

Associativity is enough (for twistorial chiral algebras)



Natalie Paquette
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Early Career Research Program

This talk is largely based on works in collaboration with
Kevin Costello & Victor Fernandez

[2201.02595, 2204.05301, to appear]

We all know to use supersymmetric or integrable quantities as a handle on certain physical systems, to compute BPS quantities, to connect to mathematics, etc.

In this talk, we will present some new twists:

- **Explain how certain loop-level form factors in non-SUSY/integrable 4d theories can be computed using 2d chiral algebra techniques**
- **Study asymptotic symmetry algebras of soft modes, identify quantum subtleties, and determine quantum deformations**

A simple algebraic computation

$$J^a(z)J^b(0) \sim \frac{1}{z} f_c^{ab} J^c(0)$$

$$\tilde{J}^a(z)J^b(0) \sim \frac{1}{z} f_c^{ab} \tilde{J}^c(0)$$

Suppose we could compute a correlation function for this algebra in a chosen conformal block, and found that the only nonvanishing such 2-pt function at tree-level was:

$$\langle \mathbf{tr}(B^2) | \tilde{J}^a(z_1) \tilde{J}^b(z_2) \rangle = (z_1 - z_2)^2 \mathbf{Tr}(t^a t^b) =: z_{12}^2 K^{ab}$$

$$\begin{aligned} \left\langle \mathbf{tr}(B^2) \left| \tilde{J}^a(z_1) \tilde{J}^b(z_2) J^c(z_3) \right. \right\rangle &= f_d^{cb} \frac{1}{z_{23}} \left\langle \mathbf{tr}(B^2) \left| \tilde{J}^a(z_1) \tilde{J}^d(z_2) \right. \right\rangle \\ &\quad + f_d^{ca} \frac{1}{z_{13}} \left\langle \mathbf{tr}(B^2) \left| \tilde{J}^d(z_1) \tilde{J}^b(z_2) \right. \right\rangle \\ &= \frac{z_{12}^3}{z_{13} z_{23}} f^{abc} \end{aligned}$$

Using the OPE,
it is then easy
to prove by
induction:

$$\langle \mathbf{tr}(B^2) | J^{a_1}(z_1) \dots \tilde{J}^{a_i}(z_i) \dots \tilde{J}^{a_j}(z_j) J^{a_n}(z_n) J^{a_{n+1}}(z_{n+1}) \rangle = \frac{z_{ij}^4}{z_{12} z_{23} \dots z_{n1}} \mathbf{tr}(t^{a_1} \dots t^{a_n})$$

$$\begin{aligned} P^{\alpha\dot{\alpha}} &=: \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \\ \lambda^\alpha &\equiv (1, z) \end{aligned}$$

Identifying $z_{ij} \leftrightarrow \langle ij \rangle$, this is the functional form of the color-ordered Parke-Taylor formula.

We will see where this comes from and how to generalize this sort of computation.

Twistor space is a complex manifold

It democratizes the $\mathbb{CP}^1$ -family of ways to complexify $\mathbb{R}^4$

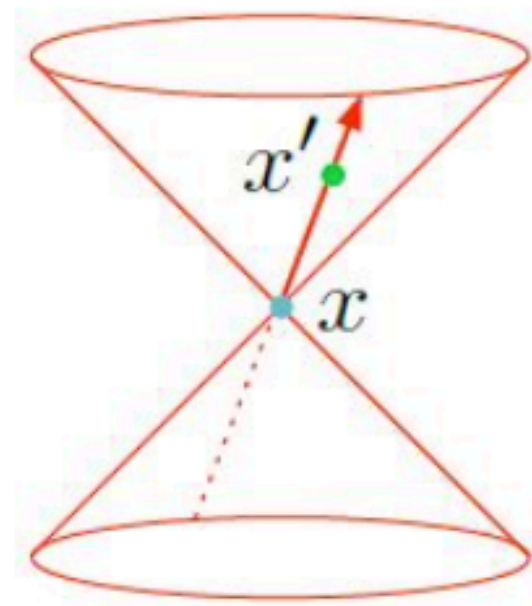
$$\mathbb{PT} = \mathcal{O}(1) \oplus \mathcal{O}(1) \simeq \mathbb{R}^4 \times \mathbb{CP}^1$$

$$\downarrow$$

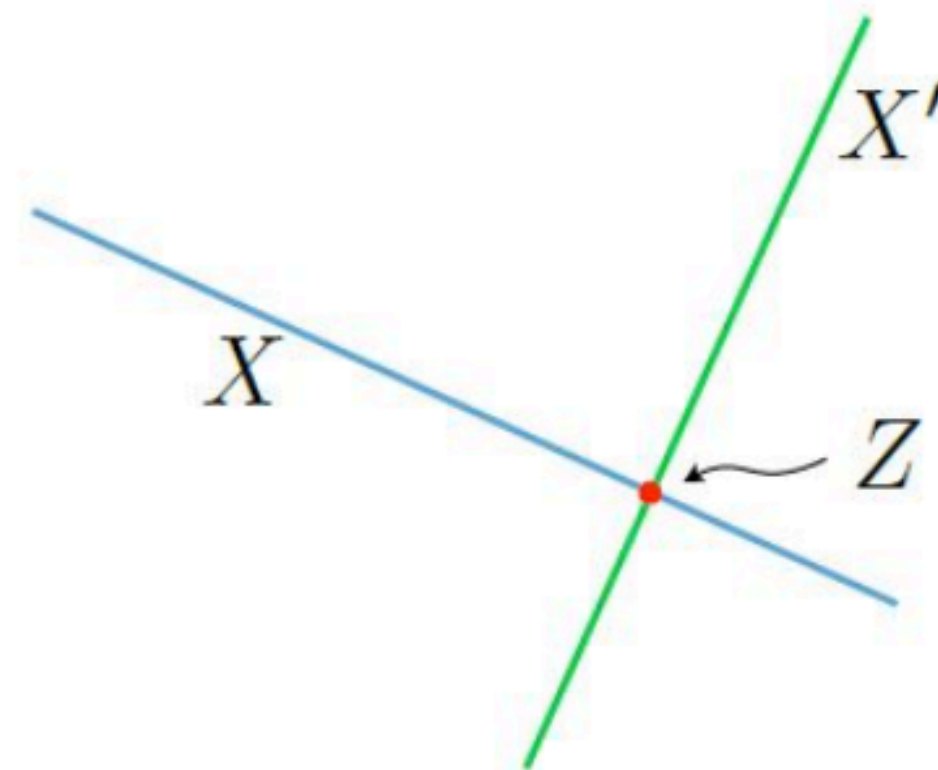
$$\mathbb{CP}^1$$

$$z \in \mathbb{CP}^1$$

$$v_1, v_2 \in \mathcal{O}(1) \oplus \mathcal{O}(1)$$



$$u \in \mathbb{C}^4$$



$$\mathbb{CP}_u^1 \in \mathbb{PT}$$

$$v_1 = u_1 + iu_2 + z(u_3 - iu_4)$$

$$v_2 = u_3 + iu_4 - z(u_1 - iu_4)$$

- $\mathbb{CP}_x^1, \mathbb{CP}_y^1$ intersect iff $||x - y||^2 = 0$
- $\therefore$ correlation fns rational fns
- singularities on lightcone
- no anomalous dimensions
- integrability (cf. motivations)

Penrose transform

$$H^{0,1}(\mathbb{PT}, \mathcal{O}(2h - 2))$$

hol'c massless fields on $\mathbb{C}^4$

solutions to massless field equations

harmonic functions

entire analytic functions, can pass to any signature

[Penrose, Atiyah, Ward, Woodhouse, Mason, Lebrun,...]

Twistor space has been studied for a long time.

It is useful for, e.g.,

- computing classical solutions to nonlinear (massless) field equations
- making symmetries manifest
- computing amplitudes (esp. integrands, so we don't have to worry about symmetry-breaking regulators)

Reminder: spinor helicity variables

$$SO(4, \mathbb{C}) \simeq (SL(2, \mathbb{C}) \times SL(2, \mathbb{C})) / \mathbb{Z}_2$$

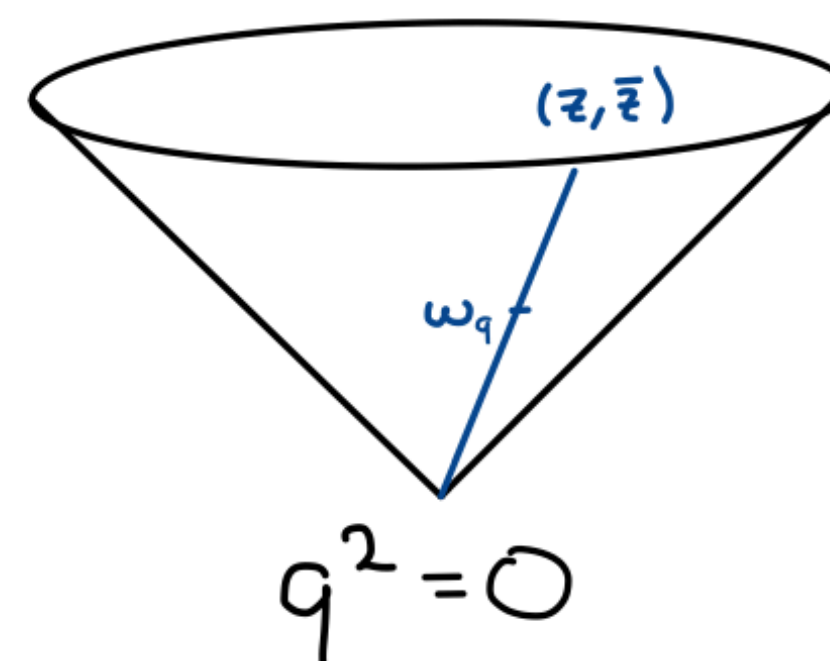
$$(P^0)^2 - (P^1)^2 - (P^2)^2 - (P^3)^2 = 0 \quad \rightarrow \quad P^\mu \quad \rightarrow \quad P^{\alpha\dot{\alpha}} = \begin{pmatrix} P^0 + P^3 & P^1 - iP^2 \\ P^1 + iP^2 & P^0 - P^3 \end{pmatrix} =: \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

$$P^\mu(\omega, z, \bar{z}) \propto \omega(-v_1(z) + \bar{v}_2(z)\bar{z}), \quad \bar{z} = v_2/v_1$$

