

Equivariant birational geometry

joint with A. Kresch

Introduction

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Addressed by

- Clemens-Griffiths – intermediate Jacobians (cubic threefolds)
- Manin-Iskovskikh – birational rigidity (quartic threefolds)
- Artin-Mumford – Brauer groups (conic bundles over rational surfaces)

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This question emerged from Hilbert's work on the **Inverse Galois Problem**. Working over a nonclosed field k , rationality of the quotient implies existence of Galois extensions of k with Galois group G .

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Unramified cohomology

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A key definition is that of **unramified cohomology**:

$$H_{\text{nr}}^i(K) := \bigcap_{v \in \mathcal{D}\text{Val}_K} \text{Ker}(\partial_v)$$

where

$$\partial_v : H^i(K) \rightarrow H^{i-1}(K_v), \quad v \in \mathcal{D}\text{Val}_K,$$

is the **residue homomorphism**.

Basic properties (see, e.g., Colliot-Thélène-Ojanguren, or the recent survey by S. Schreieder):

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- Vanishes when $i > \dim(X)$,
- Stable birational invariant, **trivial** for rational varieties over algebraically closed k ,
- Functoriality, etc.
- If X is **unirational** then $H_{\text{nr}}^i(k(X))$ is **finite** for $i = 2, 3$, over $k = \bar{k}$.

Conjecture: Finite for unirational varieties, for all i (over $k = \bar{k}$).

The first interesting case is the **unramified Brauer group**

$$\mathrm{Br}_{\mathrm{nr}}(K) := H_{\mathrm{nr}}^2(K) = \mathrm{Br}(X) = H^2(X, \mathbb{G}_m),$$

i.e., the Brauer group of a smooth projective model X of $K = k(X)$.

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It is accessible, in many cases, and has become a standard tool.

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These examples inspired major developments in **birational** and **anabelian** geometry.

One of the surprising results was the construction of **universal spaces**:

Theorem (Bogomolov-T. 2017)

Let X be an algebraic variety of dimension ≥ 2 over $k = \bar{\mathbb{F}}_p$, with function field $K = k(X)$, $\ell \neq p$ a prime, and

$$\alpha \in H_{\text{nr}}^i(K, \mathbb{Z}/\ell^n).$$

Then there exist finite-dimensional V_j over k , depending on α , such that α is induced, via a rational map, from an unramified class in the cohomology of the quotient

$$\left(\prod_j \mathbb{P}(V_j) \right) / G$$

by a finite abelian group G acting projectively on each factor.

Let X be a **geometrically** rational variety over a nonclosed k , with $X(k) \neq \emptyset$. Colliot-Thélène–Sansuc introduced and studied **universal torsors**,

$$\pi : \mathcal{T} \rightarrow X,$$

i.e., torsors for **tori** (Neron-Severi tori), serving as **descent** varieties for various arithmetic problems (Hasse-Principle, Weak Approximation, over $k = \mathbb{Q}$).

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Warning: There exist geometrically rational **threefolds**, over $\mathbb{R}$, such that their universal torsors fail stable rationality over $\mathbb{R}$ (Colliot-Thélène 1977).

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Moral: Torsors **untwist** unramified cohomology.

A variety X over a field k admits a **rational tower** if there exists a sequence of dominant rational maps

$$X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots X_1 \xrightarrow{\pi_1} X_0 = X,$$

over k , such that

- the **source** of the tower, X_n is rational over k ,
- the generic fiber of π_i is geometrically rational and irreducible over the function field $k(X_{i-1})$, for all i .

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In particular, **all** unramified cohomology of the source X_n is trivial.

Evidence:

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 - **New** unramified classes can appear in the process and we cannot guarantee triviality of unramified cohomology of the source X_n ,
 - Even when that cohomology is trivial, there is no information about **rationality** of the source.
- We prove the existence of a toric rational tower for $X = V/G$, for a large class of G (**special groups**), which includes **all** G appearing in the theory of **universal spaces**.

