

On Elliptic units for almost totally real fields.

Simons Symposium, Schloss Elmau 13/6/2024
Pierre Charollois, Sorbonne Université

joint work with Nicolas Bergeron and Luis García



based on arxiv 2311.04110

Hilbert's 12th problem (1900)

(Hilbert, transl. Bull. AMS 1902). " It will be seen that in the problem just sketched the three fundamental branches of mathematics, number theory, algebra and function theory, come into closest touch with one another, and I am certain that the theory of analytical functions of several variables in particular would be notably enriched if one should succeed in finding and discussing those functions which play the part for algebraic number field corresponding to that of the exponential function in the field of rational numbers and of the elliptic modular functions in the imaginary quadratic number field."

Very much an open problem. To set the stage for our discussion of the case of **complex cubic fields K** , we proceed by analogy and first review the positive results that motivated this problem (**when $K = \mathbb{Q}$ and K imaginary quadratic**).

— | —

**Some classical results relevant to Hilbert's
12th problem :**

Cyclotomic units, elliptic units

Cyclotomic units : If $(a, f) = 1$ are coprime integers, then

$$u_a(f) = (1 - e^{\frac{2i\pi a}{f}}) = 1 - q_{\frac{a}{f}}$$

is a f -unit in the ring $\mathbf{Z}[e^{\frac{2i\pi}{f}}]$, and a unit as soon as f is divisible by two distinct primes.

It is rather these kind of units that we are after.

Kronecker's Jugendtraum (1880)

Let α be a *modular unit*. It is a nowhere vanishing function on the modular curve $\mathcal{H}/\Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : N \text{ divides } c \right\}.$$

Example. The quotient

$$\Delta(\tau)/\Delta(N\tau) \quad \text{where} \quad \Delta(\tau) = q_\tau \prod_{n=1}^{\infty} (1 - q_\tau)^{24},$$

or more generally the expressions

$$\Delta_\delta(\tau) = \prod_{d|N} \Delta(d\tau)^{n_d} \quad \text{with} \quad \sum_{d|N} n_d = 0,$$

are modular units.

Kronecker's Jugendtraum (1880)

Complex Multiplication (CM) theory by Shimura and Taniyama implies that if τ_0 is in $\mathcal{H} \cap K$, with K imaginary quadratic, then $\alpha(\tau_0)$ belongs to an abelian extension of K .

Elliptic units : We consider more generally the **theta function** on $\mathbf{C} \times \mathcal{H}$

$$\theta_0(z, \tau) = (1 - q_z) \prod_{n=1}^{\infty} (1 - q_{\tau}^n q_z)(1 - q_{\tau}^n q_{-z}),$$

Symmetries :

Kronecker Jugendtraum (1880), theta function

Elliptic units : We consider the **theta function** on $\mathbf{C} \times \mathcal{H}$

$$\theta_0(z, \tau) = (1 - q_z) \prod_{n=1}^{\infty} (1 - q_{\tau}^n q_z)(1 - q_{\tau}^n q_{-z}),$$

Symmetries :

$$\begin{aligned} \text{i) periodicities: } \theta_0(z, \tau) &= -\theta_0(z + 1, \tau) \\ &= \underbrace{-e^{2i\pi z}} \theta_0(z + \tau, \tau). \end{aligned}$$

Kronecker Jugendtraum (1880), theta function

Elliptic units : We consider the **theta function** on $\mathbf{C} \times \mathcal{H}$

$$\theta_0(z, \tau) = (1 - q_z) \prod_{n=1}^{\infty} (1 - q_{\tau}^n q_z)(1 - q_{\tau}^n q_{-z}),$$

Symmetries :

i) periodicities: $\theta_0(z, \tau) = -\theta_0(z + 1, \tau)$
 $= \underbrace{-e^{2i\pi z}}_{\text{factor}} \theta_0(z + \tau, \tau).$

ii) distribution : $\theta_0(z, \tau) = \prod_{k, \ell=1}^N \theta_0\left(\frac{z + k + \ell\tau}{N}, \tau\right).$

Kronecker Jugendtraum (1880), theta function

Elliptic units : We consider the **theta function** on $\mathbf{C} \times \mathcal{H}$

$$\theta_0(z, \tau) = (1 - q_z) \prod_{n=1}^{\infty} (1 - q_{\tau}^n q_z)(1 - q_{\tau}^n q_{-z}),$$

Symmetries :

$$\begin{aligned} \text{i) periodicities: } \theta_0(z, \tau) &= -\theta_0(z + 1, \tau) \\ &= \underbrace{-e^{2i\pi z}} \theta_0(z + \tau, \tau). \end{aligned}$$

ii) distribution

iii) modularity under $SL_2(\mathbf{Z})$, a sample :

$$\theta_0(z, \tau + 1) = \theta_0(z, \tau),$$

$$\boxed{\theta_0\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = e^{i\pi Q_2(z, \tau)} \theta_0(z, \tau)}, \text{ where } Q_2 \in \mathbf{Q}(\tau)[z]_{\leq 2}.$$

Kronecker's Jugendtraum (1880), CM specialisations

When τ_0 is CM and $\mathfrak{a}$ is an integral ideal of K of norm $N = N(\mathfrak{a})$ coprime to 6, the ratio

$$\theta_{\mathfrak{a}}(z, \tau_0) = \theta_0(z, \tau_0)^N / \theta_0(Nz, N\tau_0)$$

becomes $\mathbf{Z} + \tau_0\mathbf{Z}$ -periodic in z .

