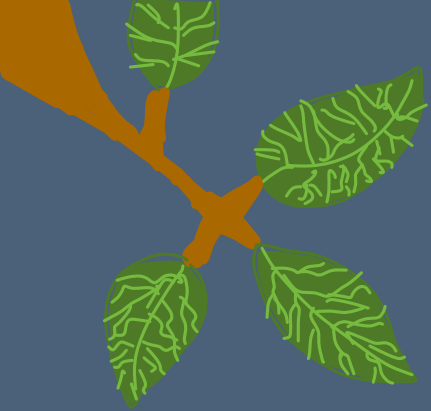




A Celestial Dual for MHV Amplitudes

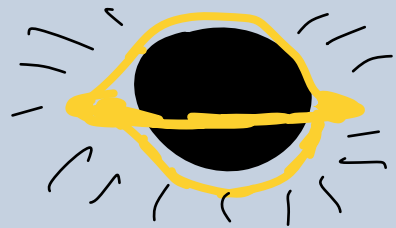
Atul Sharma

Black Hole Initiative, Harvard

Work with **Walker Melton**, **Andy Strominger** and **Tianli Wang**

[2403.18896] [2402.04150]

[2312.07820] [2310.15104]



Light/shadow
transforms

$w_{1+\infty}$ & S algebras

**Leaf
amplitudes**

Liouville

BFSS

Integrable systems

Twistors

Twisted holography

Celestial OPE

Burns

QCD

IR dressings

Soft theorems

Memory effects

Conformal blocks