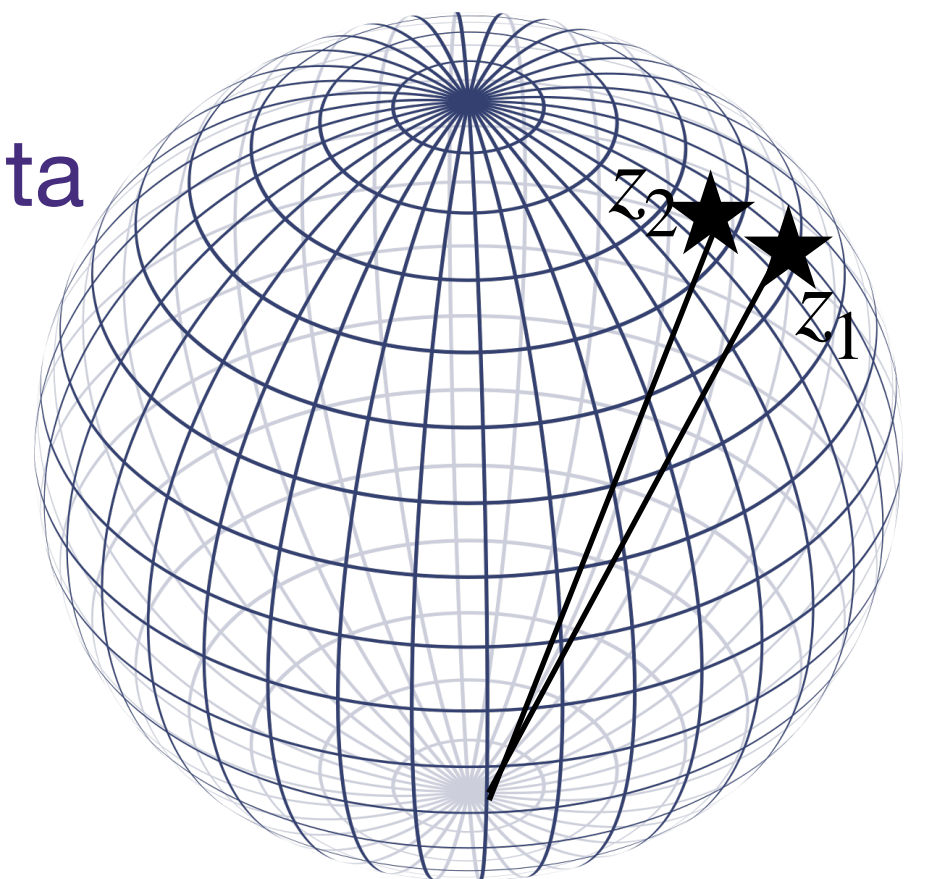
$$\lambda^\alpha = \begin{pmatrix} 1, \frac{P^1 + iP^2}{P^0 + P^3} \end{pmatrix} \equiv (1, z)$$

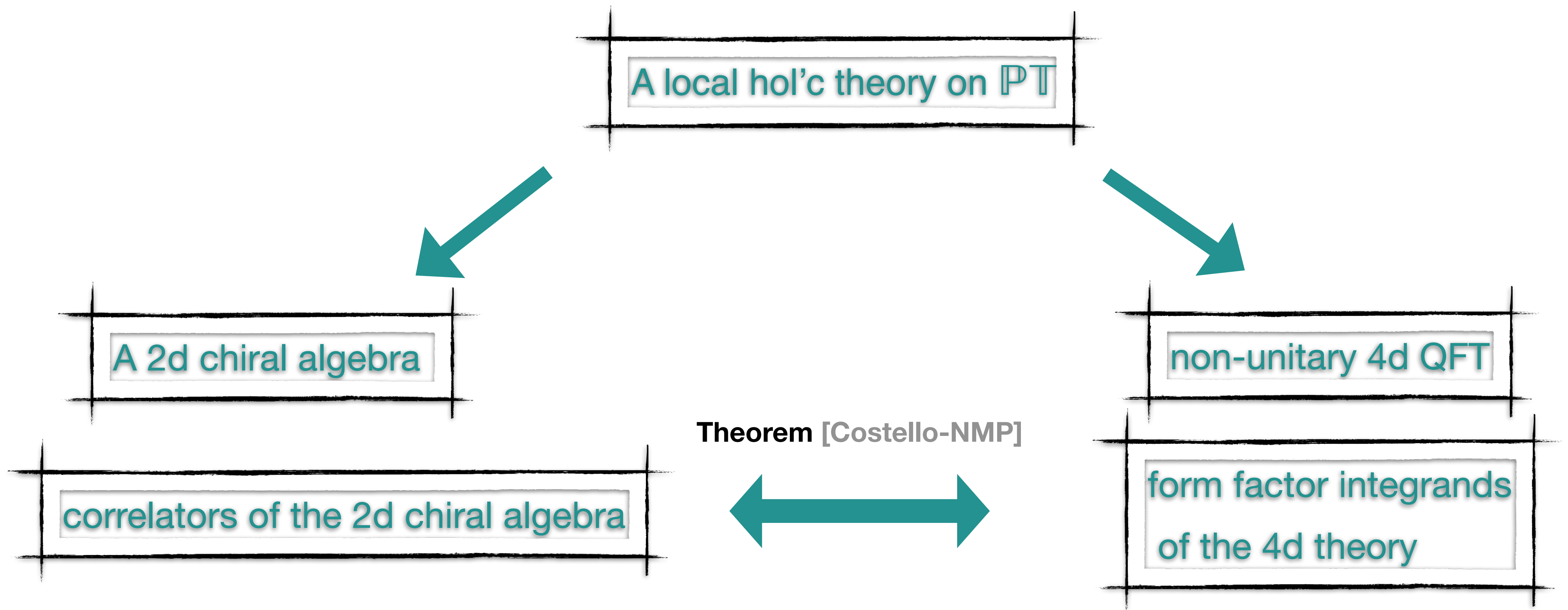
**What kind of local QFTs
can be placed
on twistor space?**

hol'c coord. of
celestial sphere, "space" of null momenta



$$S^2 \simeq \mathbb{CP}^1_z$$





“Chiral algebra bootstrap”: For the theories to which this applies, trade loop-level amplitude computations for algebraic manipulations

- Judiciously chosen quantities are equivalent to observables for Yang-Mills, QCD w/ special choices of matter, etc.
- Form factors are rational, with poles only in $\langle ij \rangle$
- polar part is itself a form factor at lower loop-order or insertion #, determined by OPE

Local holomorphic field theories on twistor space are integrable, self-dual, etc.

But many classical examples of such 6d theories *suffer from gauge anomalies* when you quantize them

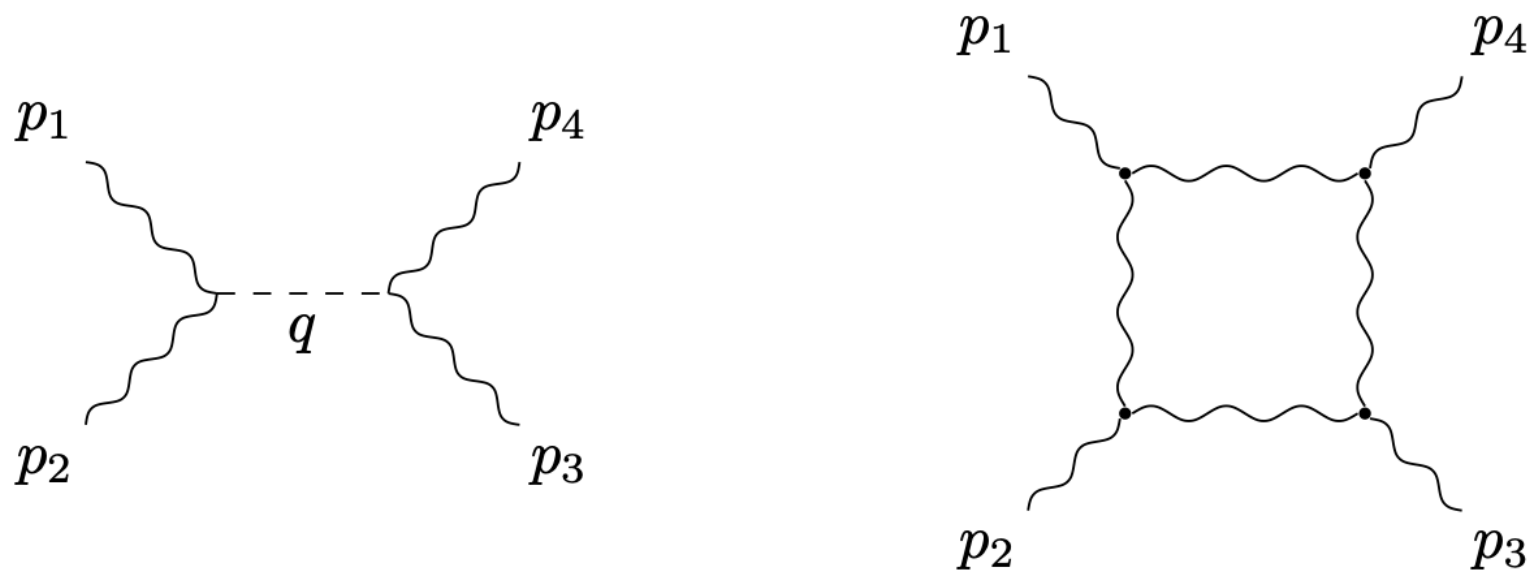
Here are some 4d theories for which the anomaly of their 6d lifts
can be cancelled *without supersymmetrizing*:

- **SDYM w/ $\mathfrak{g} = sl_2, sl_3, so(8), e_{6,7,8}$ + periodic scalar with fourth-order kinetic term (“axion”)**
- **SDGR + a periodic scalar with fourth-order kinetic term [B-S-S]**
- **SDYM w/ $G = SU(N) + (N = N_f)$ fundamental flavors + “axion”**
- **SDYM w/ $G = SO(N) + (N_f = N - 8)$ fundamental flavors + “axion”**
- **SDYM w/ $G = SU(N) + 8F \oplus 8F^\vee \oplus \wedge^2 F \oplus \wedge^2 F^\vee$**

[Costello '22] for more examples

- $G = Sp(N) + R = 16F \oplus \wedge_0^2 F$ [Costello, Bittleston]
- $SU(2), N_f = 8, SU(3), N_f = 9$

These “twistorial” theories will be our focus of study.



$$\lambda_{\mathfrak{g}}^2 = \frac{10(h^\vee)^2}{\dim \mathfrak{g} + 2}$$

$$\mathrm{Tr}_{\mathrm{adj}}(X^4) - \mathrm{Tr}_R(X^4) = \lambda_{\mathfrak{g},R}^2 \mathrm{Tr}_{\mathrm{fun}}(X^2)^2$$

For concreteness, focus on the example of pure SDYM gauge theory + anomaly-cancelling scalar field

I will discuss both the 4d and the 2d sides in more detail shortly.

Both sides can be obtained directly from the 6d theory on twistor space.

The chiral algebras can be thought of as an asymptotic symmetry algebra in the 4d theory.

[Costello-NMP '22] Main theorems that relate the two:

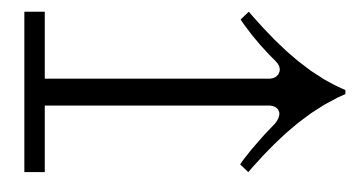
2d chiral algebra	4d theory
conf. primary generators	massless on-shell states (boost eigenbasis)
OPEs	holomorphic collinear limits
conformal blocks (cf. CS/WZW)	local operators
correlation functions	form factors

A twistorial theory: Part 1. Self-dual Yang-Mills theory

$$S[\mathcal{A}, \mathcal{B}] = \left(\frac{1}{2\pi i} \right) \int_{\mathbb{PT}} \text{Tr}(\mathcal{B} F^{0,2}(\mathcal{A})) = \left(\frac{1}{2\pi i} \right) \int_{\mathbb{PT}} \text{Tr}(\mathcal{B} \bar{\partial} \mathcal{A} + \frac{1}{2} \mathcal{B} [\mathcal{A}, \mathcal{A}])$$

$$\mathcal{B} \in \Omega^{3,1}(\mathbb{PT}, \mathfrak{g})$$

$$\mathcal{A} \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$$



[Penrose, Ward, Mason]

$$S[A, B] = \int_{\mathbb{R}^4} \text{tr} (B \wedge F(A)_-) \quad \text{[Chalmers-Siegel]}$$

$$B \in \Omega_-^2(\mathbb{R}^4, \mathfrak{g})$$

Notice if we add to the SDYM a term like $\frac{1}{2} g_{YM}^2 \int \text{tr}(B \wedge B)$ and integrate out B , we obtain YM (+ some theta angle)

Terms in YM at order $2n$ in g_{YM} $\longleftrightarrow$ Insert $\text{tr}(B^2)(x_i)$ at $x_1, \dots, x_n \in \mathbb{R}^4$
& integrate over positions

A twistorial theory: Part 2. The fourth-order axion

$$S[\mathcal{A}, \mathcal{B}, \eta] = \left(\frac{1}{2\pi i} \right) \int_{\mathbb{PT}} \left(\text{Tr}(\mathcal{B} F^{0,2}(\mathcal{A})) + \frac{1}{2} \partial^{-1} \eta \bar{\partial} \eta + \frac{1}{2} \frac{\lambda_g}{(2\pi i) \sqrt{12}} \eta \text{Tr}(\mathcal{A} \partial \mathcal{A}) \right)$$

Can cancel the anomaly by coupling to free limit of BCOV theory.