Let $v_0 \in \mathfrak{f}^{-1}L$ be a primitive $\mathfrak{f}$ -division point in L . If $(\mathfrak{a}, 6\mathfrak{f}) = 1$, then

- ① $u_{\mathfrak{a}}(\mathfrak{f}) = \theta_{\mathfrak{a}}(v_0, \tau_0)$ is a **$\mathfrak{f}$ -unit** in $K(\mathfrak{f})$, and
- ② for every integral ideal $\mathfrak{c}$ coprime to $\mathfrak{f}$,

$$\sigma_{\mathfrak{c}}(u_{\mathfrak{a}}(\mathfrak{f})) = u_{\mathfrak{a}\mathfrak{c}}(\mathfrak{f}) u_{\mathfrak{c}}(\mathfrak{f})^{-N(\mathfrak{c})}.$$

This essentially solves Hilbert's 12th problem when the base field is imaginary quadratic.

Kronecker's Jugendtraum (1880), CM specialisations

When τ_0 is CM and $\mathfrak{a}$ is an integral ideal of K of norm $N = N(\mathfrak{a})$ coprime to 6, the ratio

$$\theta_{\mathfrak{a}}(z, \tau_0) = \theta_0(z, \tau_0)^N / \theta_0(Nz, N\tau_0)$$

becomes $\mathbf{Z} + \tau_0\mathbf{Z}$ -periodic in z .

Let v_0 be a primitive f -division point in L . If $(\mathfrak{a}, 6f) = 1$, then

- ① $u_{\mathfrak{a}}(f) = \theta_{\mathfrak{a}}(v_0, \tau_0)$ is a **f -unit** in $K(f)$, and
- ② for every integral ideal $\mathfrak{c}$ coprime to f ,

$$\sigma_{\mathfrak{c}}(u_{\mathfrak{a}}(f)) = u_{\mathfrak{a}\mathfrak{c}}(f) u_{\mathfrak{c}}(f)^{-N(\mathfrak{c})}.$$

This essentially solves Hilbert's 12th problem when the base field is imaginary quadratic.

The **modulus** of these **elliptic units** encode special values (at $s = 0$) of partial zeta functions of K by means of the **Kronecker limit formula**.

In this sense they are prototype of **Stark units**.

$$K_0 = \mathbb{Q}(i), \tau_0 = \frac{2+i}{5}.$$

$$\alpha_0 = 2 + i \text{ of norm } N_0 = 5,$$

with $v_0 = \frac{1}{6}$ a 6-division point ($f = 6$).

Then

$$\theta_{\alpha_0}(v_0, \tau_0) = \frac{\theta_0(\frac{1}{6}, \frac{2+i}{5})^5}{\theta_0(\frac{5}{6}, 2+i)} \simeq 2.285570413 \dots - i \times 1.82956517 \dots.$$

By CM theory, this complex number is a root of a unitary $P \in \mathbb{Z}[x]_{\leq 8}$ defining a unit in $K_0(6)$.

Confirmed and made explicit by numerical experimentation :
given $\alpha \in \mathbf{C}$, PARI/GP's terrific command

$$\text{algdep}(\alpha, n, B)$$

(relying on LLL) finds a polynomial $P \in \mathbf{Z}_n[X]$ such that $P(\alpha) \approx 0$
up to B digits.

This is a solution to Hilbert's twelfth problem when the base field is an imaginary quadratic field. Shimura partially generalised this to CM fields. Hilbert's 12th problem over other ground fields cannot be approached using the theory of complex multiplication.

Recent proposals involve replacing complex analytic functions by p -adic analytic ones. For example, Darmon–Dasgupta give a conjectural p -adic formula for units in abelian extensions of real quadratic fields.

In a recent breakthrough, Dasgupta and Kakde have proved the Brumer–Stark conjecture using p -adic methods. This implies in particular that the conjectural formulas of Darmon and Dasgupta indeed hold.

An even more recent (and different) approach by Darmon-Pozzi-Vonk also establishes this striking result.

Beyond SL_2 ? complex cubic field case

From now on, $K \subset \mathbf{C}$ is a complex cubic field,
 $\mathcal{O}_K = \mathbf{Z} + \tau\mathbf{Z} + \sigma\mathbf{Z} \simeq \mathbf{Z}^3$ possess a $\mathbf{Z}$ -basis $(1, \tau, \sigma)$.

What is the relevant function ? Some kind of infinite product ? ...

Beyond SL_2 ? complex cubic field case

From now on, $K \subset \mathbf{C}$ is a complex cubic field,
 $\mathcal{O}_K = \mathbf{Z} + \tau\mathbf{Z} + \sigma\mathbf{Z} \simeq \mathbf{Z}^3$ possess a $\mathbf{Z}$ -basis $(1, \tau, \sigma)$.

