

Periods of automorphic forms and motivic periods

Joint work in progress with Grobner, Lin, Raghuram, Kobayashi, and Speh

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Outline

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The aim of this talk

I will be presenting two overlapping projects.

The first is a new approach to the project with Grobner and Lin I discussed the last time I was at Schloss Elmau, in 2018. At the time we had results relating automorphic and motivic periods that depended on a strong hypothesis of non-vanishing of twists of Rankin-Selberg L -functions. The work of S.-Y. Chen on L -functions of symmetric powers of modular forms introduces a new method that should allow us to remove that hypothesis. This is joint work in progress of the three of us with Raghuram.

The second is the characterization of the automorphic periods in the Ichino-Ikeda-N. Harris identity that can be interpreted as cup products in coherent cohomology. This is joint with Kobayashi and Speh.

Notation

We let F be a CM field, F^+ its maximal totally real subfield, $c \in \text{Gal}(F/F^+)$.
(I'll assume $F^+ = \mathbb{Q}$ to save notation; there is a more general version.) Let $H = H_n = GL(n)_F$.

Π a cuspidal automorphic representation of H_n .

We always assume

- Π cohomological (with coefficients in $W = W(\Pi)$)
- $\Pi^\vee \xrightarrow{\sim} \Pi^c$

Notation

Let $H' = H_m$, $m < n$, let Π' , $W' = W(\Pi')$ on H' with same hypotheses, not necessarily cuspidal.

W and W' : finite-dimensional irreducible representations of H, H' .

Highest weights $a_1 \geq a_2 \cdots \geq a_n$; $a'_1 \geq a'_2 \cdots \geq a'_m$ (more generally, for each real place of F^+).

Fact

*This implies that the associated motive $M(\Pi)$, of weight $n - 1$, is **regular**: for any p, q with $p + q = n - 1$, $\dim H^{p,q}(M(\Pi)) \leq 1$ (and likewise for $M(\Pi')$ of weight $m - 1$).*

Fontaine-Mazur conjecture

The motives are also **polarized**:

$$M(\Pi) \otimes M(\Pi)^c \rightarrow \mathbb{Q}(1-n); \quad M(\Pi') \otimes M(\Pi')^c \rightarrow \mathbb{Q}(1-m).$$

The modularity conjecture in this setting (roughly):

Conjecture (Fontaine-Mazur)

Any irreducible polarized regular motive over F of rank n over its coefficient field is of the form $M(\Pi)$ for some cohomological cuspidal automorphic Π of H_n .

Polarization plus regularity gives a simplified calculation of the *Deligne period*.

Deligne conjecture for Rankin-Selberg motives

Associated motives $M(\Pi), M(\Pi')$ – over F – of rank n, m with coefficients in $E(\Pi), E(\Pi')$. Set $E = E(\Pi) \otimes E(\Pi')$

$$L(s, R_{F/\mathbb{Q}}[M(\Pi) \otimes M(\Pi')]) = L(s - \frac{n+m-2}{2}, \Pi \times \Pi')$$

(Rankin-Selberg on right for $GL(n)_F \times GL(m)_F$).

Conjecture (Deligne conjecture)

For any critical value s_0 of $L(s, M(\Pi) \otimes M(\Pi'))$ there is an element

$$c(s_0, \Pi, \Pi') \in E \otimes \mathbb{C}^\times / E^\times$$

such that

$$L(s_0, M(\Pi) \otimes M(\Pi')) \sim c(s_0, \Pi, \Pi')$$

(i.e., $L(s_0, M(\Pi) \otimes M(\Pi')) / c(s_0, \Pi, \Pi') \in E$.)

Deligne period for Rankin-Selberg motives

The Deligne period $c(s_0, \Pi, \Pi')$ can be expressed in terms of inner products, depending on the relative position of (a_i) and (a'_j) .

Example (Standard position)

$$m = n - 1, a_i \geq a'_i \geq a_{i+1} \quad \forall i.$$

Let M be any polarized regular motive, for example $M = M(\Pi)$. For $0 \leq i \leq n - 1$ there are (motivic) invariants $Q_i(M)$, defined via cohomological structures M_B, M_{dR} that are free E -modules of the appropriate rank.

Define $P^i(M) = \prod_{k=0}^{i-1} Q_k(M)$, $0 \leq i \leq n - 1$

Also $Q_j(M'), P^j(M')$, $0 \leq j \leq m - 1$.

(For $\tau : F^+ \rightarrow \mathbb{R}$, get $P_\tau^r(M)$, $Q_{r,\tau}(M)$, etc.)