This is an avatar of 6d $\mathcal{N} = (1,0)$ gauge + tensor multiplet being anomaly-free (holomorphic twist)

$$S[B, A, \rho] = S[B, A]_{SDYM} + \int \frac{1}{2} (\Delta \rho)^2 + \frac{\lambda_g}{\sqrt{6}(2\pi i)} \rho \text{tr}(F \wedge F)$$

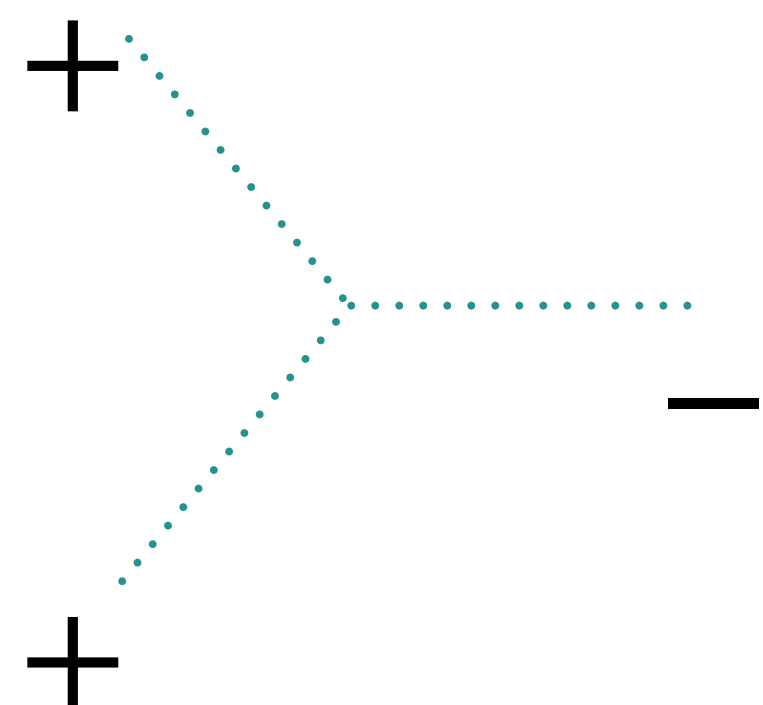
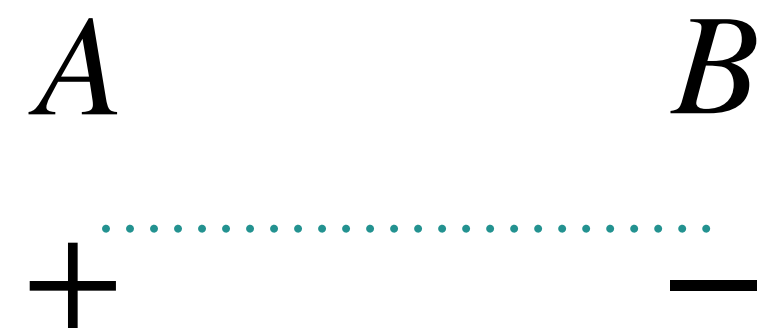
You may have seen such fourth-order scalars in 4d before [Fradkin/Tseytlin, Komargodski/Schwimmer, Riegert,...]

All told, our 4d theory is conformal and non-unitary

The 6d theory can be treated quantum mechanically.

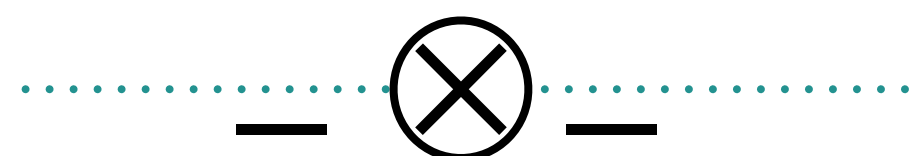
Useful bc geometry of twistor space implies rationality of 4d form factors
(scattering amplitudes vanish in this theory)

Self-dual YM

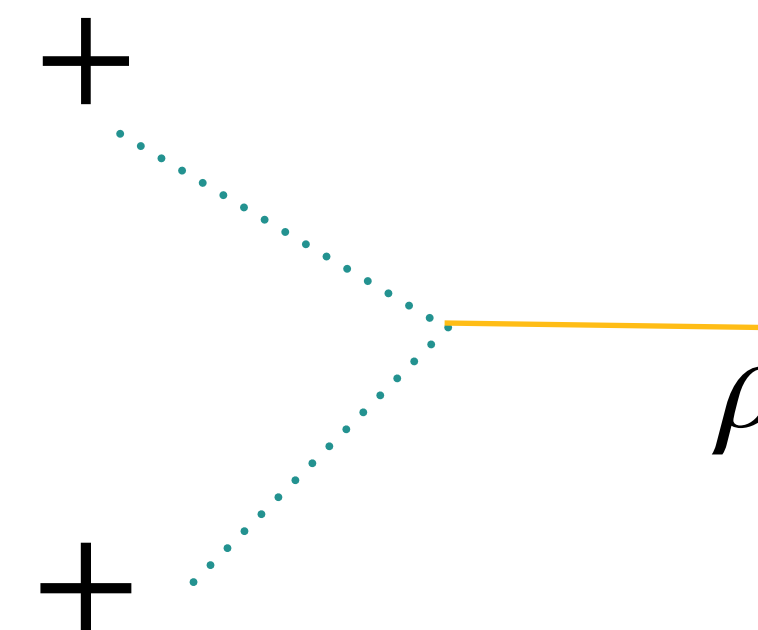


form factors:

$$\text{tr}(B^2)(0)$$



+ axion



effectively 1-loop
by Green-Schwarz

SDYM form factors w/ L loops, N insertions of $\text{tr}(B^2) \rightarrow$
amplitudes of $N-L+1$ (-) helicity, arbitrary (+) helicity gluons
in YM [$N=1$, $L=0$ is Parke-Taylor.]

1-loop example:

N=1, L=1 : 1 negative-helicity, arbitrary positive-helicity
(+ anomaly-cancelling ordinary matter): [Costello-NMP]

$$\left\langle \frac{1}{2} \text{tr}(B^2) \middle| \tilde{J}^{a_1}(z_1) J_{a_2}[\mu](z_2) \dots J_{a_n}[\mu](z_n) \right\rangle$$
$$= -\frac{1}{n96\pi^2} \sum_{\sigma \in S_n} \sum_{2 \leq i < j \leq n} \frac{[ij] \langle 1i \rangle^2 \langle 1j \rangle^2}{\langle ij \rangle \langle \sigma_1 \sigma_2 \rangle \langle \sigma_2 \sigma_3 \rangle \dots \langle \sigma_n \sigma_1 \rangle} \text{Tr}_{\mathfrak{g} \oplus \Pi R} (\mathbf{t}_{a_{\sigma_n}} \dots \mathbf{t}_{a_{\sigma_1}})$$

For our special matter choices, can be matched with: [Bern-Kosower '92 (4 pts), Bern-Dixon-Kosower (n-pts) '05]

(Contrast w/ non-rational expressions from QCD proper [Bern-Dixon-Dunbar-Kosower, Mahlon])

See also Costello '23 for N=1, L=2 all-positive-helicity amplitude (+ anomaly-cancelling ordinary matter)

What is this good for/where do we want to go with this?

- Consider ordinary anomaly-cancelling matter. In SDYM, the n -loop contribution of a form factor with a $\text{tr}(B^n)$ operator insertion & scattering of pos. helicity gluons is the same quantity in YM. $n = 2$ case computed by Costello & matched by Dixon-Morales. *$n = 3$ case is WIP using some new loop-level results I'll present later [Fernandez-NMP].*
- In $SU(N) + N$ flavors+ axion, n -loop "axion" exchanges contribute to $(n + 1)$ -loop processes. Cancels non-rational contributions. May provide a new window to extracting transcendental pieces of QCD diagrams.
- What about form factors with n operator insertions? Costello and I proved that the 4d form factors then take the form of sums of products of 2d chiral correlators x 4d OPE coefficients. We used this to reproduce CSW formula at tree-level, but the 4d OPEs have not been studied systematically. How to relate to YM/QCD amplitudes?

$$\text{Tr} B^2(0) \text{Tr} B^2(x_1) \dots \text{Tr} B^2(x_{n-1}) \sim \sum_i F_i(x_1, \dots, x_{n-1}) \mathcal{O}^i(0)$$

$$\mathcal{A}(0, x_1, \dots, x_{n-1}) = \sum F^i(x_1, \dots, x_{n-1}) \left\langle \mathcal{O}_i(0) \mid J^{a_1}(\mu_1, z_1) \dots J^{a_n}(\mu_n, z_n) \right. \\ \left. \tilde{J}^{b_1}(\mu'_1, z'_1) \dots \tilde{J}^{b_m}(\mu'_m, z'_m) \right\rangle.$$

e.g.

$$\text{tr}(B^2)(0) \text{tr}(B^2)(x) \sim \frac{1}{\|x\|^2} B_{\alpha_1 \beta_1}^a B_{\alpha_2 \beta_2}^b B_{\alpha_3 \beta_3}^c f_{abc} \epsilon^{\beta_1 \alpha_2} \epsilon^{\beta_2 \alpha_3} \epsilon^{\beta_3 \alpha_1}.$$

2d conformal blocks $\simeq$ 4d local operators

see also [Bu-Casali]

$$\mathbb{PT} \setminus \mathbb{CP}_0^1 = S^3 \times \mathbb{CP}^1 \times \mathbb{R}_{>0}$$

**Formal theorems in paper
using, e.g. factorization homology**

$$\mathbb{R}^4 \setminus 0 = \mathbb{R}_{>0} \times S^3$$

$$\mathbb{R}_{>0} \times \mathbb{CP}_z^1$$

include all KK modes!