What is the relevant function ? Some kind of infinite product ? ...

Obstruction : It is classical that there are no meromorphic functions on $\mathbf{C}$ with 3 lin. independent periods.

The beautiful elliptic Gamma function

A major role in this talk will be played by the **elliptic Gamma function**, a youthful special function introduced in 1997 by Ruijsenaars; it is defined and meromorphic on the domain $\mathbf{C} \times \mathcal{H} \times \mathcal{H}$, where it is given by the c.v. infinite double product

$$\Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i(j\tau + k\sigma + z)}}$$

It appears in a formula for the free energy of the eight-vertex model in statistical physics.

The beautiful elliptic Gamma function

A major role in this talk will be played by the **elliptic Gamma function**, a youthful special function introduced in 1997 by Ruijsenaars; it is defined and meromorphic on the domain $\mathbf{C} \times \mathcal{H} \times \mathcal{H}$, where it is given by the c.v. infinite double product

$$\Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i(j\tau + k\sigma + z)}}$$

It appears in a formula for the free energy of the eight-vertex model in statistical physics.

Rather than discuss its origin in physics, I'd like to motivate this definition by providing some historical perspective as to why it is indeed a natural candidate to Kronecker/ Hilbert's quest. I'd like to argue that some avatar of this function, and maybe even some of its algebraic properties, might have been anticipated by Eisenstein as early as 1844.



21 years old G. Eisenstein (1844) observes :

$$\sum_{a,b,c \in \mathbb{Z}^3} \frac{1}{(a\omega_1 + b\omega_2 + c\omega_3 + z)^k} \text{ diverges.}$$

In this very prospective paper, Eisenstein actually considers infinite triple products rather than series. We follow closely his discussion.

a) Simple infinite products (Euler)

$$\prod_{\lambda \in \mathbb{Z}}^{(\text{eis})} \left(1 - \frac{z}{\lambda + B} \right) = \frac{\sin \pi(B - z)}{\sin \pi B} = e^{i\pi z} \left(\frac{1 - e^{2i\pi(B-z)}}{1 - e^{2i\pi B}} \right).$$

In this very prospective paper, Eisenstein actually considers infinite triple products rather than series. We follow closely his discussion.

a) Simple infinite products (Euler)

$$\prod_{\lambda \in \mathbb{Z}}^{(\text{eis})} \left(1 - \frac{z}{\lambda + B} \right) = \frac{\sin \pi(B - z)}{\sin \pi B} = \underbrace{e^{i\pi z}} \left(\frac{1 - e^{2i\pi(B-z)}}{1 - e^{2i\pi B}} \right).$$

In this very prospective paper, Eisenstein actually considers infinite triple products rather than series. We follow closely his discussion.

a) Simple infinite products (Euler)

$$\prod_{\lambda \in \mathbb{Z}}^{(\text{eis})} \left(1 - \frac{z}{\lambda + B} \right) = \frac{\sin \pi(B - z)}{\sin \pi B} = e^{i\pi z} \left(\frac{1 - e^{2i\pi(B-z)}}{1 - e^{2i\pi B}} \right).$$

b) Double infinite products (Jacobi, Eisenstein). $B = \tau\lambda' + C$

$$\prod_{\lambda, \lambda' \in \mathbb{Z}}^{(\text{eis})} \left(1 - \frac{z}{\lambda + \tau\lambda' + C} \right) = e^{i\pi z} \frac{\theta_0(z - C, \tau)}{\theta_0(C, \tau)},$$



In the same paper, about the higher degree case he indicates :
"We should consider as an analog of higher degree *quotients of quotients of infinite triple products* of the shape

$$\prod_{\lambda, \lambda', \lambda''} \left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A'} \right),$$

where A et A' are constants. One cannot assign all values to the indices independently, from $-\infty$ to $+\infty$ ".



In the same paper, about the higher degree case he indicates :

"We should consider as an analog of higher degree *quotients of quotients of infinite triple products* of the shape

$$\prod_{\lambda, \lambda', \lambda''} \left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A'} \right),$$

where A et A' are constants. One cannot assign all values to the indices independently, from $-\infty$ to $+\infty$ ". **He further suggests :**



In the same paper, about the higher degree case he suggests :

"However no such inconvenience arises if we impose some restrictions on the indices such as inequalities conditions....
... There is a large class of such functions that is closely connected to Number Theory.... These functions possess very remarkable properties; they lead to the most beautiful researches, and seem to lie at the crossroads where the most difficult parts of analysis and number theory meet."



In the same paper, about the higher degree case he suggests :

"However no such inconvenience arises if we impose some restrictions on the indices such as inequalities conditions....
... There is a large class of such functions that is closely connected to Number Theory.... These functions possess very remarkable properties; they lead to the most beautiful researches, and seem to lie at the crossroads where the most difficult parts of analysis and number theory meet."

("... number theory, algebra and function theory come into closest touch with one another" would be Hilbert's phrasing 56 years later. Note that Eisenstein seems to possess something more already, since he is being rather specific).