The Q -periods for automorphic motives

The motive $M(\Pi)$ is realized in the cohomology of the Shimura variety $Sh(U(V_1))$ for signature $(1, n-1)$.

Fix a π_f in the (finite adelic) L -packet on $U(V_1)$ attached to Π .

Hodge decomposition:

$$\begin{aligned} M(\Pi)_{dR} \otimes \mathbb{C} &= \bigoplus_{k=1}^n H_!^{k-1}(Sh(U(V_1)), \mathcal{V}^k)[\pi_f] \otimes \mathbb{C} \\ &:= \bigoplus_{k=1}^n H^{p_k, q_k}(M(\Pi)) \otimes \mathbb{C} \end{aligned}$$

(coherent cohomology) for specific automorphic vector bundles $\mathcal{V}^k$ in bijection with the L -packet for $U(1, n-1)$ attached to Π_∞ . Each space on the right has rank 1 over $E \otimes \mathbb{C}$.

The Q -periods

For a general regular polarized motive M of rank n and weight $n - 1$, one has

$$M_{dR} \otimes \mathbb{C} = \bigoplus_{k=1}^n H^{p_k, q_k}(M),$$

with each H^{p_k, q_k} generated by a (rational) ω_k .

Then we let

$$Q_k(M) = \langle \omega_k, F_\infty(\omega_k) \rangle$$

where F_∞ is archimedean Frobenius and $\langle \bullet, \bullet \rangle$ is the polarization. And we define $P^i(M)$ to be the product of Q_k , $0 \leq k \leq i - 1$ as before.

Automorphic Q and P periods

In the automorphic case, let $Q_k(M(\Pi))$ to be the (normalized) Petersson norm of an F -rational generator $\omega_k(\Pi) \in H^{p_k, q_k}(M(\Pi)) \subset H^{p_k, q_k}(M(\Pi)) \otimes \mathbb{C}$.

On the other hand, for each r , Π descends to an L -packet of some $U(V_r)$ of signature $(r, n - r)$ (this is a hypothesis).

We let $P^r(\Pi)$ be the (normalized) Petersson norm of a (rational) *holomorphic* form $\varphi^r(\Pi)$ in this L -packet. Thus $\varphi^1(\Pi) = \omega_1(\Pi)$.

The Factorization Conjecture

Conjecture (Factorization Conjecture)

$P^r(\Pi)$ (the automorphic period) $\sim P^r(M(\Pi))$ (the motivic period).

In other words, up to (rational) multiples,

$$P^r(\Pi) = \langle \varphi^r(\Pi), \varphi^r(\Pi) \rangle \sim \prod_{k=1}^r \langle \omega_k(\Pi), \omega_k(\Pi) \rangle .$$

Then the Factorization Conjecture is a concrete period relation between automorphic periods on inner forms of $U(V_1)$.

It follows from the Tate conjecture (comparison of Galois representations on the cohomology of Shimura varieties).

More to the point, automorphic methods express critical values of L -functions in terms of the $P^r(\Pi)$ (Petersson norms of holomorphic forms on different Shimura varieties) and not the $P^r(M(\Pi))$ (products of Petersson norms of cohomological forms on the fixed $Sh(U(V_1))$).

Deligne period for Rankin-Selberg motives

The following is an application of linear algebra to the motive $R_{F/\mathbb{Q}}M \otimes M'$, where M and M' are regular polarized.

Proposition

For $0 \leq i \leq n$, $0 \leq j \leq m$, there are integers $a(s_0)$, $sp(i, M, M)$, $sp(j, M', M)$ depending only on the Hodge structures, such that

$$c(s_0, M, M') \sim (2\pi i)^{a(s_0)} \prod_i P^i(M)^{sp(i, M, M')} \prod_j P^j(M')^{sp(j, M', M)}$$

Example (Standard position)

$$\begin{aligned} sp(i, \Pi, \Pi') &= sp(j, \Pi', \Pi) = 1 \quad \forall i, j \quad \text{except} \\ sp(0, \Pi, \Pi') &= sp(n, \Pi, \Pi') = 0. \end{aligned}$$

Automorphic version of the Deligne period

Thus the Deligne conjecture for Rankin-Selberg L -functions of cohomological polarized cuspidal automorphic representations comes down to two points:

- The automorphic version of the Deligne conjecture:

$$L(s_0, \Pi \times \Pi') \sim (2\pi i)^{a(s_0)} \prod_i P^i(\Pi)^{sp(i, M, M')} \prod_j P^j(\Pi')^{sp(j, M', M)},$$

with $M = M(\Pi)$, $M' = M(\Pi')$; in other words, an expression in terms of Petersson norms of holomorphic forms on different Shimura varieties;

- The Factorization Conjecture, which identifies the automorphic periods $P^i(\Pi)$, $P^j(\Pi')$ with the motivic periods $P^i(M(\Pi))$, $P^j(M(\Pi'))$.