$\mathcal{H}(S^3) :=$ space of local operators

$\mathcal{H}(\mathbb{CP}^1) :=$ space of conformal blocks

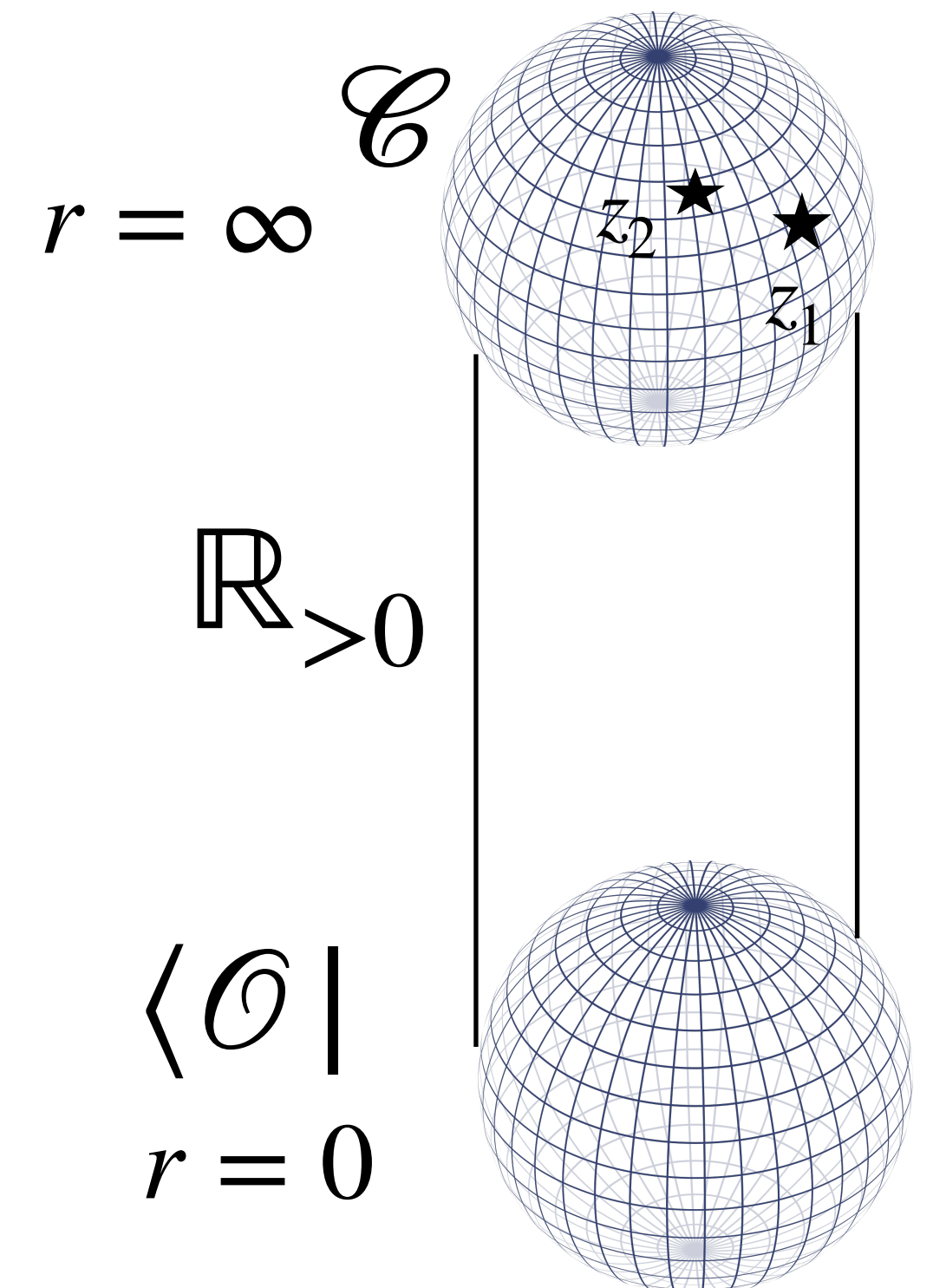
$$\mathcal{O}(0) \leftrightarrow \langle \mathcal{O} |$$

Chiral algebra as 3d bdy algebra:

hol'c-top'l theory (twist of 3d $\mathcal{N} = 2$)

- chiral algebra a boundary condition at ∞
- conformal block a state at 0
- correlator in 3d bulk/boundary system

$$\langle \mathcal{O} | J(z_2)J(z_1) \rangle$$



Much Ado About Chiral Algebras

There is a 2d chiral algebra associated to any 4d twistorial theory

For general 4d QFTs, classical chiral algebras from holomorphic collinear limits of scattering amplitudes
[Strominger, Guevara et. al.] “Asymptotic symmetry algebra of conformally soft modes”

momentum eigenstate
for massless field
(positive helicity gluon in 4d)

$$J(\tilde{\lambda}, z) = \sum_{r,s} \omega^{r+s} \frac{(\tilde{\lambda}^{\dot{1}})^r (\tilde{\lambda}^{\dot{2}})^s}{r!s!} J[r, s](z)$$

chiral algebra generators

$$J^a[r, s](z) J^b[t, u](0) \sim \frac{1}{z} f_c^{ab} J^c[r+t, s+u](0)$$

Quantum mechanically, the usual *local, associative* chiral algebra structure persists for *twistorial theories*.

Also for $\tilde{J} = 0$ [Ball-Narayanan-Salzer-Strominger]

More generally, there are violations of associativity for local non-twistorial theories [Costello-NMP, PRL '22], see also
[BLRSS, RSSV]

However, note: Proposed nonlocal repairs from multicollinear limits [Ball-Hu-Pasterski]

The 2d chiral algebra from a local holomorphic theory on $\mathbb{P}^1$

Technique developed in twisted holography, connections to mathematics (Koszul duality) [Costello-NMP '21]

Generator	Field	Scaling Dimension	Spin	Combined Dilatation	Weight
$J[t_1, t_2], t_i \geq 0$	$\mathcal{A}$	$-(t_1 + t_2)$	$1 - \frac{t_1+t_2}{2}$	1	0
$\tilde{J}[t_1, t_2], t_i \geq 0$	$\mathcal{B}$	$-(t_1 + t_2 + 2)$	$-1 - \frac{t_1+t_2}{2}$	0	1
$E[t_1, t_2], t_1 + t_2 \geq 1$	η	$-(t_1 + t_2)$	$-\frac{t_1+t_2}{2}$	0	1/2
$F[t_1, t_2], t_i \geq 0$	η	$-(t_1 + t_2 + 2)$	$-\frac{t_1+t_2}{2}$	1	1/2

Table 1: Local operators of the 2d chiral algebra, the 6d fields they source, and their quantum numbers. By scaling dimension, we mean charge under scaling of Euclidean 4d spacetime $\mathbb{R}^4$. Here, spin refers to the holomorphic 2d conformal weight, and combined dilatation corresponds to the charge of the operator under simultaneous dilatations $z \rightarrow \frac{z}{r}$ on the celestial sphere and $x \rightarrow \sqrt{r}x$ on 4d spacetime. Finally, weight describes how the operator transforms under a rescaling of $\hbar$.

The 2d chiral algebra from a local holomorphic theory on $\mathbb{PT}$

Penrose transform:

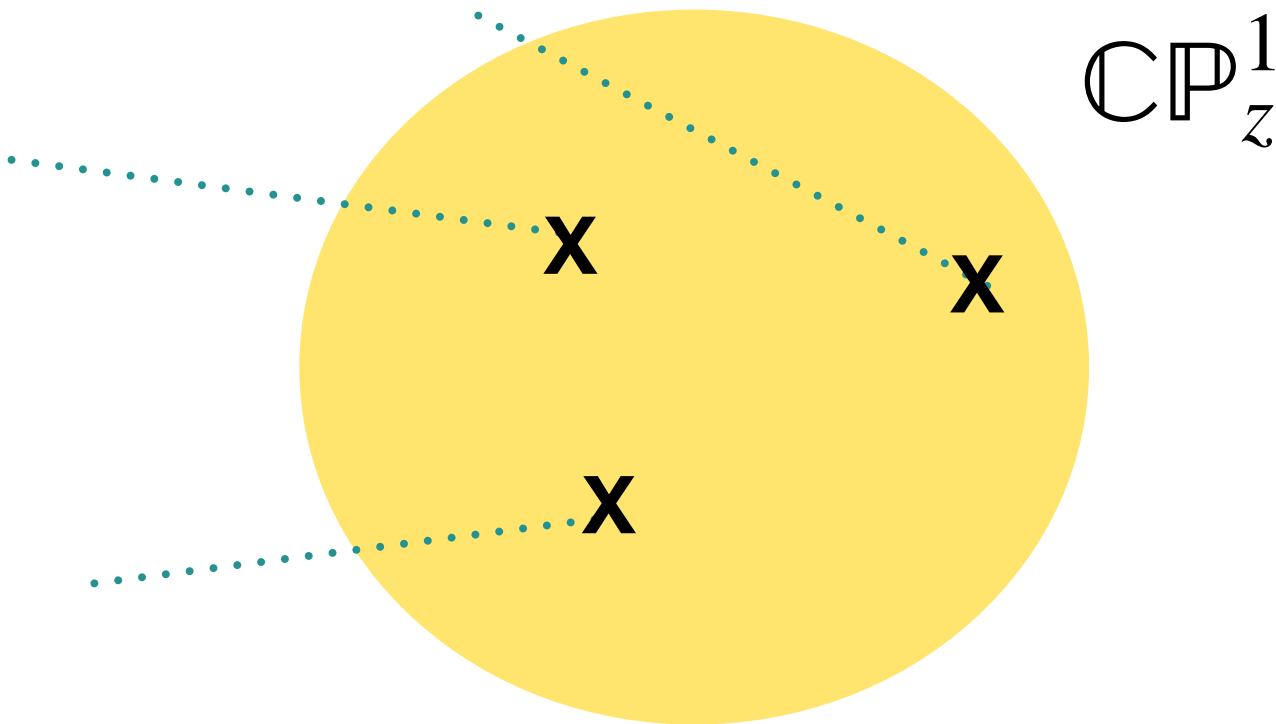
massless on-shell states on $\mathbb{R}^4$ lift to $(0, 1)$ forms

$$\delta_{z=z_i} e^{\tilde{\lambda}_{\dot{\alpha}} \nu^{\dot{\alpha}}}$$

the modes have mostly-negative integer conformal weights

$$J[r, s](z_i) \leftrightarrow \mathcal{A} = \delta_{z=z_i} (\tilde{\lambda}^1)^r (\tilde{\lambda}^2)^s$$

state in vacuum module = on-shell background field localized on $\mathbb{CP}^1$



Couple to a 2d chiral theory along $\nu^i = 0 \subset \mathbb{PT}$

$$\text{Exp} \left[\frac{1}{2\pi i} \left(\int_{\mathbb{CP}^1} J_a[m, n] \partial_{\nu_1}^m \partial_{\nu_2}^n \mathcal{A}_{\bar{z}}^a \right) \right]$$

Demanding gauge-invariance of this coupling constrains the OPEs of the currents.

We can expand the exponential in the path integral, and study BRST invariance order-by-order in perturbation theory.

Including analogous couplings to $\mathcal{B}, \eta$ produces the four towers of operators:

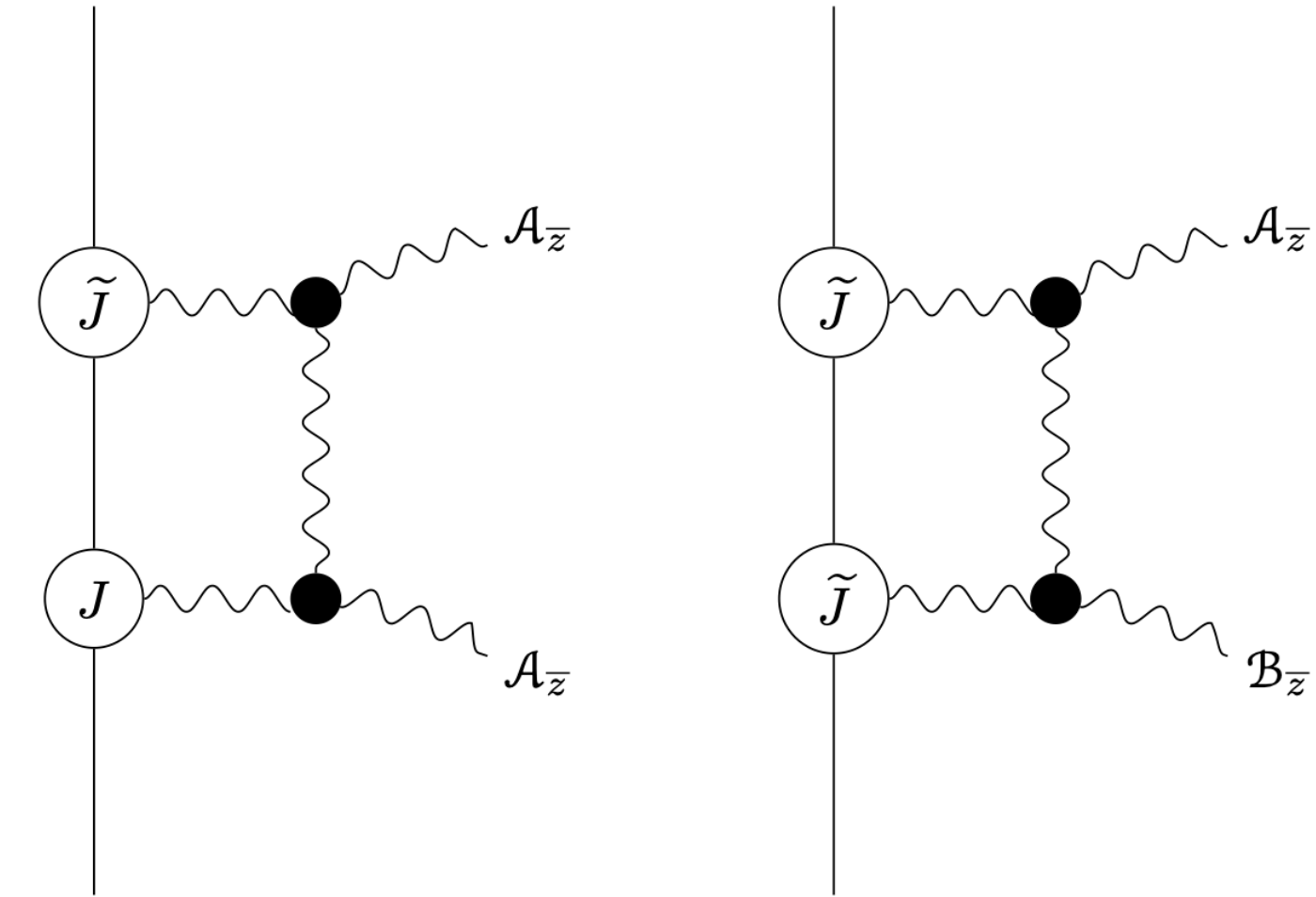
$$J[r, s] \quad \tilde{J}[r, s] \quad E[r, s] \quad F[r, s]$$