According to Eisenstein, of particular interest are the inequality conditions arising from "geometric progression". Our thesis is that he had in mind the action of the fundamental unit of a complex cubic field.

- Historical aside :

G. Eisenstein (1823-1852), and

L. Kronecker (1823-1891) were students and friends in Berlin.

- Historical aside :
G. Eisenstein (1823-1852), and
L. Kronecker (1823-1891) were students and friends in Berlin.
- This 1844 paper of Eisenstein has been ridiculed by Jacobi**, and subsequently has been largely ignored (quoted less than 10 times in the next 150 years). It certainly deserves to be re-evaluated.

- Historical aside :
G. Eisenstein (1823-1852), and
L. Kronecker (1823-1891) were students and friends in Berlin.
- This 1844 paper of Eisenstein has been ridiculed by Jacobi**, and subsequently has been largely ignored (quoted less than 10 times in the next 150 years). It certainly deserves to be re-evaluated.

** Jacobi, oct. 1845 : "Mr Eisenstein dit que, par analogie, dans la théorie des intégrales abéliennes, il faudrait considérer *des quotients de quotients*. Mais qu'est-ce que c'est que des *quotients de quotients* ? C'est tout simplement des quotients."

- Historical aside :
G. Eisenstein (1823-1852), and
L. Kronecker (1823-1891) were students and friends in Berlin.
 - This 1844 paper of Eisenstein has been ridiculed by Jacobi, and subsequently has been largely ignored (quoted less than 10 times in the next 150 years). It certainly deserves to be re-evaluated.
-

• **Our aim :** guided by this predictive (and cryptic) assertion of Eisenstein, we first bridge the gap between quotients of quotients of infinite triple products and the elliptic Gamma function.

Connecting $\Gamma(z, \tau, \sigma)$ with an Eisenstein triple product

Following Eisenstein's *two advices*.

Theorem (BCG, 2023) : For $N \geq 2$, consider the smoothed quotient

$$Q_{++} = \prod_{\lambda \in \mathbf{Z}, \lambda', \lambda'' > 0}^{(eis)} \frac{\left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A' + v}\right)^N}{\left(1 - \frac{Nz}{\lambda + N\lambda' A + N\lambda'' A' + Nv}\right)}$$

Connecting $\Gamma(z, \tau, \sigma)$ with an Eisenstein triple product

Following Eisenstein's two advices altogether.

Theorem (BCG, 2023) : For $N \geq 2$, consider the smoothed quotient

$$Q_{++} = \prod_{\lambda \in \mathbf{Z}, \lambda', \lambda'' > 0}^{(\text{eis})} \frac{\left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A' + v}\right)^N}{\left(1 - \frac{Nz}{\lambda + N\lambda' A + N\lambda'' A' + Nv}\right)}$$

Recall from Euler that

$$\prod_{\lambda \in \mathbf{Z}}^{(\text{eis})} \left(1 - \frac{z}{\lambda + B}\right) = \underbrace{e^{i\pi z}} \left(\frac{1 - e^{2i\pi(B-z)}}{1 - e^{2i\pi B}} \right).$$

Connecting $\Gamma(z, \tau, \sigma)$ with an Eisenstein triple product

Following Eisenstein's two advices altogether.

Theorem (BCG, 2023) : For $N \geq 2$, consider the smoothed quotient

$$\begin{aligned} Q_{++} &= \prod_{\lambda \in \mathbf{Z}, \lambda', \lambda'' > 0}^{(\text{eis})} \frac{\left(1 - \frac{z}{\lambda + \lambda' A + \lambda'' A' + \nu}\right)^N}{\left(1 - \frac{Nz}{\lambda + N\lambda' A + N\lambda'' A' + N\nu}\right)} \\ &= \prod_{\lambda', \lambda'' > 0} \frac{(1 - e^{2\pi i(\lambda' A + \lambda'' A' + \nu - z)})^N (1 - e^{2\pi i(N\lambda' A + N\lambda'' A' + N\nu)})}{(1 - e^{2\pi i(\lambda' A + \lambda'' A' + \nu)})^N (1 - e^{2\pi i(N\lambda' A + N\lambda'' A' + N\nu - Nz)})}. \end{aligned}$$

Connecting $\Gamma(z, \tau, \sigma)$ with an Eisenstein triple product

Theorem (BCG, 2023) : For $N \geq 2$, consider the smoothed quotient

$$\begin{aligned}
 Q_{++} &= \prod_{\lambda \in \mathbf{Z}, \lambda', \lambda'' > 0}^{(\text{eis})} \frac{\left(1 - \frac{z}{\lambda + A\lambda' + A'\lambda'' + v}\right)^N}{\left(1 - \frac{Nz}{\lambda + NA\lambda' + NA'\lambda'' + Nv}\right)} \\
 &= \prod_{\lambda', \lambda'' > 0} \frac{(1 - e^{2\pi i(A\lambda' + A'\lambda'' + v - z)})^N (1 - e^{2\pi i(NA\lambda' + NA'\lambda'' + Nv)})}{(1 - e^{2\pi i(A\lambda' + A'\lambda'' + v)})^N (1 - e^{2\pi i(NA\lambda' + NA'\lambda'' + Nv - Nz)})}.
 \end{aligned}$$