(The motivic periods are themselves products of periods of the automorphic motive in the cohomology of $Sh(U(V_1))$).

Relative positions and GGP

Assume $m = n - 1$ and recall the coefficient spaces $W = W(\Pi)$, $W' = W(\Pi')$ (which determine the Hodge types) To $(\Pi_\infty, \Pi'_\infty)$ associate a discrete series Vogan L -packet $\mathbb{DP}(W, W')$ for $\{U(V) \times U(V')\}$, $\dim V = \dim V' + 1 = n$, $V' \subset V$.

Fact (GGP, Beuzart-Plessis, H. He)

To each relative position of W, W' there is a unique configuration

$$r(W, W') = (\pi_\infty, \pi'_\infty) \in \mathbb{DP}(W, W')$$

such that $\mathrm{Hom}_{U(V')_\infty}(\pi_\infty \otimes \pi'_\infty, \mathbb{C}) \neq 0$.

Thus

(π, π') in the global L -packet for (Π, Π') has a non-zero period

$$\Rightarrow (\pi_\infty, \pi'_\infty) = r(W, W') \in \mathbb{DP}(W, W').$$

The IINH formula

Fix a finite set S of places of $\mathbb{Q}$, including ∞ , and define

$$\mathcal{L}^S(\pi, \pi') := \frac{L^S(\frac{1}{2}, \Pi \otimes \Pi')}{L^S(1, \pi, Ad)L^S(1, \pi', Ad)}. \quad (5)$$

Then there is an integer β , depending on the L -packets containing π and π' , such that, for $f \in \pi, f' \in \pi'$

$$\mathcal{P}(f, f') = 2^\beta \Delta_H \mathcal{L}^S(\pi, \pi') \prod_{v \in S} I_v^*(f_v, f'_v).$$

Here Δ_H is an elementary L -function, each $I_v^*(f_v, f'_v)$ is a normalized local integral of matrix coefficients, and

$$\mathcal{P}(f, f') := \frac{|P_H(f, f')|^2}{\langle f, f \rangle \langle f', f' \rangle}.$$

The periods

Here

$$P_H(f, f') = \int_{[H(\mathbf{A})]} f(\iota(h)) f'(h) dh$$

and $\langle f, f \rangle, \langle f', f' \rangle$ are Petersson norms. The integrals are taken with respect to Tamagawa measure.

For each $v \in S$, the GGP conjecture picks out inner forms G_v^*, H_v^* , and unique members $\pi_v^*, \pi_v^{'*}$ of the local L -packets defined by Π_v, Π_v' such that

$$I_v^*(\bullet, \bullet) \neq 0 \Leftrightarrow G_v = G_v^*, H_v = H_v^*, \pi_v = \pi_v^*, \pi_v' = \pi_v^{'*}.$$

We assume we have picked $H \subset G, \pi, \pi'$ as on the RHS.

(There may be a parity obstruction, in which case the sign of the functional equation of $L(s, \Pi \otimes \Pi')$ is -1 and the central value vanishes; but we assume this is not the case.)

Main Theorems (work in progress)

Theorem (Grobner-H-Lin-Raghuram)

Assume W sufficiently regular. ($a_i - a_{i+1} > n$ suffices.). Then the automorphic version of Deligne's conjecture for critical values $L(s, M(\Pi) \otimes M(\Pi'))$ holds up to constants that depend only on Π_∞, Π'_∞ .

Theorem (Grobner-H-Lin)

Under the regularity assumption of the previous theorem, the Factorization Conjecture holds up to constants that depend only on Π_∞, Π'_∞ .

A formula of S.-Y. Chen

Let Π, Π' be on $GL(n), GL(m)$, as before. Let Σ, Σ' be a second pair of cohomological automorphic representations, also of $GL(n)$ and $GL(m)$, but *not assumed to be cuspidal*.

Assume $\Pi_\infty = \Sigma_\infty, \Pi'_\infty = \Sigma'_\infty$. We expect to be able to prove the following version of a result proved by Shih-Yu Chen for $GL(n)_\mathbb{Q} \times GL(m)_\mathbb{Q}$.