**Classically, BRST invariance of the coupling produces the OPE
which includes the expected Kac-Moody algebra of gauge modes as a subalgebra**

For quantum corrections, study BRST-variation of, e.g.

$$J^a[r, s](0)J^b[t, u](z) \sim \frac{1}{z}f_c^{ab}J^c[r+t, s+u](0)$$

$$J^a[r, s](0)\tilde{J}^b[t, u](z) \sim \frac{1}{z}f_c^{ab}\tilde{J}^c[r+t, s+u](0)$$



$$J^a[r, s](0)E[t, u](z) \sim \frac{1}{z}\frac{(ts-ur)}{t+u}\tilde{J}^a[t+r-1, s+u-1](0)$$

$$J^a[r, s](0)F[t, u](z) \sim -\frac{1}{z}\partial_z\tilde{J}^a[r+t, s+u](0) - \frac{1}{z^2}\left(1 + \frac{r+s}{t+u+2}\right)\tilde{J}^a[r+t, s+u](0)$$

$$J^a[r, s](0)J^b[t, u](z) \sim \frac{1}{z}K^{ab}(ru-st)F[r+t-1, s+u-1](0)$$

$$- \frac{1}{z}K^{ab}(t+u)\partial_z E[r+t, s+u](0) - \frac{1}{z^2}K^{ab}(r+s+t+u)E[r+t, s+u](0).$$

(N.B. The axion towers are subleading in $\hbar$, even though their OPEs come from studying gauge variation of tree-diagrams)

Associativity at the quantum level is guaranteed for *twistorial* theories

[Costello-NMP: 2204.05301]

$$\text{Split}_+^{[1]}(a^+, b^+) = -\frac{N_c}{96\pi^2} \frac{[ab]}{\langle ab \rangle^2} \quad [\text{Bern-Dixon-Kosower}]$$

$$J_a[1, 0](z_1) J_b[0, 1](z_2) \sim \frac{N_c}{48\pi^2} \frac{1}{\langle 12 \rangle^2} f_{ab}^c \tilde{J}_c[0, 0](z_1) - \frac{N_c}{96\pi^2} \frac{1}{\langle 12 \rangle} f_{ab}^c \frac{1}{2\pi i} \partial_z \tilde{J}_c[0, 0](z_1).$$

**Failure of associativity in $J, \tilde{J}$ in one-loop
Axion field necessary for its (local) restoration**

$$C = \frac{3}{2(2\pi i)^3 12}$$

$$D = -\frac{h^\vee}{(2\pi i)^3 12}$$

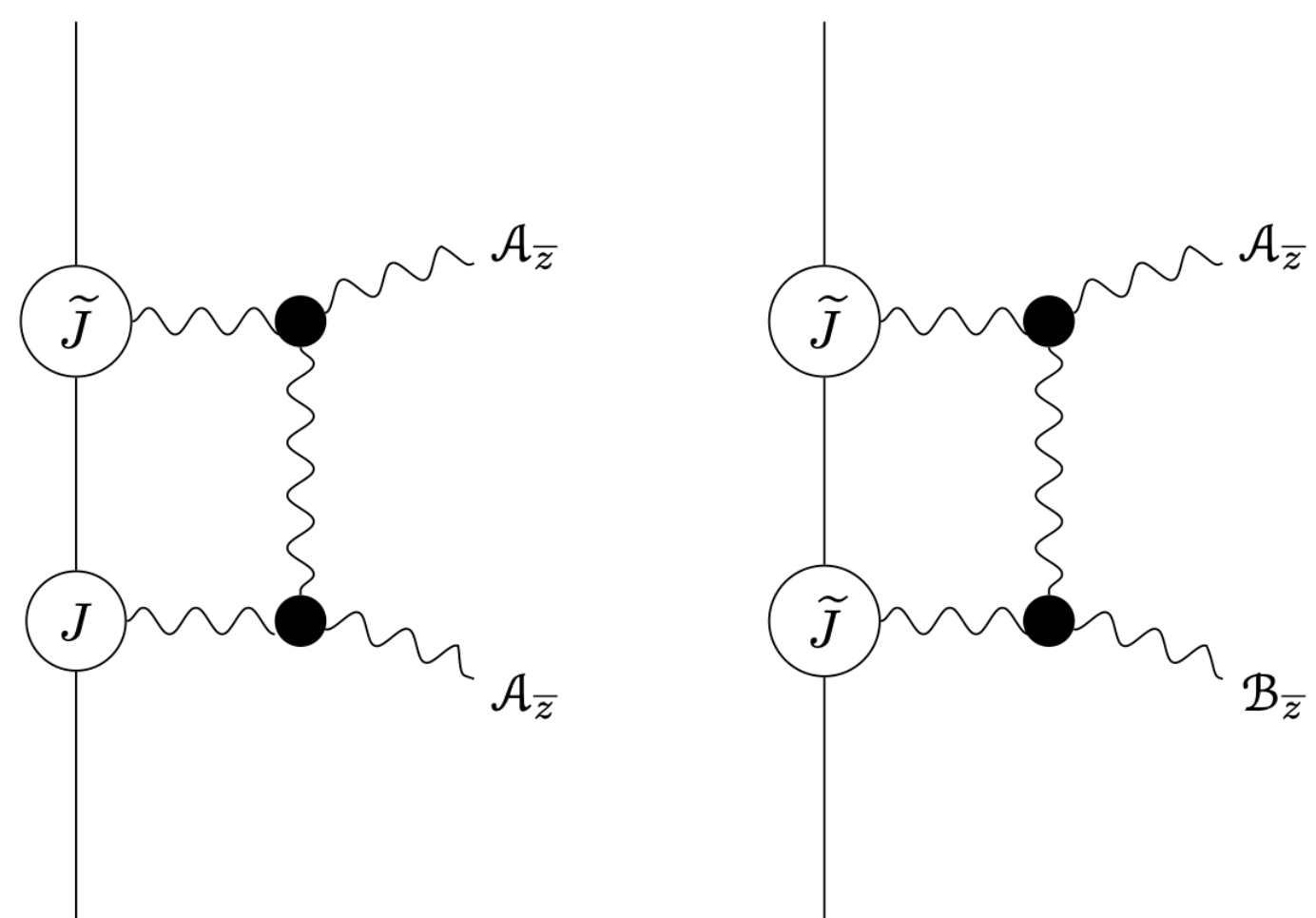
Some of the one-loop OPEs can be obtained by demanding associativity. Coefficients fixed by λ_g

The complete one-loop chiral algebra was obtained by my student, Victor Fernandez. [Fernandez, '23]

$$J_a[1, 0](0) J_b[0, 1](z) = -\frac{1}{2\pi i z} C K^{fe} (f_{ae}^c f_{bf}^d + f_{ae}^d f_{bf}^c) : J_c[0, 0] \tilde{J}_d[0, 0] : + \frac{1}{2\pi i z} \frac{1}{2} D f_{ab}^c \partial_z \tilde{J}_c(0) + \frac{1}{2\pi i z^2} D f_{ab}^c \tilde{J}_c(0).$$

$$J_a[1, 0](0) \tilde{J}_b[0, 1](z) \simeq C \frac{1}{z} K^{fe} f_{ae}^c f_{bf}^d : \tilde{J}_c[0, 0] \tilde{J}_d[0, 0] :$$

$$J_a[0, 1](0) \tilde{J}_b[1, 0](z) \simeq -C \frac{1}{z} K^{fe} f_{ae}^c f_{bf}^d : \tilde{J}_c[0, 0] \tilde{J}_d[0, 0] :$$



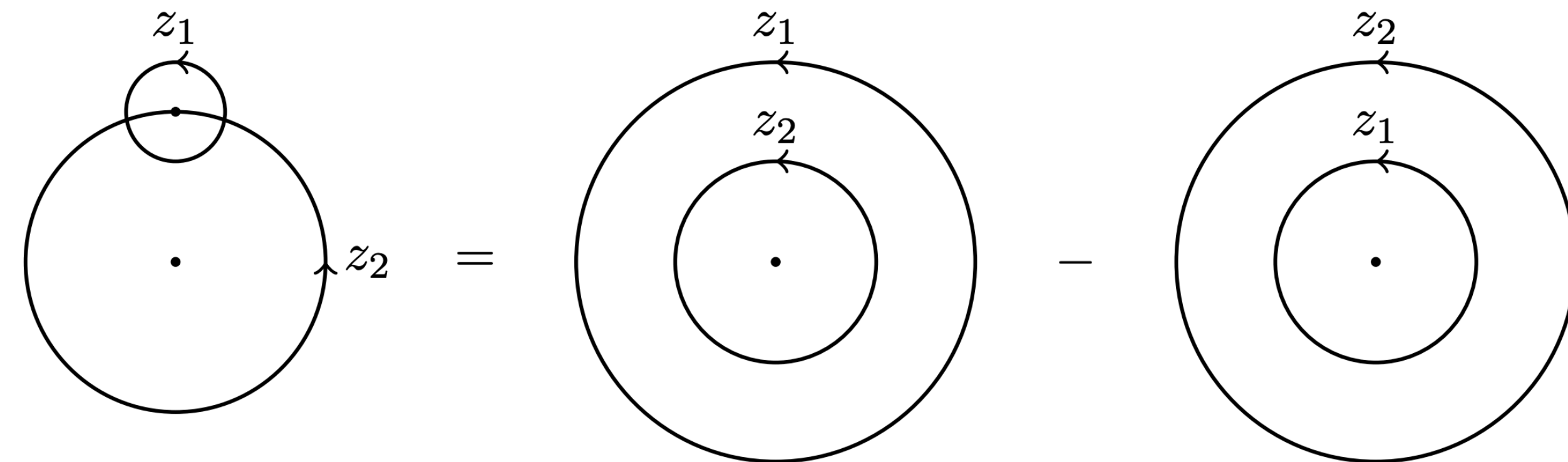
In fact, we now know the chiral algebra in its entirety at arbitrary loop order using constraints from symmetry and associativity