Construct Q_{--} similarly, now with $\lambda', \lambda'' \in \mathbf{Z}$ satisfying the **condition** $\lambda', \lambda'' \leq 0$. It is straightforward to derive the following formula for the quotient of quotients

$$\frac{Q_{++}}{Q_{--}} = \frac{\Gamma(z - v, A, A')^N / \Gamma(-v, A, A')^N}{\Gamma(Nz - Nv, NA, NA') / \Gamma(-Nv, NA, NA')}. \quad (1)$$

The smoothed infinite product

$$(z, \tau, \sigma) \mapsto \frac{\Gamma(z, \tau, \sigma)^N}{\Gamma(Nz, N\tau, N\sigma)}$$

can formally be considered as a higher analog of

$$(z, \tau) \mapsto \frac{\theta_0(z, \tau)^N}{\theta_0(Nz, N\tau)}.$$

The smoothed infinite product

$$(z, \tau, \sigma) \mapsto \frac{\Gamma(z, \tau, \sigma)^N}{\Gamma(Nz, N\tau, N\sigma)}$$

can be formally considered as a higher analog of

$$(z, \tau) \mapsto \frac{\theta_0(z, \tau)^N}{\theta_0(Nz, N\tau)}.$$

We now provide the first numerical evidence supporting our belief that, in the setting of complex cubic fields, it is one of the "transcendental functions" that Hilbert was asking for.

Choose

$$K = \mathbf{Q}(\beta) \subset \mathbb{C}, \quad \beta = \sqrt[3]{7} \cdot e^{-\frac{2i\pi}{3}}, \quad \beta^3 = 7.$$

Consider the basis $\mathbf{Z} + \mathbf{Z}\tau_0 + \mathbf{Z}\sigma_0$, with

$$\tau_0 = \frac{2\beta + \beta^2}{15} \in \overline{\mathcal{H}} \quad \text{and} \quad \sigma_0 = -\frac{2 + \beta}{15} \in \mathcal{H},$$

of a fractional ideal of K .

Together with $\nu_0 = \frac{1}{3} \in \mathbf{Q}$, we obtain the complex number

$$\Gamma(\nu_0, \tau_0, \sigma_0)^5 \cdot \Gamma(5\nu_0, 5\tau_0, 5\sigma_0)^{-1} \approx -4.024029 \dots - i \cdot 41.85595 \dots$$

...

... the complex number

$$\Gamma(v_0, \tau_0, \sigma_0)^5 \cdot \Gamma(5v_0, 5\tau_0, 5\sigma_0)^{-1} \approx -4.024029 \dots - i \cdot 41.85595 \dots$$

coincides (up to 1000 digits at least) with a complex root of the polynomial

$$\begin{aligned} Q = x^6 &+ (6\beta^2 - 14\beta + 2)x^5 + (4\beta^2 - 6\beta + 2)x^4 \\ &+ (106\beta^2 - 152\beta - 103)x^3 + (4\beta^2 - 6\beta + 2)x^2 \\ &+ (6\beta^2 - 14\beta + 2)x + 1. \end{aligned}$$

This polynomial is irreducible over K and its splitting field is a cyclic totally complex abelian extension of degree 6 ramified only at 3.

In the example above, with $\beta^3 = 7$, the other roots of Q are:

$$\Gamma\left(\frac{1}{3}, \frac{\beta^2 + 2\beta + 75}{345}, -\frac{\beta + 32}{345}\right)^5 \cdot \Gamma\left(\frac{5}{3}, \frac{\beta^2 + 2\beta + 75}{69}, -\frac{\beta + 32}{69}\right)^{-1}$$

and

$$\Gamma\left(\frac{1}{3}, \frac{\beta^2 + 2\beta + 15}{150}, -\frac{\beta - 43}{150}\right)^5 \cdot \Gamma\left(\frac{5}{3}, \frac{\beta^2 + 2\beta + 15}{30}, -\frac{\beta - 43}{30}\right)^{-1}.$$

The action of the relative Galois group translates into the evaluation at various points in K .

In the example above, with $\beta^3 = 7$, the other roots of Q are:

$$\Gamma\left(\frac{1}{3}, \frac{\beta^2 + 2\beta + 75}{345}, -\frac{\beta + 32}{345}\right)^5 \cdot \Gamma\left(\frac{5}{3}, \frac{\beta^2 + 2\beta + 75}{69}, -\frac{\beta + 32}{69}\right)^{-1}$$

and

$$\Gamma\left(\frac{1}{3}, \frac{\beta^2 + 2\beta + 15}{150}, -\frac{\beta - 43}{150}\right)^5 \cdot \Gamma\left(\frac{5}{3}, \frac{\beta^2 + 2\beta + 15}{30}, -\frac{\beta - 43}{30}\right)^{-1}.$$

The action of the relative Galois group translates into the evaluation at various points in K .