Theorem

Assume W and W' are sufficiently regular. Then for some critical value s_0 the following ratio

$$\frac{L(s_0, \Pi \times \Pi') L(s_0, \Sigma \times \Sigma')}{L(s_0, \Pi \times \Sigma') L(s_0, \Sigma \times \Pi')} \quad (9)$$

is an algebraic number that transforms as expected under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Chen's formula is consistent with Deligne's Conjecture

Recall the formula for the Deligne period:

$$c(s_0, M, M') \sim (2\pi i)^{a(s_0)} \prod_i P^i(M)^{sp(i, M, M')} \prod_j P^j(M')^{sp(j, M', M)}.$$

We rewrite it

$$c(s_0, M, M') \sim (2\pi i)^{a(s_0)} C_{W, W'}(M) \cdot C_{W', W}(M').$$

The power $a(s_0)$ depends only on the archimedean data, and the periods of M and M' are separated. Then Chen's ratio is obviously algebraic:

$$\frac{C_{W, W'}(M(\Pi)) \cdot C_{W', W}(M(\Pi')) C_{W, W'}(M(\Sigma)) \cdot C_{W', W}(M(\Sigma'))}{C_{W, W'}(M(\Pi)) \cdot C_{W', W}(M(\Sigma')) C_{W, W'}(M(\Sigma)) \cdot C_{W', W}(M(\Pi'))},$$

Proving Chen's formula

Chen's proof (for $GL(n)_{\mathbb{Q}} \times GL(m)_{\mathbb{Q}}$) is based on constructions of Eisenstein cohomology, as in the book of Harder-Raghuram. Raghuram has generalized this to CM fields and expects the proof to work in the same way.

Using Chen's formula

In the application, one takes Σ and Σ' to be cohomological Eisenstein series for the Borel subgroups of $GL(n)$ and $GL(n-1)$, respectively. Then

$$L(s_0, \Pi \times \Sigma') = \prod_{i=1}^{n-1} L(s_0, \Pi \otimes \chi'_i)$$

for certain algebraic Hecke characters, determined by W' . The right hand side was computed in my 1997 Crelle paper as a product of the $P^i(\Pi)$, multiplied by a product of periods of CM elliptic curves. A similar formula holds for $L(s_0, \Sigma \times \Pi')$.

Finally, by Deligne's conjecture for Hecke characters (Damarell, Shimura, Blasius), $L(s_0, \Sigma \times \Sigma')$ is a product of CM periods. All of the CM periods cancel and we are left with the automorphic version of Deligne's conjecture.

General critical values

To deduce the result for general critical values, one uses the method of Harder-Raghuram.

Rational cup products

Assuming the automorphic Deligne conjecture, we obtain the Factorization conjecture by considering the pair $U(V_1) \supset U(V'_1)$ (signatures $(1, n-1)$ and $(1, n-2)$).

For certain $\pi \times \pi'$ in the corresponding L -packet, the period integral can be interpreted as a cup product in coherent cohomology.

$$P(\pi, \pi') = \int_{[U(V'_1)]} f_\pi \cdot f'_{\pi'} = \text{Tr}(\omega_k \cup \omega'_{n-1-k}) \in \overline{\mathbb{Q}}$$

if f_π and $f'_{\pi'}$ define arithmetic coherent cohomology classes

Under the regularity hypothesis, one finds enough examples of such π using work of Salamanca-Riba and the Burger-Sarnak method.

IINH conjecture, rank one

The IINH conjecture then yields

$$\frac{|P(\pi, \pi')|}{Q_k(M(\Pi))Q_{n-1-k}(M(\Pi'))} = (*) \frac{L(\frac{1}{2}, \Pi \times \Pi')}{L(1, \Pi \times \Pi', As^\pm)}$$

The numerator on the left is algebraic. The expression on the left can be written in terms of automorphic periods $P_i(\Pi), P_j(\Pi')$ by the Deligne conjecture except that the factor $(*)$ contains an unknown archimedean period integral.

Using a theorem of Burger-Sarnak, we can assume the left hand side doesn't vanish.

Comparing the expressions on the two sides gives the factorization by induction on n .