To fix the OPE corrections

to appear soon, [Fernandez-NMP]

- Step 1: Determine the general form of the OPE corrections (bulk/defect couplings, symmetries)
- Step 2: Determine conditions on the undetermined numerical coefficients from associativity
- Step 3: Solve those equations to fix the coefficients

$$\oint_{|z_2|=2} dz_2 z_2^l \left(\oint_{|z_1|=1} \phi_1(z_1) \phi_2(z_2) \right) \phi_3(0) = \oint_{|z_1|=2} dz_1 \phi_1(z_1) \left(\oint_{|z_2|=1} dz_2 z_2^l \phi_2(z_2) \phi_3(0) \right) - (-1)^{F_1 F_2} \oint_{|z_2|=2} dz_2 z_2^l \phi_2(z_2) \left(\oint_{|z_1|=1} dz_1 \phi_1(z_1) \phi_3(0) \right)$$



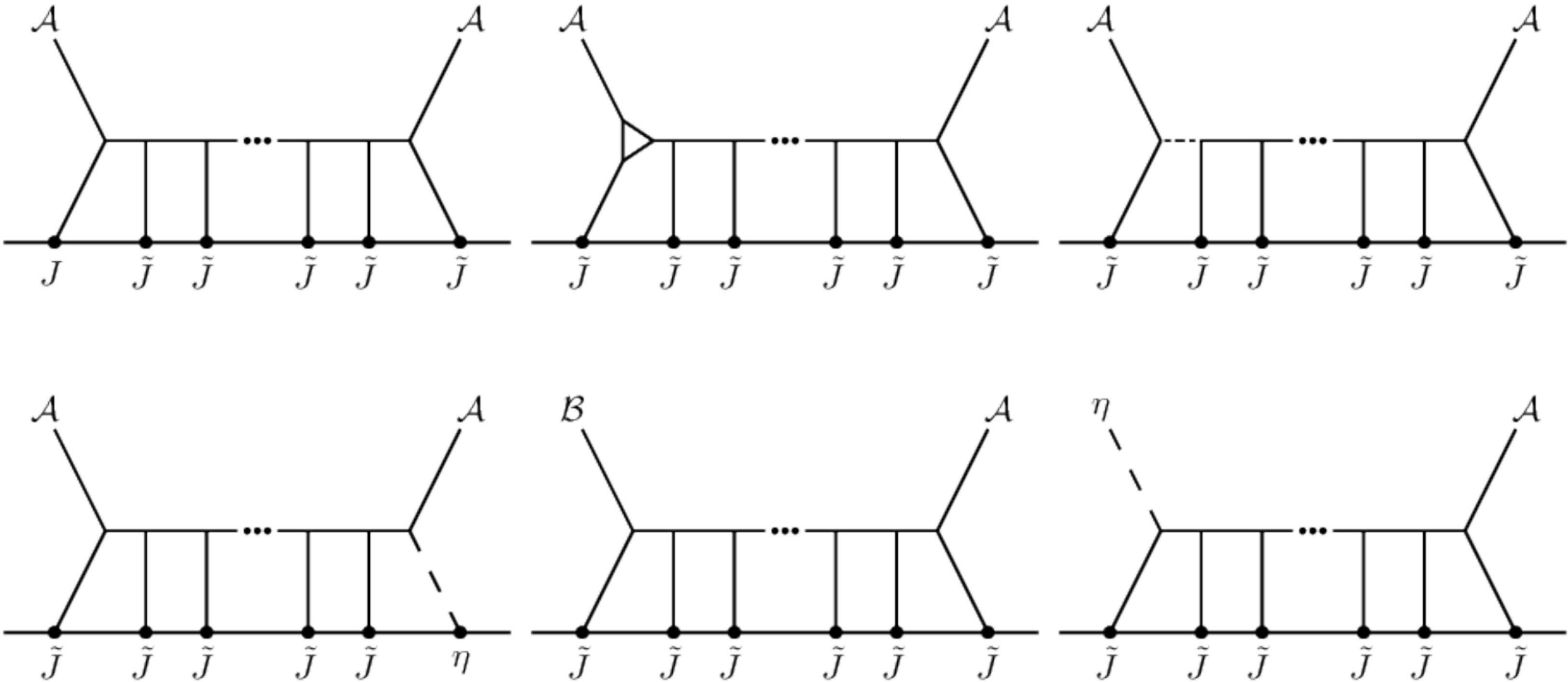
$$\phi_i(z) \phi_j(w) \sim \sum_{n \geq 1} \frac{\{\phi_i \phi_j\}_n(w)}{(z - w)^n}$$

$$\{\{\phi_1 \phi_2\}_1 \phi_3\}_{l+1} = \{\phi_1 \{\phi_2 \phi_3\}_{l+1}\}_1 - (-1)^{F_1 F_2} \{\phi_2 \{\phi_1 \phi_3\}_1\}_{l+1}$$

SDYM + axion is invariant under the following scaling symmetry: $\hbar \rightarrow \alpha \hbar \qquad \mathcal{B} \rightarrow \alpha \mathcal{B} \qquad \eta \rightarrow \sqrt{\alpha} \eta$

To determine which diagrams contribute to a given OPE, we use:

- Interactions in the twistorial Lagrangian are only cubic in form
- Only two type of vertices: $\mathcal{B}\mathcal{A}^2$ and $\eta\mathcal{A}^2$
- Invariance under rescaling of $\hbar$



Here are some of the terms in terms of the unknown coefficients.

Pole order: fixed by matching combined dilatation symmetry

$$\frac{1}{z} \text{ and } \partial_z \text{ have combined dilatation} = 1$$

$J, \tilde{J}, E, F$ have combined dilatation = 1, 0, 0, 1 respectively

$$\tilde{J}_a[t](z)J_b[r](0) \sim \frac{1}{z} \sum_{m \geq 1}^{\sum_{j=1}^{m+1} k_j = t+r-m} \sum_{k_j^i \geq 0} \hbar^m \overset{(m)}{f}_{(t,r)}[k_1, \dots, k_{m+1}]_{ab}^{i_1 \dots i_{m+1}} : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

$$E[t](z)J_b[r](0) \sim \frac{1}{z} \sum_{m \geq 1}^{\sum_{j=1}^{m+1} k_j = t+r-m-1} \sum_{k_j^i \geq 0} \hat{\lambda}_{\mathfrak{g}} \hbar^{m+\frac{1}{2}} \overset{(m)}{j}_{(t,r)}[k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

$$F[t](z)J_b[r](0) \sim \sum_{m \geq 1}^{\sum_{j=1}^{m+1} k_j = t+r-m} \sum_{k_j^i \geq 0} \hat{\lambda}_{\mathfrak{g}} \hbar^{m+\frac{1}{2}} \left(\frac{1}{z^2} \overset{(m)}{k}_{(t,r)}[k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} + \frac{1}{z} \overset{(m)}{l}_{(t,r)}[k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} \hat{\partial}_1 \right) : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

- Use associativity to fix OPE coefficients in terms of the $f^{(m)}$ coefficients. $\{\tilde{J}_a[t]\{J_b[r]J_c[s]\}_1\}_1 = \{J_b[r]\{\tilde{J}_a[t]J_c[s]\}_1\}_1 - \{J_c[s]\{\tilde{J}_a[t]J_b[r]\}_1\}_1$
 $\{\{J_a[t]J_b[r]\}_1J_c[s]\}_1 = \{J_a[t]\{J_b[r]J_c[s]\}_1\}_1 - \{J_b[r]\{J_a[t]J_c[s]\}_1\}_1$
- We obtained a recursion relation for $f^{(m)}$ at arbitrary m . $\{F[t]\{J_b[r]J_c[s]\}_1\}_2 = \{J_b[r]\{F[t]J_c[s]\}_2\}_1 - \{J_c[s]\{J_b[r]F[t]\}_1\}_2$
- We used the recursion relation to find a closed-form expression for $f^{(1)}$, and a recursive expression for $f^{(m)}$ with $m > 1$.

$$K_{ab}^{i_1 \cdots i_{m+1}} = -f_{aj_1}^{i_1} K^{j_1 j_2} f_{j_2 j_3}^{i_2} \cdots f_{j_{2m-2} j_{2m-1}}^{i_m} K^{j_{2m-1} j_{2m}} f_{j_{2m} b}^{i_{m+1}}$$

$$\alpha(t, k) = t^2(k^1 + 1) - t^1(k^2 + 1) \quad \beta(t) = t^1 + t^2$$

$$\binom{(m)}{j}_{(t,r)} [k_1, \dots, k_{m+1}]_b^{i_1 \cdots i_{m+1}} = - \left(\frac{\alpha(t, k_1)}{\beta(t)} \right) \binom{(m)}{f}_{(t-1,r)} (k_1, \dots, k_{m+1}) K^{i_1 j} K_{j b}^{i_2 \cdots i_{m+1}}$$

$$\binom{(m)}{k}_{(t,r)} [k_1, \dots, k_{m+1}]_b^{i_1 \cdots i_{m+1}} = - \left(\frac{\beta(k_1 + 1)}{\beta(t + 1)} \right) \binom{(m)}{f}_{(t,r)} (k_1, \dots, k_{m+1}) K^{i_1 j} K_{j b}^{i_2 \cdots i_{m+1}}$$

$$\binom{(m)}{l}_{(t,r)} [k_1, \dots, k_{m+1}]_b^{i_1 \cdots i_{m+1}} = - \binom{(m)}{f}_{(t,r)} (k_1, \dots, k_{m+1}) K^{i_1 j} K_{j b}^{i_2 \cdots i_{m+1}}.$$

Some terms with single poles were also determined more formally by homotopy transfer methods [Zeng]

$f^{(1)}$ **two ways**

$$\sum_{ac} = \left(\sum_{a=0}^{\min(m,l_1)} \sum_{c=1}^{\min(n,l_2+1)} \binom{l_1}{a} \binom{l_2}{c-1} - \sum_{a=1}^{\min(m,l_1+1)} \sum_{c=0}^{\min(n,l_2)} \binom{l_1}{a-1} \binom{l_2}{c} \right) \binom{m}{a} \binom{n}{c}$$

holomorphic integral [Fernandez]

$$\mathcal{M}_1 = \sum_{ac} \frac{(m+n-a-c)!(a+c-1)!a!c!(r+m-a)!(s+n-c)!(1+k_1+k_2)!(1+l_1+l_2-a-c)!}{(m+n)!(1+m+r+n+s-a-c)!k_1!k_2!l_1!l_2!}$$

$$m(p,q;x,y;u) = \sum_{j=1}^{\text{Min}[u,x+1]} \frac{(p-j)!(1+x+y-j)!}{(1+p+q-j)!(1+x-j)!}.$$

associativity [Fernandez-NMP]

$$\begin{aligned} \overset{(1)}{f}_{(r,t)}(k,l) &= \left(\frac{1}{16\pi^2}\right) \frac{(r^2+t^2)!(1+k^1+k^2)!}{k^1!k^2!l^2!} m(r^1+t^1,r^2+t^2,l^1,l^2;t^1) \\ &\quad - \left(\frac{1}{16\pi^2}\right) \frac{t^2!(1+k^1+k^2-r^1-r^2)!}{(k^1-r^1)!(k^2-r^2)!l^2!} m(t^1,t^2,l^1,l^2;t^1) \theta(k^1-r^1) \theta(k^2-r^2) \\ &\quad - \left(\frac{1}{16\pi^2}\right) \frac{r^1!(1+k^1+k^2)!}{k^1!k^2!(l^1-t^1)!} m(r^2+t^2,r^1,l^2,l^1-t^1;t^2) \theta(l^1-t^1). \end{aligned}$$

It turns out, these two heinous expressions
are indeed equal!