Choosing the specialization $z_0 = 1/3$, $v_0 = -1/3$, we obtain that the quotient of quotient "à la Eisenstein" Q_{++}/Q_{--} introduced above seems to possess remarkable algebraic properties as well.

More on the elliptic Gamma function

To explain our evaluation procedure and set the stage for our general conjecture in Luis' talk, we need to say more about the properties of

$$\Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i(j\tau + k\sigma + z)}}$$

A) It has internal symmetry properties (elliptic and modular) that make it a higher dimensional analog of $\theta_0(z, \tau)$.

B) Together with $\theta_0(z, \tau)$, they are the first and second level in a ladder of functions $G_r(z, \tau_1, \dots, \tau_n)$.

Automorphic properties of $\Gamma(z, \tau, \sigma)$

Felder–Varchenko (2000) prove that $\Gamma(z, \tau, \sigma)$ extends to $\mathbf{C} \times (\mathbf{C} - \mathbf{R}) \times (\mathbf{C} - \mathbf{R})$ and transforms under $\mathrm{SL}_3(\mathbf{Z})$ -action.

- Periodicities: $\Gamma(z + 1, \tau, \sigma) = \Gamma(z, \tau, \sigma)$ and

$$\Gamma(z + \tau, \tau, \sigma) = \theta_0(z, \sigma)\Gamma(z, \tau, \sigma).$$

Automorphic properties of $\Gamma(z, \tau, \sigma)$

Felder–Varchenko (2000) prove that $\Gamma(z, \tau, \sigma)$ extends to $\mathbf{C} \times (\mathbf{C} - \mathbf{R}) \times (\mathbf{C} - \mathbf{R})$ and transforms under $SL_3(\mathbf{Z})$ -action.

- Periodicities: $\Gamma(z + 1, \tau, \sigma) = \Gamma(z, \tau, \sigma)$ and

$$\Gamma(z + \tau, \tau, \sigma) = \theta_0(z, \sigma) \Gamma(z, \tau, \sigma).$$

- Modularity for $SL_3(\mathbf{Z}) \curvearrowright \mathbb{P}^2(\mathbf{C})$ (a sample):

$$(*) \quad \Gamma\left(\frac{z}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \Gamma\left(\frac{z}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right) \Gamma\left(\frac{z}{x_2}, \frac{x_3}{x_2}, \frac{x_1}{x_2}\right) = e^{-i\pi B_3(z, x)}$$

where $B_3 \in \mathbf{Q}(x)[z]$ is of degree 3 in z .

Should be thought of as a mult. 1-cocycle identity for $SL_3(\mathbf{Z})$ (splitting a 2-cocycle). More on this in Luis's talk.

Up the ladder, and computational aspects

There is a rather uniform 1-dim. series formula particularly efficient for the computation of all of $\theta_0(z, \tau)$, $\Gamma(z, \tau, \sigma)$, $\Gamma(z, \tau, \sigma, \rho)$, $\dots$ (with symmetry group SL_2 , SL_3 , SL_4 resp.), namely

$$\theta_0(z, \tau) = \exp \left(-i \sum_{j \geq 1} \frac{\cos(\pi j(2z - \tau))}{j \sin(\pi j \tau)} \right),$$

(smoothing by N simply means that the index j gets restricted to be coprime to N).

Up the ladder, and computational aspects

There is a rather uniform 1-dim. series formula particularly efficient for the computation of all of $\theta_0(z, \tau)$, $\Gamma(z, \tau, \sigma)$, $\Gamma(z, \tau, \sigma, \rho), \dots$ (with symmetry group SL_2, SL_3, SL_4 resp.), namely

$$\theta_0(z, \tau) = \exp \left(-i \sum_{j \geq 1} \frac{\cos(\pi j(2z - \tau))}{j \sin(\pi j \tau)} \right),$$

$$\Gamma(z, \tau, \sigma) = \exp \left(-\frac{i}{2} \sum_{j \geq 1} \frac{\sin(\pi j(2z - \tau - \sigma))}{j \sin(\pi j \tau) \sin(\pi j \sigma)} \right),$$

Up the ladder, and computational aspects

There is a rather uniform 1-dim. series formula particularly efficient for the computation of all of $\theta_0(z, \tau)$, $\Gamma(z, \tau, \sigma)$, $\Gamma(z, \tau, \sigma, \rho), \dots$ (with symmetry group SL_2, SL_3, SL_4 resp.), namely

$$\theta_0(z, \tau) = \exp \left(-i \sum_{j \geq 1} \frac{\cos(\pi j(2z - \tau))}{j \sin(\pi j \tau)} \right),$$

$$\Gamma(z, \tau, \sigma) = \exp \left(-\frac{i}{2} \sum_{j \geq 1} \frac{\sin(\pi j(2z - \tau - \sigma))}{j \sin(\pi j \tau) \sin(\pi j \sigma)} \right),$$

(computational complexity comparable to that of θ_0)