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There are many examples of such G with V/G not stably rational.

This discussion explains our interest in

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G),$$

when V is a representation of G , and more generally, in

$$\mathrm{Br}_{\mathrm{nr}}(k(X)^G),$$

when X is a rational G -variety with generically free action.

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For example,

G	$H^2(G, k^\times)$	$H^3(G, k^\times)$
$\mathbb{Z}/n$	0	$\mathbb{Z}/n$
$\mathbb{Z}/n \oplus \mathbb{Z}/m$	$\mathbb{Z}/\gcd(n, m)$	$\mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\gcd(n, m)$

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$$\mathrm{Br}_{\mathrm{nr}}(k(X)^G) = \{\alpha \in \mathrm{Br}(k(X)^G) \mid \alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(X)^A), \quad \forall A \in \mathcal{B}_G\}.$$

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E.g., when $X = V$, a representation of an **abelian** group A , then V/A is rational, and

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0.$$

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Let V be a G -representation and $K = k(V/G)$. Then

$$\mathrm{Br}_{\mathrm{nr}}(K) \cong \ker \left(H^2(G) \rightarrow \bigoplus_{A \in \mathcal{B}_G} H^2(A) \right) =: B_0(G).$$

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- there exist G with $|G| = p^5$ and nonvanishing $B_0(G)$ (Hoshi, Kang, Moravec, ... 2012)
- $B_0(G) = 0$ for almost simple groups G , with a few exceptions (Kunyavski 2007).

Similar formulas have been derived in other situations:

Let T be an algebraic torus over k , with G -action arising from an injective homomorphism $G \rightarrow \operatorname{Aut}(M)$, where $M = \mathfrak{X}^*(T)$, and $K = k(T/G)$.

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Let T be an algebraic torus over k , with G -action arising from an injective homomorphism $G \rightarrow \operatorname{Aut}(M)$, where $M = \mathfrak{X}^*(T)$, and $K = k(T/G)$.

Then

$$\operatorname{Br}_{\text{nr}}(K) \cong \ker \left(H^2(G, \mathbb{Q}/\mathbb{Z} \oplus M) \rightarrow \bigoplus_{A \in \mathcal{B}_G} H^2(A, \mathbb{Q}/\mathbb{Z} \oplus M) \right).$$

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It is not known whether or not such quotients are rational.

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- Computation of $\mathrm{Br}([X/G])$,
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- Residue computations, on a nice model, to identify

$$\mathrm{Br}_{\mathrm{nr}}(k(X)^G) \subset \mathrm{Br}([X/G]).$$

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Applying Leray spectral sequence to $[X/G] \rightarrow BG$, we obtain

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}(G, k^\times) \rightarrow \mathrm{Pic}(X, G) \rightarrow \mathrm{Pic}(X)^G &\xrightarrow{\delta_2} H^2(G, k^\times) \\ \rightarrow \mathrm{Br}([X/G]) \rightarrow H^1(G, \mathrm{Pic}(X)) &\xrightarrow{\delta_3} H^3(G, k^\times) \rightarrow H^3([X/G], \mathbb{G}_m), \end{aligned}$$

where $\mathrm{Pic}(X, G)$ is the group of isomorphism classes of G -linearized line bundles.

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where $\mathrm{Pic}(X, G)$ is the group of isomorphism classes of G -linearized line bundles. Note that

$$\delta_2, \quad \delta_3 = 0$$

when $X^G \neq \emptyset$.

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- nonvanishing of $\text{Im}(\delta_3)$.

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Bogomolov–Prokhorov (2013)

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$$H^1(G, \text{Pic}(X)) = (\mathbb{Z}/p)^{2g},$$

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This used **classification**, i.e., equivariant MMP. Shinder generalizes this to $G = \mathbb{Z}/m$, when **all** fixed points have maximal stabilizers.