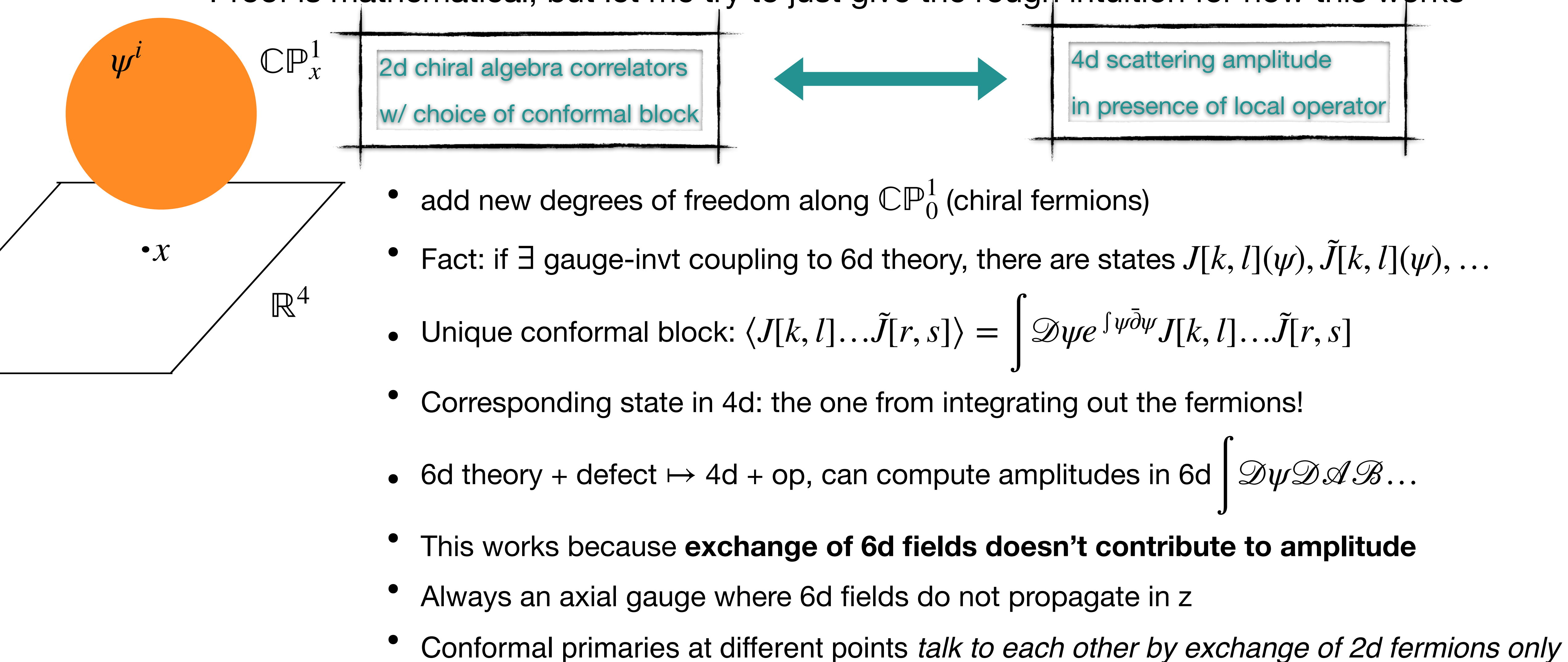
Recursive Expression for $f^{(m>1)}$

$$\begin{aligned}
 f_{(r^1, r^2)(t^1, t^2)}^{(m)}[k_1; \dots; k_{m+1}] = & - \sum_{j=1}^{t^1} f_{(r^1, r^2)(t^1-j, t^2)}^{(m-1)}[k_1; \dots; k_{m-1}; l] f_{(l^1, l^2)(1,0)}^{(1)}[k_m; (k_{m+1}^1 + 1 - j, k_{m+1}^2)] \\
 & + \sum_{j=1}^{t^1} f_{(r^1, r^2)(l^1, l^2)}^{(m-1)}[k_1; \dots; k_m] f_{(t^1-j, t^2)(1,0)}^{(1)}[l; (k_{m+1}^1 + 1 - j, k_{m+1}^2)] \\
 & - \sum_{j=1}^{t^2} f_{(r^1, r^2)(0, t^2-j)}^{(m-1)}[k_1; \dots; k_{m-1}; l] f_{(l^1, l^2)(0,1)}^{(1)}[k_m; (k_{m+1}^1 - t^1, k_{m+1}^2 + 1 - j)].
 \end{aligned}$$

Also a nice closed form expression to this recursion relation [Zeng] in terms of Clebsch-Gordon coefficients, Wigner 6j symbols

We wanted these OPEs because they can help us compute loop-level 4d form factors

Proof is mathematical, but let me try to just give the rough intuition for how this works



Known: 0, 1, 2-loop level quantities with these methods (e.g. Parke-Taylor formula, 1-loop & 2-loop all-+ form factor for $SU(N)$ + matter). [C-P, C] 3+ loops are WIP

We got a lot of new mileage from thinking about integrable 4d theories at the quantum level + their 6d uplifts to twistor space, but there is much to do.

Chiral algebra bootstrap for 4d form factors,
many loop-level investigations underway or yet to be
studied. Other connections to amplitudes program?
SDGR case also accessible with these methods

Thank you!

**There is one other twistorial theory that we have studied recently,
that is of interest for a different (but related) application of these ideas**

[Costello-NMP-Sharma x2]

Consider two coupled 6d holomorphic theories on twistor space:

Kodaira-Spencer theory of complex structure deformations

&

Holomorphic Chern-Simons theory

These theories can be thought of as a twist of **closed** + **open** type IIB or I strings,
and are anomaly free for the same reasons.

If we take the type I example and reduce to 4d, the resulting coupled theories are
G=SO(8) 4d WZW model [Losev-Moore-Nekrasov-Shatashvili] and a theory of a fluctuating
periodic scalar field governed by “Mabuchi functional”.

$$\text{e.o.m: } R(K + \rho) = 0$$

$$K \mapsto K + \rho$$

One can add N D1 branes on $\mathbb{CP}_0^1$.

Their worldvolume theory is a chiral algebra given by the BRST reduction (by $\mathrm{Sp}(N)$) of free symplectic bosons

We also compute how the presence of these branes deforms the complex structure of $\mathbb{PT}$ (the holomorphic remnant of backreaction)

The twistor space of $\mathbb{R}^4$ gets deformed to the twistor space $\simeq \mathrm{EAdS}_3 \times S^3$ of Burns space (asymptotically Euclidean, scalar-flat Kahler)

$$u^\alpha = (u^1, u^2) \in \mathbb{C}^2, \quad ||u||^2 = |u^1|^2 + |u^2|^2 \quad ds^2 = ||du||^2 + N \frac{|u^1 du^2 - u^2 du^1|}{||du||^4}$$

Here is the metric in coordinates on $\mathbb{C}^2 \setminus 0$, but it can be smoothly extended over the origin.
Burns space as a complex manifold is a blow-up of $\mathbb{C}^2$ at the origin

This gives a toy holographic duality for a 4d asymptotically flat spacetime. Scattering of the QFT on a curved 4d spacetime is controlled by a 2d chiral algebra.

We computed some large-N 3 pt functions (dually, collinear limits) as a check

[Beem-Rastelli...]

$$X_{\alpha ij} \in \mathbb{C}^2 \otimes \bigwedge^2_{\text{t.f.}} \mathbb{C}^{2N}$$

$$I_{ri} \in \mathbb{C}^8 \otimes \mathbb{C}^{2N}$$

+ bc ghosts for BRST reduction

$$I_{im}(z_1)I_{jn}(z_2) \sim \frac{\delta_{ij}\omega_{mn}}{z_{12}}$$

$$X_{\alpha mn}(z_1)X_{\beta rs}(z_2) \sim \frac{\epsilon_{\alpha\beta}}{z_{12}} \left(\omega_{m[r} \omega_{n|s]} - \frac{\omega_{mn}\omega_{rs}}{2N} \right)$$

$$\text{BRST invariant operators:} \quad \langle I_r, X_1^{(k} X_2^{l)} I_s \rangle(z) \quad \longleftrightarrow \quad \phi[k,l](z) \mathfrak{t}_{rs}$$

An associative, local chiral algebraic structure persists here because the 4d theory is twistorial (nice analyticity properties).

But how should we try to construct generic 4d QFTs? Einstein gravity?

$$\mathcal{L} = \frac{N}{8\pi^2} \int_M \partial\bar\partial K \wedge \text{tr}(g^{-1}\partial g \wedge g^{-1}\bar\partial g) - \frac{N}{24\pi^2} \int_{M\times[0,1]} \partial\bar\partial K \wedge \text{tr}(\tilde g^{-1}d\tilde g)^3$$

$$N \in \mathbb{Z}_+$$

$$g : M \rightarrow SO(8) \qquad g = e^{\phi \mathfrak{t}}$$

$$\triangle \phi = 0 \qquad \begin{array}{l} \text{We solved this equation exactly on Burns space.} \\ \text{When } N \rightarrow 0 \text{ we recover ordinary plane waves} \end{array}$$

$$\phi(\omega, z, \bar{z}) = \sum_{p=0}^\infty (\omega)^p \sum_{k+l=p} \frac{\bar{z}^l}{k!l!} \phi[k,l](z)$$

To start, we have checked 2 & 3-pt funs in this proposed duality when $N \rightarrow \infty$
 For example, matching states between part of chiral algebra to the WZW4 sector in the bulk:

$$g = \exp(\phi \mathfrak{t})$$

$$\Delta \phi = 0$$

“Momentum eigenstates”

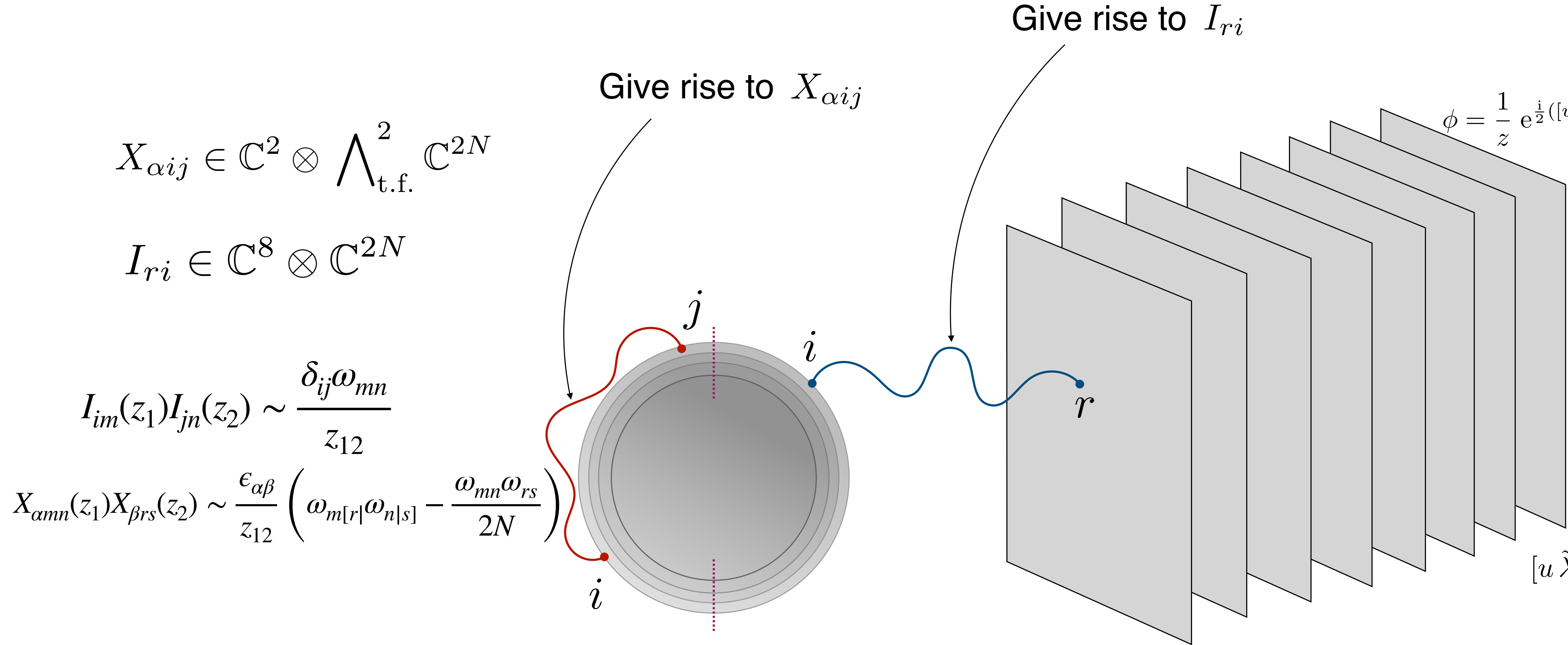
$$\phi = \frac{1}{z} e^{\frac{i}{2}([u \tilde{\lambda}] + z[\hat{u} \tilde{\lambda}])} \left(\cos \frac{\psi}{2} + \frac{i([u \tilde{\lambda}] + z[\hat{u} \tilde{\lambda}])}{\psi} \sin \frac{\psi}{2} \right)$$