Automorphic computation of coherent cohomology

Recall the formula

$$H_{dR}M(\Pi) \otimes \mathbb{C} = \bigoplus_{k=1}^n H_{!}^{k-1}(Sh(U(V)), \mathcal{V}^k)[\pi_f] \otimes \mathbb{C}.$$

More generally, for V of signature (r, s) , and W a finite-dimensional representation of $U(V)$, with associated local system $\tilde{W}/Sh(U(V))$, we have a generalized Hodge decomposition

$$H_{dR,!}^{rs}(Sh(U(V)), \tilde{W}) \xrightarrow{\sim} \bigoplus_{\pi_{\infty} \in \Pi(W)} H_{!}^{q(\pi_{\infty})}(Sh(U(V)), \mathcal{V}_{\pi_{\infty}})$$

where $\Pi(W)$ runs over the archimedean L -packet of W , with $\frac{n!}{r!s!}$ members, and an automorphic vector bundle $\mathcal{V}_{\pi_{\infty}}$ attached to each $\pi_{\infty} \in \Pi(W)$, with preferred degree $q(\pi_{\infty})$.

$(\mathfrak{P}^-, K_\infty)$ -cohomology

The notation $H_!^{q(\pi_\infty)}(Sh(U(V)), \bullet)$ refers to cohomology computed on certain toroidal compactifications. Let $K_\infty \subset U(V)(\mathbb{R})$ be maximal compact; it is the Levi component of a certain (complex) maximal parabolic, with Lie algebra $\mathfrak{P}^-$. Let V_{π_∞} be the representation of K_∞ defining $\mathcal{V}_{\pi_\infty}$.

There is an isomorphism (in most cases)

$$\begin{aligned} H_!^q(Sh(U(V)), \mathcal{V}_{\pi_\infty}) &\xrightarrow{\sim} H^q(\mathfrak{P}^-, K_\infty; \mathcal{A}_{cusp}(U(V)) \otimes V_{\pi_\infty}) \\ &\xrightarrow{\sim} \bigoplus_{\Pi \subset \mathcal{A}_{cusp}} H^q(\mathfrak{P}^-, K_\infty; \Pi_\infty \otimes V_{\pi_\infty}) \otimes \Pi_f \end{aligned}$$

(Rational cup products, elementary cases)

Suppose (Π, Π') is a pair satisfying the GGP conditions, so that in particular there is a $U(V')$ -invariant pairing

$$\Pi_{\infty} \otimes \Pi'_{\infty} \rightarrow \mathbb{C}.$$

Say $(\pi, \pi') := (\Pi_{\infty}, \Pi'_{\infty})$ is an *elementary pair* if $q(\pi) + q(\pi') = \dim Sh(V')$ ($= (r-1)s$), and if there is a K'_{∞} -invariant pairing $V_{\pi} \otimes V_{\pi'} \rightarrow \mathbb{C}$, so that the induced map

$$H^{q(\pi)}(\mathfrak{P}^-, K_{\infty}; \pi \otimes V_{\pi}) \otimes H^{q(\pi')}(\mathfrak{P}'^-, K'_{\infty}; \pi' \otimes V_{\pi'}) \rightarrow \mathbb{C}$$

corresponds to the cup product

$$H_!^{q(\pi)}(Sh(U(V)), \mathcal{V}_{\pi}) \otimes H_!^{q(\pi')}(Sh(U(V')), \mathcal{V}_{\pi'}) \rightarrow \mathbb{C}.$$

(by the trace on top cohomology $H^{\dim Sh(V')}(Sh(U(V')), \Omega^{top})$ of $Sh(V')$).

Rational cup products, in general

The cup products used to prove the factorization conjecture, and their extensions to other signatures, more or less exhaust the elementary pairs.

Say (π, π') is a *coherent pair* if the GGP pairing can be interpreted as a cup product in coherent cohomology, with coefficients more general than $\mathcal{V}_\pi$.

In work in progress with Kobayashi and Speh, we have classified all the coherent pairs. We are now working on the conjecture that the cup products they give are non-trivial (in general).

Application

Recall the IINH formula, where $(*)$ is a product of elementary terms:

$$\frac{|P_H(f, f')|^2}{\langle f, f \rangle \langle f', f' \rangle} = (*) \frac{L^S(\frac{1}{2}, \Pi \otimes \Pi')}{L^S(1, \pi, Ad) L^S(1, \pi', Ad)}.$$

For a coherent pair, the numerator on the left can be assumed algebraic. Then we have

$$L^S(\frac{1}{2}, \Pi \otimes \Pi') \sim (*) \frac{L^S(1, \pi, Ad) L^S(1, \pi', Ad)}{\langle f, f \rangle \langle f', f' \rangle}.$$

Here f, f' are coherent cohomology classes defined over $\overline{\mathbb{Q}}$, so the denominator on the right is a product of generalizations of the Q -periods. Thus this gives rise to new factorization results (work in progress of Jie Lin).