Celestial amplitudes

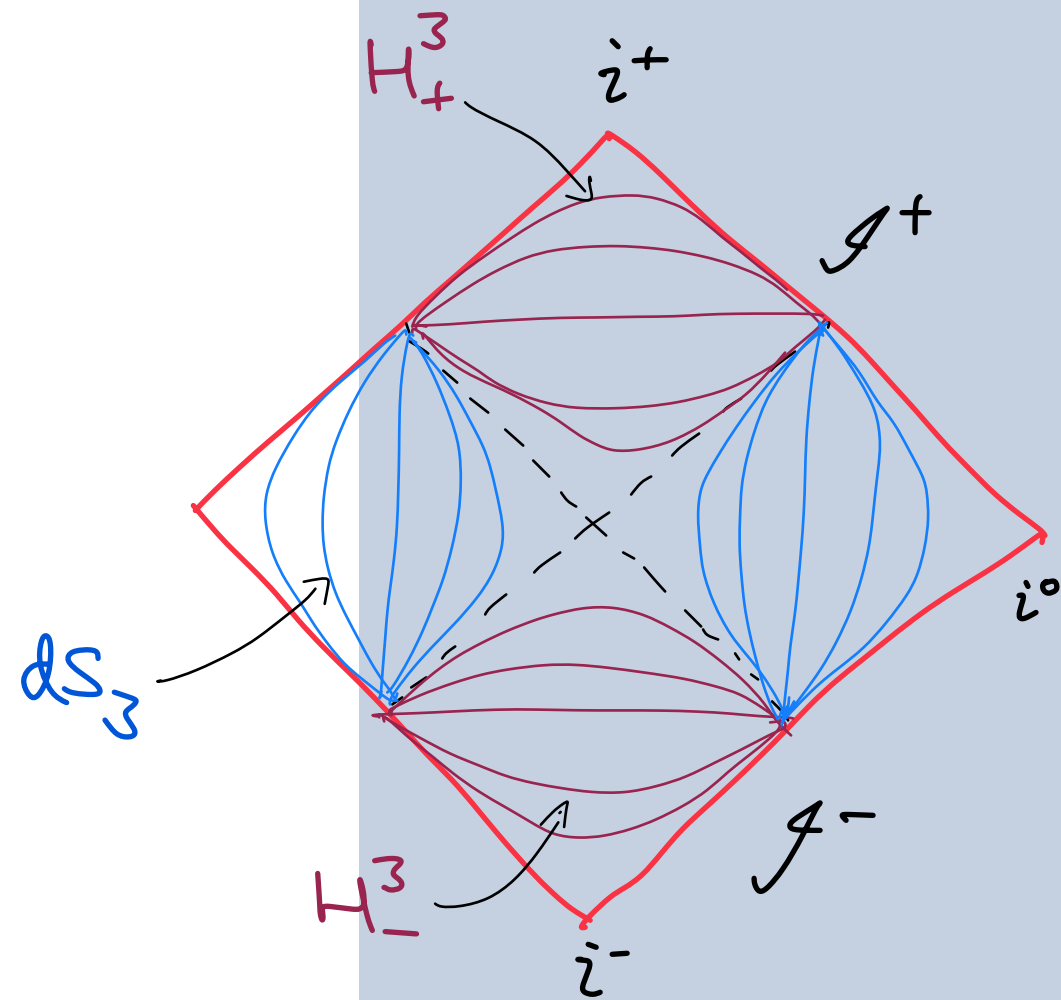


AdS/CFT / stringy roots



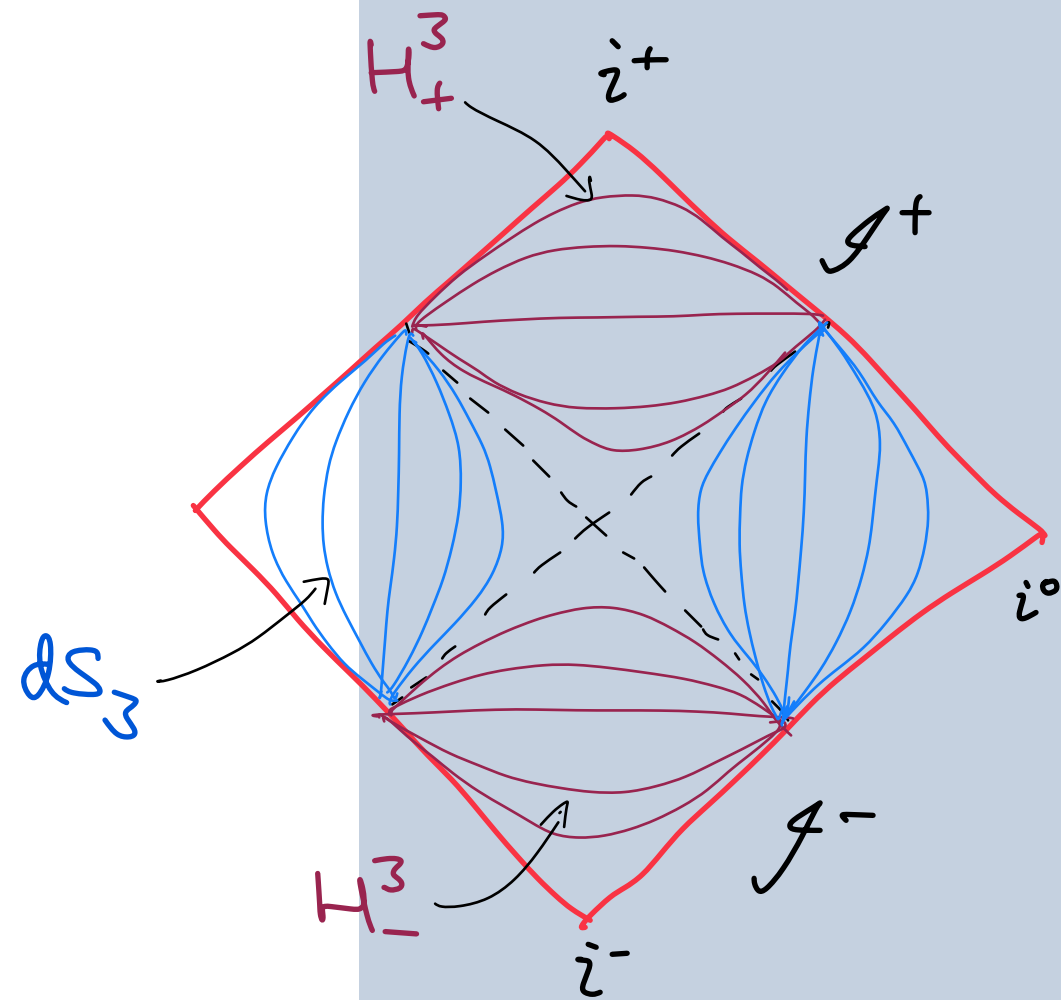
Holography for the MHV sector

- Gluon MHV *leaf amplitudes* in $\mathbb{R}^{1,3}$



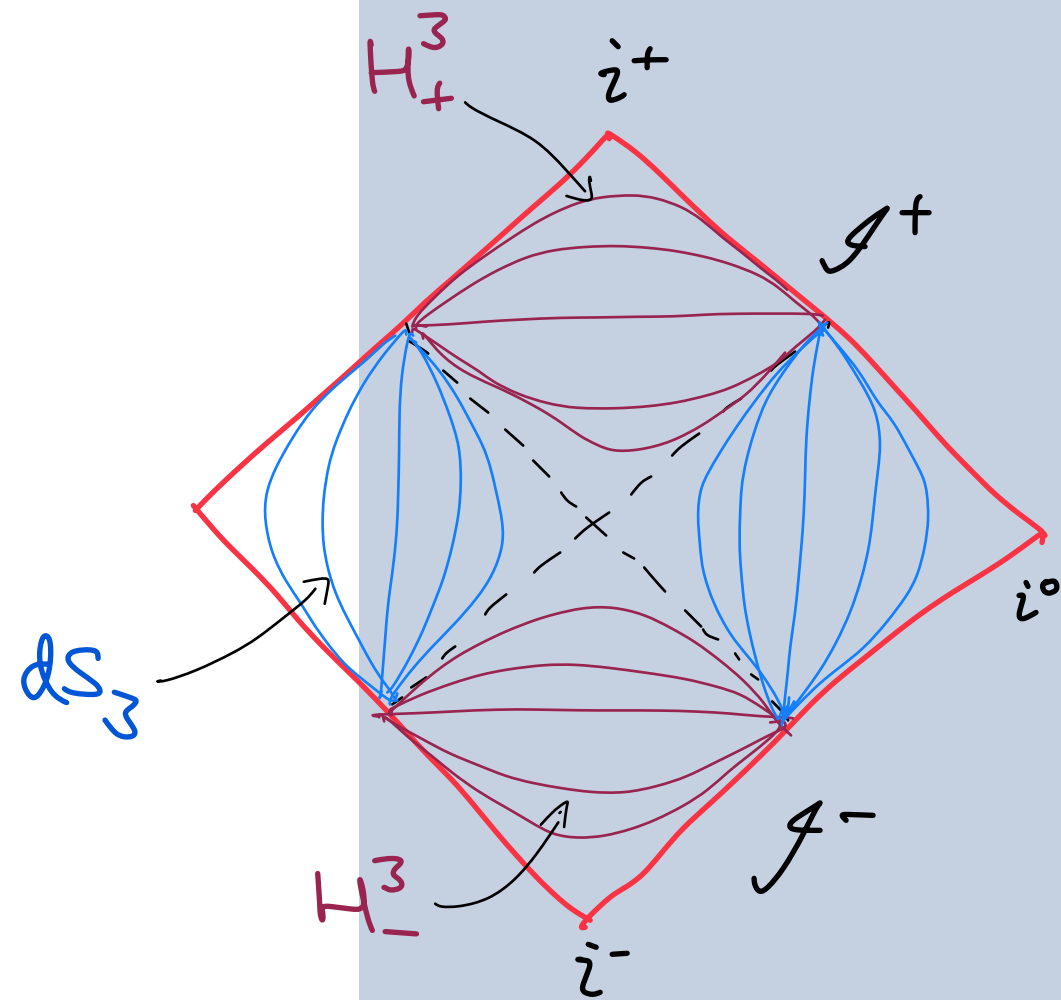