Up the ladder, and computational aspects

There is a rather uniform 1-dim. series formula particularly efficient for the computation of all of $\theta_0(z, \tau)$, $\Gamma(z, \tau, \sigma)$, $\Gamma(z, \tau, \sigma, \rho)$, $\dots$ (with symmetry group SL_2 , SL_3 , SL_4 resp.), namely

$$\theta_0(z, \tau) = \exp \left(-i \sum_{j \geq 1} \frac{\cos(\pi j(2z - \tau))}{j \sin(\pi j \tau)} \right),$$

$$\Gamma(z, \tau, \sigma) = \exp \left(-\frac{i}{2} \sum_{j \geq 1} \frac{\sin(\pi j(2z - \tau - \sigma))}{j \sin(\pi j \tau) \sin(\pi j \sigma)} \right),$$

$$\Gamma(z, \tau, \sigma, \rho) = \exp \left(-\frac{i}{4} \sum_{j \geq 1} \frac{\cos(\pi j(2z - \tau - \sigma - \rho))}{j \sin(\pi j \tau) \sin(\pi j \sigma) \sin(\pi j \rho)} \right),$$

The last series/infinite product $\Gamma(z, \tau, \sigma, \rho)$ (and its siblings in the ladder) has been introduced by Nishizawa in the 2000's.

It satisfies

- Periodicities: $\Gamma(z + 1, \tau, \sigma, \rho) = \Gamma(z, \tau, \sigma, \rho)$, and the ladder condition

$$\Gamma(z + \tau, \tau, \sigma, \rho) = \Gamma(z, \sigma, \rho) \cdot \Gamma(z, \tau, \sigma, \rho),$$

The last series/infinite product $\Gamma(z, \tau, \sigma, \rho)$ (and its siblings in the ladder) has been introduced by Nishizawa in the 2000's.

It satisfies

- Periodicities: $\Gamma(z + 1, \tau, \sigma, \rho) = \Gamma(z, \tau, \sigma, \rho)$, and the ladder condition

$$\Gamma(z + \tau, \tau, \sigma, \rho) = \Gamma(z, \sigma, \rho) \cdot \Gamma(z, \tau, \sigma, \rho),$$

together with a multiplicative 2-cocycle identity for $SL_4(\mathbf{Z})$.

These r -cocycle on $SL_{r+2}(\mathbf{Z})$, with $r = 1, 2$ can then be evaluated on a tori T of rank r arising from the unit group of a complex cubic field K ($r = 1$, in [BCG]) or a quartic ATR field ($r = 2$ in the very recent preprint of Pierre Morain (on-going PhD thesis at Sorbonne Université)).

Roughly speaking, we let these units act on a fractional ideal $\mathfrak{c} = \mathfrak{fb}^{-1} \simeq \mathbf{Z}^{r+2}$ of K . The image of a $\mathbf{Z}$ -basis of $\mathfrak{c}$ under the complex embedding of K then provides inputs for the special values of the corresponding elliptic Γ –function. Our smoothed quotients are conjecturally related to Stark units lying in the (narrow) ray class field $K(\mathfrak{f})^+$ above K . (More details by Luis).

I) live demo for complex cubic fields

Instead of working out the above example with $\mathbf{Q}(7^{\frac{1}{3}})$ as in our preprint, let's play instead with

$$K = \mathbf{Q}(\delta) \subset \mathbb{C}, \quad \delta = \sqrt[3]{10} \cdot e^{-\frac{2i\pi}{3}}, \quad \delta^3 = 10.$$

Its fund. unit is $\epsilon = \frac{2\delta^2 - \delta - 7}{3}$. Fix $\tau_0 = \frac{-\epsilon^{-1} + 1751}{495}$, $\sigma_0 = \frac{-\epsilon + 776}{495}$.

Our conjecture predicts in this particular setting that the

11-smoothed invariant $\Gamma(\frac{1}{5}, \tau_0, \sigma_0)^{11} / \Gamma(\frac{11}{5}, 11\tau_0, 11\sigma_0)$ is a unit in $K(5)^+$, as witnessed by numerical evidence.

I) live demo for complex cubic fields

Instead of working out the above example with $\mathbf{Q}(7^{\frac{1}{3}})$ as in our preprint, let's play instead with

$$K = \mathbf{Q}(\delta) \subset \mathbb{C}, \quad \delta = \sqrt[3]{10} \cdot e^{-\frac{2i\pi}{3}}, \quad \delta^3 = 10.$$

Its fund. unit is $\epsilon = \frac{2\delta^2 - \delta - 7}{3}$. Fix $\tau_0 = \frac{-\epsilon^{-1} + 1751}{495}$, $\sigma_0 = \frac{-\epsilon + 776}{495}$.

Our conjecture predicts in this particular setting that the

11-smoothed invariant $\boxed{\Gamma(\frac{1}{5}, \tau_0, \sigma_0)^{11} / \Gamma(\frac{11}{5}, 11\tau_0, 11\sigma_0)}$ is a unit in $K(5)^+$, as witnessed by numerical evidence.

There seems to be more going on. Define four values $v_k = \Gamma(\frac{k}{5}, 11\tau_0, 11\sigma_0)$, for $k = 1, 2, 3, 4$.