Equivariant geometry of surfaces: Example

Consider the $\mathbb{Z}/2$, respectively $\mathbb{Z}/3$, action on the cubic surface $X \subset \mathbb{P}^3$, given by

$$w^3 + x^3 + y^3 + xz^2 + \lambda yz^2 = 0,$$

with actions

$$(w, x, y, z) \mapsto (w, x, y, -z), \quad \text{respectively,} \quad (w, x, y, z) \mapsto (\zeta_3 w, x, y, z).$$

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We find, by Bogomolov–Prokhorov,

$$H^1(\mathbb{Z}/2, \text{Pic}(X)) = (\mathbb{Z}/2)^2, \quad H^1(\mathbb{Z}/3, \text{Pic}(X)) = (\mathbb{Z}/3)^2.$$

Let G be cyclic. Recall the basic exact sequence:

$$\dots \xrightarrow{\delta_2} H^2(G, k^\times) \rightarrow \mathrm{Br}([X/G]) \rightarrow H^1(G, \mathrm{Pic}(X)) \xrightarrow{\delta_3} \dots$$

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Thus,

$$\mathrm{Br}([X/G]) \simeq H^1(G, \mathrm{Pic}(X)).$$

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Step 2. Recall the exact sequence

$$0 \rightarrow \mathrm{Br}(k(\mathbb{P}^2)) \rightarrow \bigoplus_C H^1(k(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z},$$

where the $\mathfrak{p}$ -component of r is the sum over all curves C through $\mathfrak{p}$ of local ramification indices of points over $\mathfrak{p}$ of the normalization of C .

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where the $\mathfrak{p}$ -component of r is the sum over all curves C through $\mathfrak{p}$ of local ramification indices of points over $\mathfrak{p}$ of the normalization of C . We have

$$\mathrm{Br}([X/G]) \hookrightarrow \bigoplus_{[C] \in X^{(1)}/G} H^1(k(C)^{D_C}, \mathbb{Q}/\mathbb{Z}),$$

where D_C is the **decomposition group** at (an orbit representative) C , and the image in each summand is annihilated by $|I_C|$, the **inertia** at C .

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Given a collection

$$(\alpha_C)_{[C] \in X^{(1)}/G}$$

of

$$\alpha_C \in H^1(k(C)^{D_C}, \mathbb{Q}/\mathbb{Z}),$$

there exists a class $\alpha \in \text{Br}([X/G])$ iff the local ramification indices at points over $\mathfrak{p}$ sum to zero, for every orbit representative of $\mathfrak{p} \in X$.

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For noncyclic G , we get an interplay between δ_2, δ_3 , etc. We obtain obstructions to linearizability of G -actions on rational surfaces coming from all terms of the basic exact sequence.

The analysis in higher dimensions (Kresch–T. 2024) is more involved.

Consider the $G = \mathbb{Z}/6$ action on the cubic surface $X \subset \mathbb{P}^3$, given by

$$w^3 + x^3 + y^3 + xz^2 + \lambda yz^2 = 0,$$

with action

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There are:

- a curve of genus 1, fixed by g^2 ($w = 0$), and containing 4 G -fixed points, and
- another curve of genus 1, fixed by g^3 ($z = 0$), and containing 3 G -fixed points.

Shinder's theorem does not apply.

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Shinder's theorem does not apply. Quotients by residual action are rational. The action is in standard form. Since 2-torsion and 3-torsion classes summing to zero must be zero, we find

$$\mathrm{Br}([X/G]) \simeq H^1(G, \mathrm{Pic}(X)) = 0.$$

Return to

$$X \curvearrowright G \rightsquigarrow [X/G] \rightsquigarrow k(X/G) = k(X)^G.$$

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the free abelian group on birational equivalence classes of function fields of varieties of dimension n over k , and proved **specialization** for this group. There are many geometric applications.