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 - Negative weight fermion
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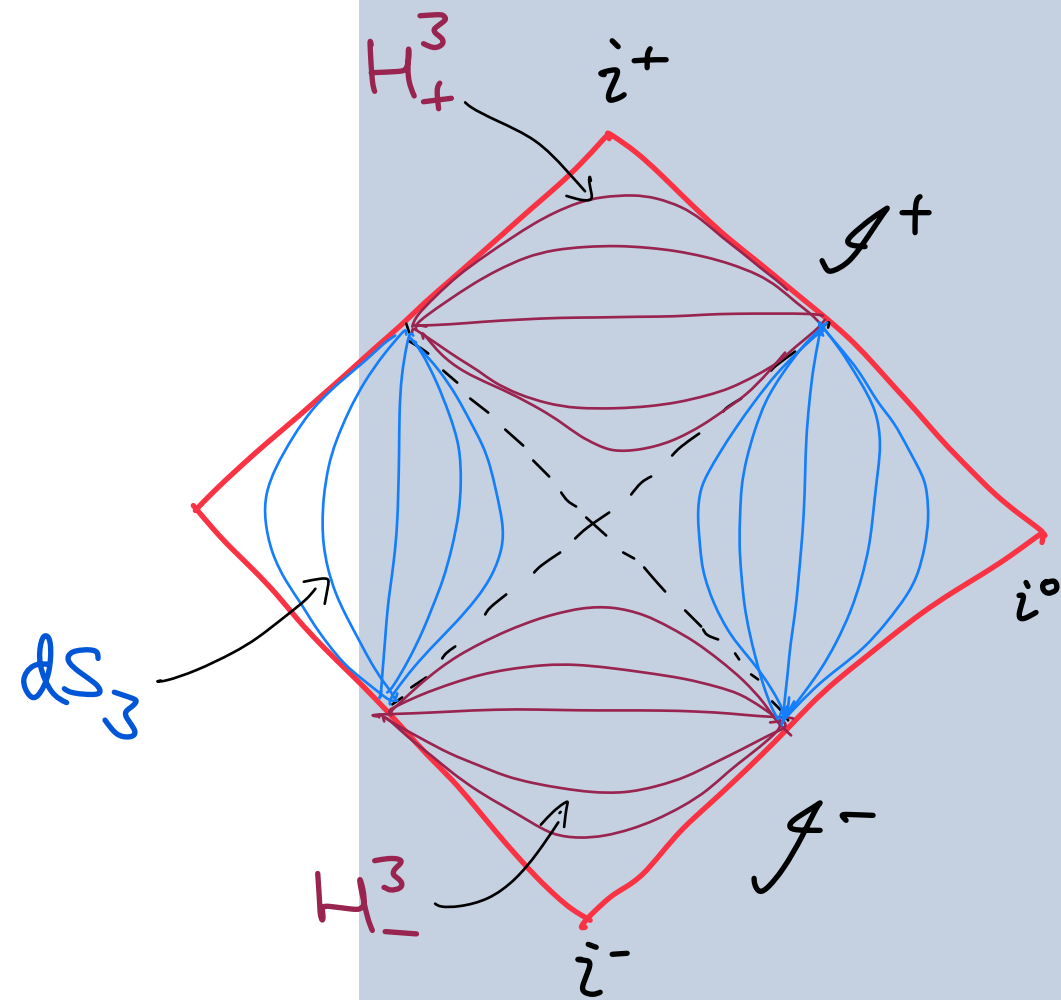
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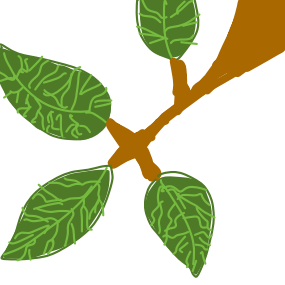