$$k \cdot x$$

$$\xrightarrow{N \rightarrow 0} \frac{1}{z} e^{ik \cdot x}$$

$$\psi = \sqrt{([u \tilde{\lambda}] + z[\hat{u} \tilde{\lambda}])^2 + \frac{4N}{||u||^2} [u \tilde{\lambda}][\hat{u} \tilde{\lambda}]}$$

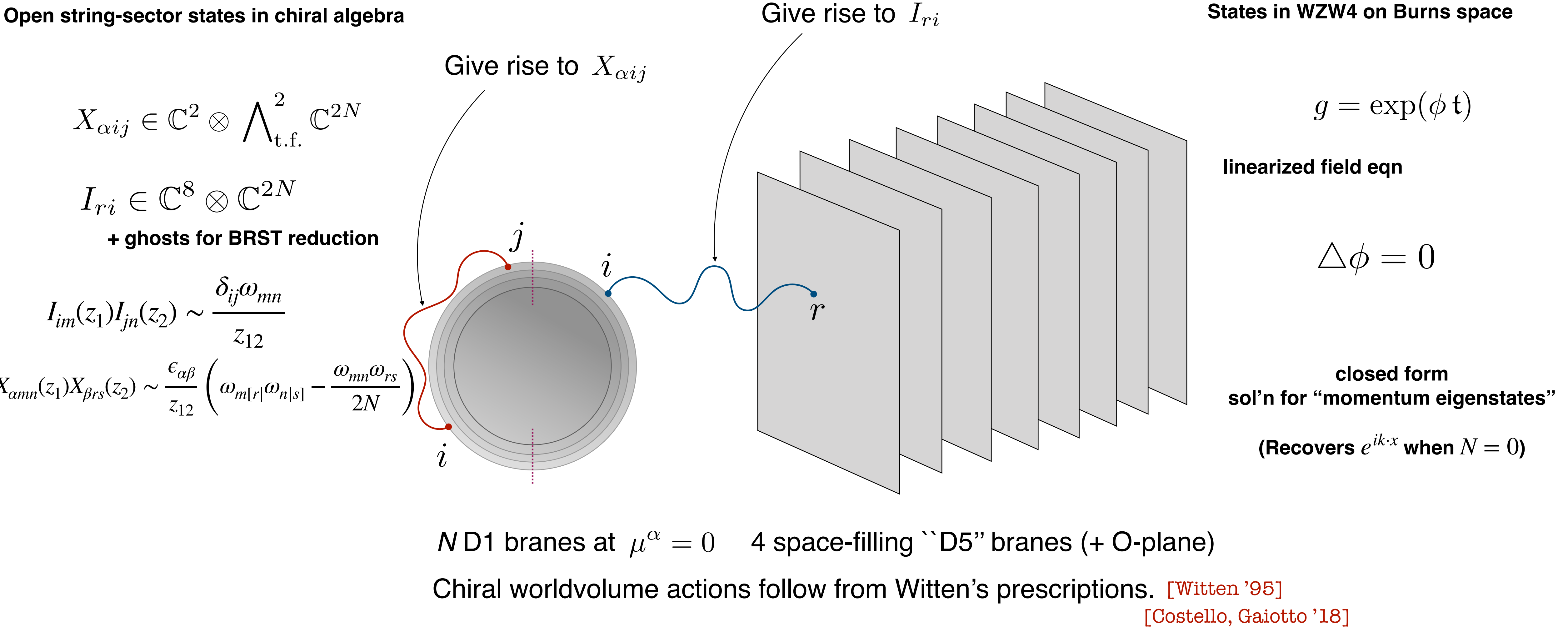
$$[u \tilde{\lambda}] = \tilde{\lambda}_1 u^1 + \tilde{\lambda}_2 u^2, \quad [\hat{u} \tilde{\lambda}] = \tilde{\lambda}_2 \bar{u}^1 - \tilde{\lambda}_1 \bar{u}^2$$



N D1 branes at $\mu^\alpha = 0$ 4 space-filling “D5” branes (+ O-plane)

Chiral worldvolume actions follow from Witten’s prescriptions. [Witten ’95]
 [Costello, Gaiotto ’18]

To start, we have checked 2 & 3-pt funs in this proposed duality when $N \rightarrow \infty$
 For example, matching states between part of chiral algebra to the WZW4 sector in the bulk:



- Dictionary between soft modes of states and symmetry currents in the dual CFT

$$\tilde{\lambda}_\alpha = \omega(1, \tilde{z})$$

Soft expansion

$$\phi(\omega, z, \tilde{z}) = \frac{1}{z} \sum_{p=0}^{\infty} (i\omega)^p \sum_{k+l=p} \frac{\tilde{z}^l}{k!l!} \phi[k, l](z)$$

Dictionary

$$\frac{1}{z} \phi[k, l](z) \text{ t}_{rs} \longleftrightarrow \langle I_r, X_1^{(k)} X_2^{(l)} I_s \rangle(z)$$

Examples of soft modes

$$\begin{aligned} \phi[0, 0] &= 1, & \phi[1, 0] &= u^1 - z \bar{u}^{\bar{2}}, & \phi[0, 1] &= u^2 + z \bar{i} \\ \phi[1, 1] &= \phi[0, 1] \phi[1, 0] + \frac{Nz}{2} \frac{|u^1|^2 - |u^2|^2}{|u^1|^2 + |u^2|^2} \end{aligned}$$

- chiral algebra OPE computations easily done in planar limit by Wick contractions
- In bulk, Euclidean amplitudes computed via on-shell effective action, as in standard AdS/CFT computations

$$\begin{aligned} J_a[\tilde{\lambda}_1](z_1) J_b[\tilde{\lambda}_2](z_2) &\sim \frac{f_{ab}^c}{z_{12}} J_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_2) \\ &- \frac{[1\,2] f_{ab}^c}{z_{12}^2} \int_0^1 d\omega_1 \int_0^1 d\omega_2 J_c[\omega_1 \tilde{\lambda}_1 + \omega_2 \tilde{\lambda}_2](z_2) \end{aligned}$$

$$\begin{aligned} \phi_1 \cdot \phi_2 &\sim \frac{f_{a_1 a_2}^c}{z_{12}} \phi_c(z_2, \tilde{\lambda}_1 + \tilde{\lambda}_2) \\ &- \frac{[1\,2] f_{a_1 a_2}^c}{z_{12}^2} \int_0^1 d\omega_1 \int_0^1 d\omega_2 \phi_c(z_2, \omega_1 \tilde{\lambda}_1 + \omega_2 \tilde{\lambda}_2) \\ &\quad + O([1\,2]^2). \end{aligned} \quad ($$

Our toy top-down example of asymptotically-flat holography has passed standard checks.

Next: Go beyond the planar limit, study states of dim'n $\mathcal{O}(\sqrt{N})$, $\mathcal{O}(N)$, fully flesh out embedding in physical string, etc.

This gives a concrete toy-model of a “celestial holography”-type correspondence, analogous to the supersymmetric sectors of AdS/CFT we have been studying in twisted holography program

Unfortunately, I only know how to build associative chiral algebras using these methods for theories that are integrable/self-dual in 4d. Relatedly, the closed-string sector of the bulk theory, which only captures Kahler potential fluctuations, means our gravitational sector is rather poor.

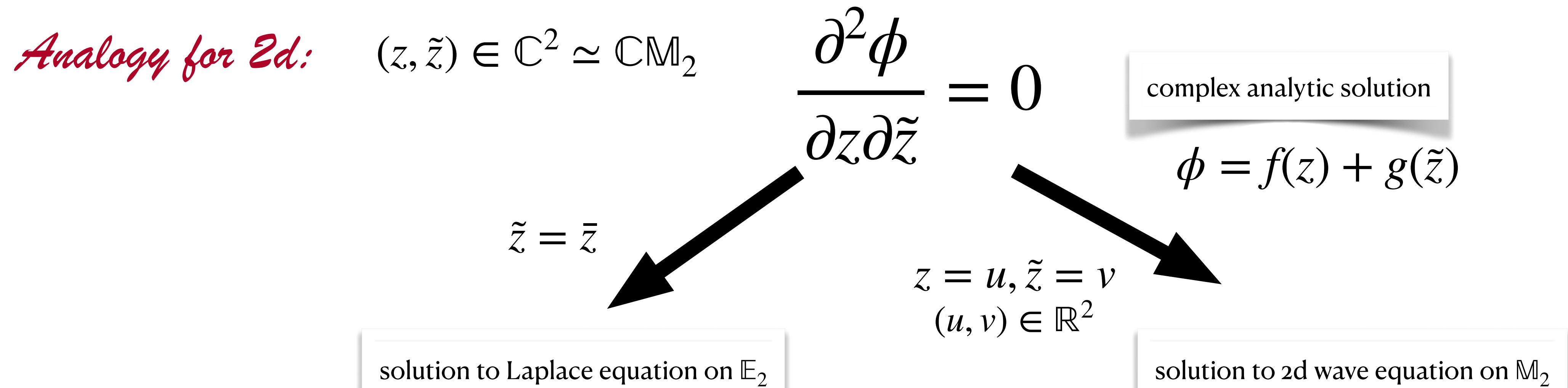
Perhaps the first step towards 4d asymptotically flat holography in more physically interesting setups would be to find a chiral algebra dual for self-dual Einstein gravity (perhaps coupled to SDYM).

We know how to cancel the twistorial anomaly there, thanks to work of Bittleston-Sharma-Skinner.

Note that Burns space can be viewed as an Einstein-Maxwell instanton...

non-unitarity, operator product associativity, integrability, etc. are all connected,
insight for how to move beyond the twisted realm?

The setting for today's story is *twistor space*
 What is twistor space good for? To motivate, a 2d analogy.



Further:

Fix $\phi, \nabla \phi$ on $S \subset \mathbb{CM}_2$, $D(S) = \{(z, \tilde{z}) : \text{both null lines through } (z, \tilde{z}) \text{ meet } S\}$

$S \subset \mathbb{E}_2$

$D(S) \subset \mathbb{CM}_2$ to which ϕ can be analytically extended

$S \subset \Sigma$

$D(S) \subset \mathbb{M}_2$ usual domain of dpdce