- Kontsevich–Pestun–T. (2021) introduced

$$\mathcal{B}_n(G) = \langle \beta \rangle / \sim, \quad \beta := [a_1, \dots, a_n], \quad a_j \in G^\vee,$$

an abelian group, capturing G -equivariant birational invariants, for
abelian G :

$$X \hookrightarrow G \quad \Rightarrow \quad [X \hookrightarrow G] = \sum_{\alpha} \beta_{\alpha} \in \mathcal{B}_n(G),$$

a sum of symbols (sequence of characters of G in the tangent bundle) over irreducible components F_{α} of X^G ,

For all $[a_1, a_2, \dots, a_n] \in \mathcal{B}_n(G)$ one has

(G) $\sum_i \mathbb{Z}a_i = A$, and

(B) for all $a_1, a_2, b_1, \dots, b_{n-2} \in A$ we have

$$[a_1, a_2, b_1, \dots, b_{n-2}] =$$

$$[a_1 - a_2, a_2, b_1, \dots, b_{n-2}] + [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] \text{ if } a_1 \neq a_2,$$

$$[a_1, 0, b_1, \dots, b_{n-2}] \quad \text{if } a_1 = a_2.$$

Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G -equivariant birational invariant.

For arbitrary finite G , there is an abelian group, generated by symbols

$$\mathrm{Burn}_n(G) = \langle (H, Z \hookrightarrow k(F), \beta) \rangle / \sim$$

which contains, as a direct summand, the free abelian group on G -equivariant birational equivalence classes of function fields of G -varieties of dimension n over k , and takes into account

- all strata with nontrivial stabilizers,
- the residual action on the stratum,
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The class

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is computed on a **standard model**. There is also specialization in this context. First geometric applications of this specialization are in joint work with Cheltsov and Zhang on singular cubic threefolds.

Consider symbols

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where

- $H \subseteq G$ is an **abelian** subgroup, $Y \subseteq Z_G(H)/H$,
- $K = k(F)$, with generically free Y -action, $\text{trdeg}_k(K) = d \leq n$,
- $\beta = (b_1, \dots, b_{n-d})$, a sequence, up to order, of **nonzero** characters generating $H^\vee$.

The symbols are subject to **conjugation** and **blowup** relations:

(C): $(H, Y \hookrightarrow K, \beta) = (H', Y' \hookrightarrow K, \beta')$, when

$$H' = gHg^{-1}, \quad Y' = \dots, \quad \text{with } g \in G,$$

and β and β' are related by conjugation by g .

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(B1): $(H, Y \hookrightarrow K, \beta) = 0$ when $b_1 + b_2 = 0$.

Burnside groups: relations

(B2): $(H, Y \hookrightarrow K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y \hookrightarrow K, \beta_1) + (H, Y \hookrightarrow K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \dots, b_{n-d}),$$

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and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\bar{H}, \bar{Y} \hookrightarrow \bar{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

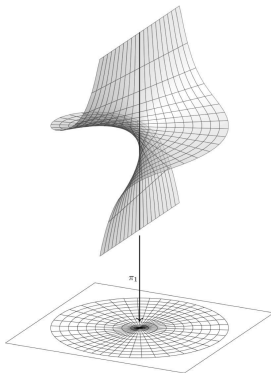
with

$$\bar{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-d}), \quad \bar{b}_i \in \bar{H}^\vee.$$

Equivariant Burnside group: relations

Model case: Blowing up a point (with abelian stabilizer) on a surface.

It will explain the action of $\bar{Y}$ on $\bar{K} = K(t)$.



$$X \hookrightarrow G \rightsquigarrow [X/G]$$

There is a formalism in this context as well:

$$\mathcal{B}urn_n = \langle (\mathcal{K}, \beta) \rangle / \sim$$

is defined by generators and relations.