Holography for the MHV sector

- Gluon MHV *leaf amplitudes* in $\mathbb{R}^{1,3}$
- Leaf CFT
 - Current algebra
 - Negative weight fermion
 - Liouville
- Analytic continuation to $\mathbb{R}^{2,2}$
- Reconstruction of MHV celestial amplitudes

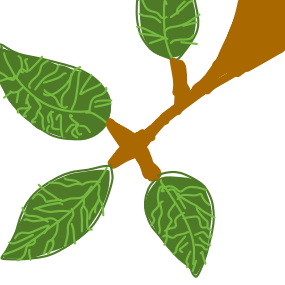


Leafwise holography



- **Dictionary:** Tree-level gluon MHV sector.

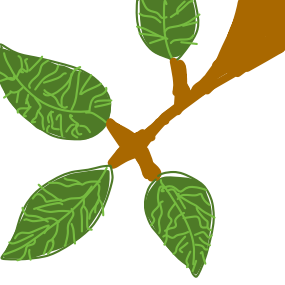
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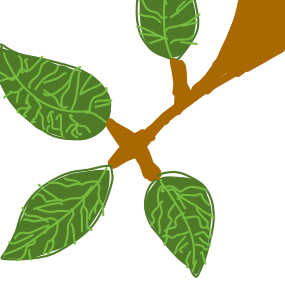


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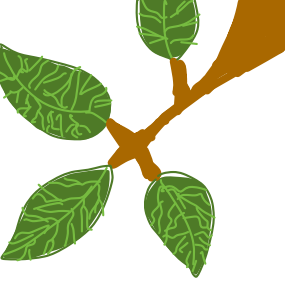
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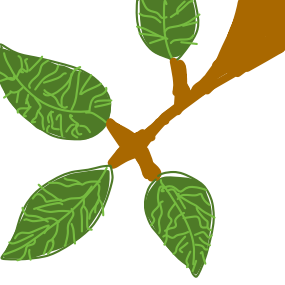
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$$\begin{aligned} (2\pi)^4 \delta^4(P) &= \int_{\mathbb{R}^{1,3}} d^4X e^{iP \cdot X} \\ &= \int_0^\infty d\tau \tau^3 \left(\underbrace{\int_{\hat{x}^2=-1} d^3\hat{x}}_{H_+^3 \cup H_-^3} + \underbrace{\int_{\hat{x}^2=1} d^3\hat{x}}_{dS_3} \right) e^{i\tau P \cdot \hat{x}} \end{aligned}$$

$$X^\mu = \tau \hat{x}^\mu$$

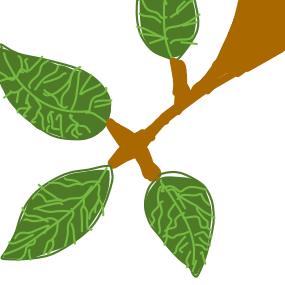
Celestial Parke-Taylor



$$A_{\text{MHV}} = \frac{z_{12}^4}{z_{12}z_{23}\cdots z_{n1}} \int_0^\infty \prod_{i=1}^n d\omega_i \omega_i^{2\bar{h}_i-1} e^{-\epsilon\omega_i} \delta^4(P)$$

[Pasterski, Shao, Strominger '17]

Celestial Parke-Taylor

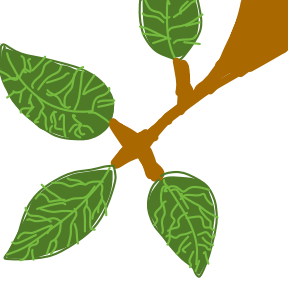


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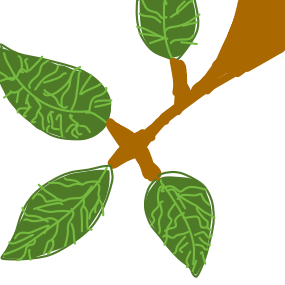
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$$P = \omega_1 \hat{p}_1 + \cdots + \omega_n \hat{p}_n$$

$$\hat{p}_i^\mu = \varepsilon_i \left(\frac{1 + |z_i|^2}{2}, \text{Re } z_i, \text{Im } z_i, \frac{1 - |z_i|^2}{2} \right)$$

$$\varepsilon_i = \pm \quad z_i \in S^2 \quad h_i = \frac{\Delta_i + J_i}{2} \quad \bar{h}_i = \frac{\Delta_i - J_i}{2}$$

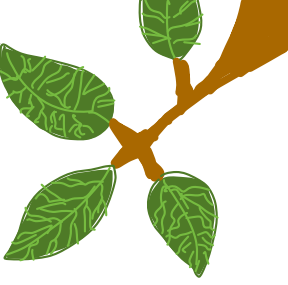
Leaf amplitudes



$$A_{\text{MHV}} = \frac{\delta(\beta)}{(2\pi)^3} (L_{H_+^3} + L_{H_-^3} + L_{\text{dS}_3})$$

$$\beta = \Delta_1 + \cdots + \Delta_n - n$$

Leaf amplitudes



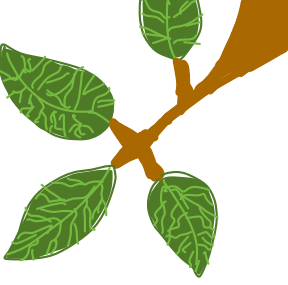
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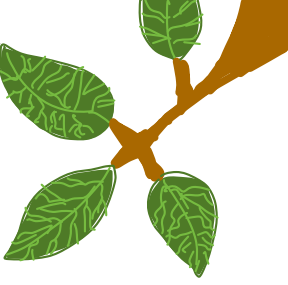
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$$L_{H_-^3} = L_{H_+^3} \Big|_{\hat{p}_i \mapsto -\hat{p}_i}$$

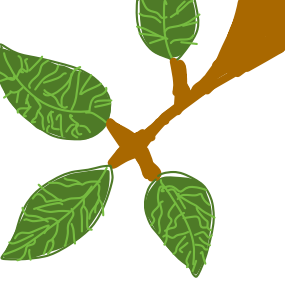
Question!



