + \frac{1}{2}\{H, g\}) \wedge H) \wedge D^3 Z$$

- We can compute the commutators

$$\{\mathcal{H}_{g_1, h_1}, \mathcal{H}_{g_2, h_2}\}_\Omega = \mathcal{H}_{\{h_1, g_2\}_\Lambda - \{h_2, g_1\}_\Lambda, \{h_1, h_2\}_\Lambda}.$$

- 'Large' gauge transformations are only symmetries if holomorphic at  $\partial\Sigma \subset \infty$ ,  $\bar{\partial}_H h = 0 = \bar{\partial}_{Hg}$ .
- On-shell, Hamiltonians given by boundary charges

$$Q_{g,h} = \int_{\partial\Sigma^5} hG + gH D^3 Z.$$

Here  $\partial\Sigma^5 = \mathbb{CP}^1 \times \mathcal{S}$  for  $M \supset \mathcal{S}$ , some 2-surface at  $\infty$ .

- This represents the gauge algebra

$$\{Q_{g_1,h_1}, Q_{g_2,h_2}\}_\Omega = Q_{\{h_1,g_2\}_\Lambda - \{h_2,g_1\}_\Lambda, \{h_1,h_2\}_\Lambda}.$$

- Here to be symmetries,  $(g, h)$  are holomorphic near  $\partial\Sigma$ .
- If global on  $\partial\Sigma^5$ , then  $h = \text{quadratic} \leftrightarrow \text{AdS symmetries}$ , but  $g = 0$ .
- For  $\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$ , need  $(G, H) = 0$  near  $\lambda_0 = 0$  &  $\lambda_1 = 0$ .

- $\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$  etc. arises from AdS symmetries by multiplication by  $\zeta = \lambda_1/\lambda_0$  etc..
- For scalar field standard Penrose transform gives

$$\phi_0(x) = \oint f(x^{0\dot{\alpha}} + x^{1\dot{\alpha}}\zeta, \zeta) \frac{d\zeta}{\zeta}$$

- Define *recursion operator*  $R$  by  $\phi \rightarrow R\phi$  when  $f \rightarrow \zeta f$ .
- Laurent expand  $f(x^{0\dot{\alpha}} + x^{1\dot{\alpha}}\zeta, \zeta) = \sum_{i=-\infty}^{\infty} \phi_i(x) \zeta^{-i}$ .
- Clearly  $\phi_0$  agrees with above and  $\phi_{i+1} = R\phi_i$ .
- It follows from  $\zeta \frac{\partial f}{\partial x^{0\dot{\alpha}}} - \frac{\partial f}{\partial x^{1\dot{\alpha}}} = 0$  that can define  $R$  by

$$\frac{\partial R\phi_i}{\partial x^{0\dot{\alpha}}} := \frac{\partial \phi_{i+1}}{\partial x^{0\dot{\alpha}}} = \frac{\partial \phi_i}{\partial x^{1\dot{\alpha}}}.$$

# Recursion operators

Integrable hierarchies [M. & Woodhouse 1996], [Dunajski, M. 2002,3]

- $\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)$  etc. arises from AdS symmetries by multiplication by  $\zeta = \lambda_1/\lambda_0$  etc..
- For scalar field standard Penrose transform gives

$$\phi_0(x) = \oint f(x^{0\dot{\alpha}} + x^{1\dot{\alpha}}\zeta, \zeta) \frac{d\zeta}{\zeta}$$

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Extends to SDYM [M. & Woodhouse 1996] and SD gravity [Dunajski & M. 2001/2]

## Conclusions:

- Ward's extension of Penrose's nonlinear graviton realizes:  
 $\{\text{space of SD gravitons with } \Lambda\} = \{\mathfrak{ham}_\Lambda(\mathbb{C}^2 \times \mathbb{C}^*)\}.$
- Underpins Taylor-Zhu  $\Lambda$ -extension of  $L\mathfrak{ham}(\mathbb{C}^2)$  at  $\Lambda = 0$ .
- Twistor action yields charge & charge algebra.  
 $\leadsto$  recursion operator and hierarchies for SD sector.

SD sector gives canonical treatment at  $\mathcal{J}$  wip [Kmec, M., Ruzziconi, Yelleshpur]

- At  $\Lambda = 0$  via Penrose's asymptotic twistor spaces.
- For  $\Lambda \neq 0$  use 'Heaven on earth construction' [Lebrun 1983].
- Beyond SD need full twistor actions cf. [M., Boels, Sharma, Skinner].

## Further questions:

- Connections with conventional  $\text{AdS}_4/\text{CFT}_3$  e.g., ABJM?
- Space-time formulations cf [Friedel, Pranzetti, Raclariu].
- Connections with worldsheet methods?

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Thank You!