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There are natural homomorphisms

$$\begin{aligned}\mathrm{Burn}_n(G) &\rightarrow \mathcal{B}urn_n, \\ [X \hookrightarrow G] &\mapsto [\mathcal{X}],\end{aligned}$$

where $\mathcal{X} = [X/G]$.

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where $\mathcal{X} = [X/G]$.

But there is **no** specialization!

$$(\mathcal{K}, \alpha),$$

where

- $\mathcal{K}$ is a **projective linearizable** DM stack of dimension $d \leq n$ (i.e., there is a dense open substack isomorphic to $U \times BH$, for some **abelian** group H),
- $\alpha = (a_1, \dots, a_{n-d})$, a sequence of **nonzero** characters of the generic stabilizer H of $\mathcal{K}$, generating $H^\vee$, up to permutation.

- (I) $(\mathcal{K}, \alpha) = (\mathcal{K}', \alpha')$ if there exists a birational equivalence (blowups and blowdowns) of projective linearizable stacks mapping α to α' ,

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- (V) if $a_1 + a_2 = 0$ then
$$(\mathcal{K}, \alpha) = 0,$$

(B) for $n - d \geq 2$, set $a := a_1 - a_2$, then

$$(\mathcal{K}, \alpha) = \Theta_1 + \Theta_2,$$

where, for

$$\alpha_1 := (a_1 - a_2, a_2, \dots, a_{n-d}), \quad \alpha_2 := (a_1, a_2 - a_1, \dots, a_{n-d}),$$

$$\Theta_1 = \begin{cases} 0 & a = 0 \\ (\mathcal{K}, \alpha_1) + (\mathcal{K}, \alpha_2) & a \neq 0, \end{cases}$$
$$\Theta_2 = \begin{cases} 0 & a_i \in \langle a \rangle, \text{ for some } i, \\ (\mathcal{Y}, (b_2, \dots, b_{n-d})), & \text{otherwise,} \end{cases}$$

here the generic stabilizer of $\mathcal{Y}$ is naturally a subgroup of the generic stabilizer of $\mathcal{K}$, and b_i are the **restrictions** of a_i .

The subtlety is in the definition of relations on symbols:

$$(\mathcal{K}, \beta) \sim (\mathcal{K}', \beta),$$

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There is specialization for a further quotient,

$$\overline{\text{Burn}}_n,$$

when one keeps only the generic points i.e., one has relation

(I')

$$(\mathcal{K}, \beta) \sim (\mathcal{K}', \beta),$$

if there is an isomorphism on dense open substacks.

Let K be the fraction field of a DVR with residue field k (of characteristic zero and containing all roots of unity) and fixed uniformizer π .

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$$\begin{array}{ccc}
 X \hookrightarrow G & & \text{Burn}_{n,K}(G) \xrightarrow{\rho_\pi^G} \text{Burn}_{n,k}(G) \\
 \downarrow & & \downarrow \bar{\kappa}^G \quad \downarrow \bar{\kappa}^G \\
 [X/G] & & \overline{\text{Burn}}_{n,K} \xrightarrow{\bar{\rho}_\pi} \overline{\text{Burn}}_{n,k}
 \end{array}$$

$$H_{\text{nr}}^i(k(X)^G)$$

versus

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- Unramified cohomology is a **stable** birational invariant,

$$H_{\text{nr}}^i(k(X)^G) = H_{\text{nr}}^i(k(X \times \mathbb{P}^m)^G), \quad \forall G, X, m.$$

Comparisons: Unramified cohomology / Burnside invariants

$$H_{\text{nr}}^i(k(X)^G) \quad \text{versus} \quad [X \curvearrowright G] \in \text{Burn}_n(G)$$

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- Burnside invariants trivialize, eventually:

$$[X \times \mathbb{P}^m \curvearrowright G] = 0, \quad \forall G, X, \text{ with } m = \max(\exp(H)), \text{ abelian } H \subseteq G,$$

giving an interesting filtration.

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