Even though the v_k 's do not appear to be algebraic individually, intriguing algebraic properties among the ratios v_k/v_h also seem to happen (another legit type of quotients of quotients in order to get rid of the factor $e^{2i\pi z}$). Some are in $K(5)^+$, others don't.

II) Excerpt from Morain's quartic computations

Let $P = x^4 - 6x^3 - x^2 - 3x + 1$ and let $\beta \in \mathcal{H}$ be a complex root of P . The field $K = \mathbf{Q}(\beta)$ is *quartic ATR*.

Morain considers a certain 13-smoothing of $\Gamma(z, \tau, \sigma, \rho)$ with arguments in K .

The ratio takes the shape

$$\frac{\Gamma(\frac{1}{2}, \tau_0, \sigma_0, \rho_0)^{13} \cdot \Gamma(-\frac{13}{2}, 13\tau'_0, 13\sigma'_0, 13\rho'_0)}{\Gamma(-\frac{1}{2}, \tau'_0, \sigma'_0, \rho'_0)^{13} \cdot \Gamma(\frac{13}{2}, 13\tau_0, 13\sigma_0, 13\rho_0)} \simeq 4.121 \dots - 5.061 \dots$$

Up to at least 1000 digits, it coincides with a root of $x^8 - 7x^7 + 33x^6 + 49x^5 + 17x^4 + 49x^3 + 33x^2 - 7x + 1$, defining a unit in the narrow class field of K .

III) Connection to other works, ATR number fields

All constructions as above require the ground field K (be it CM, cubic complex) to have exactly one complex embedding. The general setting to formulate these kinds of approaches to Hilbert's 12-th problem that emerges seems to be that of ATR (almost totally real) number fields of arbitrary degree n over $\mathbf{Q}$.

III) Connection to other works, ATR number fields

All constructions as above require the ground field K (be it CM, cubic complex) to have exactly one complex embedding. The general setting to formulate these kinds of approaches to Hilbert's 12-th problem that emerges seems to be that of ATR (almost totally real) number fields of arbitrary degree n over $\mathbf{Q}$. Other works in the direction of formulating refinements of the Stark conjecture :

- 1 Ren-Sczech (2009) for complex cubic fields K . Used Shintani fundamental domains and multiple Barnes Gamma functions.

III) Connection to other works, ATR number fields

All constructions as above require the ground field K (be it CM, cubic complex) to have exactly one complex embedding. The general setting to formulate these kinds of approaches to Hilbert's 12-th problem seems to be that of ATR (almost totally real) number fields of arbitrary degree n over $\mathbf{Q}$. Other works in the direction of formulating refinements of the Stark conjecture :

- 1 Ren-Sczech (2009) for complex cubic fields K . Used Shintani fundamental domains and multiple Barnes Gamma functions.
- 2 Charollois-Darmon (2008) for quartic ATR fields K containing a real quadratic subfield F . Used 1-cocycle and periods of weight 2 Eisenstein series for the Hilbert modular group $SL_2(\mathcal{O}_F)$

III) Connection to other works, ATR number fields

All constructions as above require the ground field K (be it CM, cubic complex) to have exactly one complex embedding. The general setting to formulate these kinds of approaches to Hilbert's 12-th problem seems to be that of ATR (almost totally real) number fields of arbitrary degree n over $\mathbf{Q}$. Other works in the direction of formulating refinements of the Stark conjecture :

- ① Ren-Sczech (2009) for complex cubic fields K . Used Shintani fundamental domains and multiple Barnes Gamma functions.
- ② Charollois-Darmon (2008) for quartic ATR fields K containing a real quadratic subfield F . Used 1-cocycle and periods of weight 2 Eisenstein series for the Hilbert modular group $SL_2(\mathcal{O}_F)$
- ③ Pierre Morain (Sorbonne Université) on-going PhD work for quartic ATR fields K . Uses 2-cocycle for $SL_4(\mathbf{Z})$ and higher dimensional elliptic Gamma functions. Very recent preprint <https://arxiv.org/pdf/2406.06094> (including a quintic example).

III) Connection to other works, ATR number fields

All constructions as above require the ground field K (be it CM, cubic complex) to have exactly one complex embedding. The general setting to formulate these kinds of approaches to Hilbert's 12-th problem seems to be that of ATR (almost totally real) number fields of arbitrary degree n over $\mathbf{Q}$. Other works in the direction of formulating refinements of the Stark conjecture :

- ① Ren-Sczech (2009) for complex cubic fields K . Used Shintani fundamental domains and multiple Barnes Gamma functions.
- ② Charollois-Darmon (2008) for quartic ATR fields K containing a real quadratic subfield F . Used 1-cocycle and periods of weight 2 Eisenstein series for the Hilbert modular group $SL_2(\mathcal{O}_F)$
- ③ Pierre Morain (Sorbonne Université) on-going PhD work for quartic ATR fields K . Uses 2-cocycle for $SL_4(\mathbf{Z})$ and higher dimensional elliptic Gamma functions. Very recent preprint <https://arxiv.org/pdf/2406.06094> (including a quintic example)