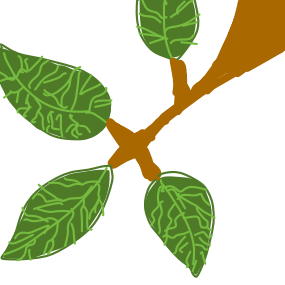
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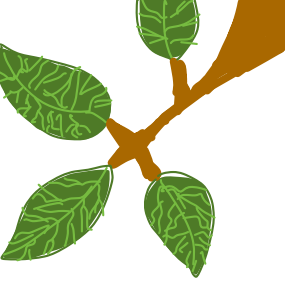


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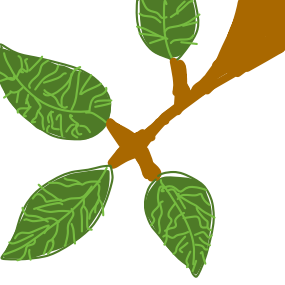


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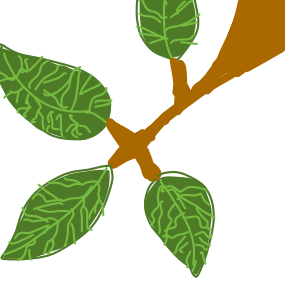
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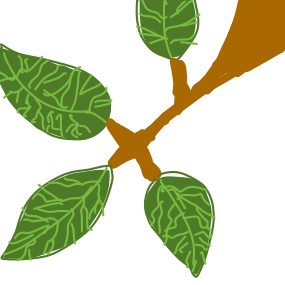
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Liouville

How to build your own Leaf CFT



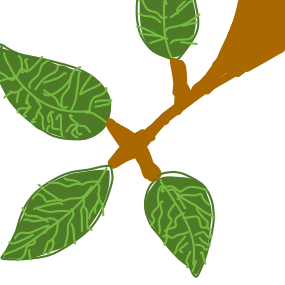


How to build your own Leaf CFT

- Planar limit of level 1, rank N current algebra

[Nair '88] [Witten '03]
[Costello, Paquette, AS '22]

$$J_a = \frac{1}{2} \mathbf{t}_a^{ij} \psi_i \psi_j \qquad \psi_i(z) \psi_j(0) \sim \frac{1}{z} \delta_{ij}$$



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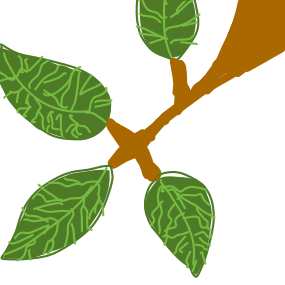
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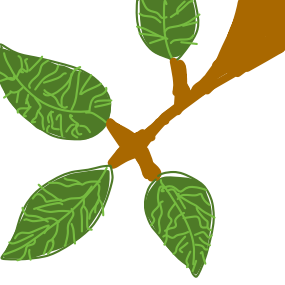
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- Semiclassical limit of Liouville theory (see also [Stieberger, Taylor, Zhu '22, '23])

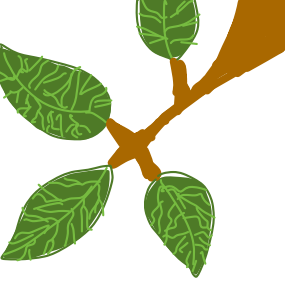
Holographic dictionary



$$O_a^+ = J_a \mathcal{V}_{\bar{h}}$$

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Holographic dictionary

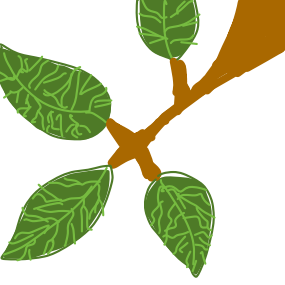


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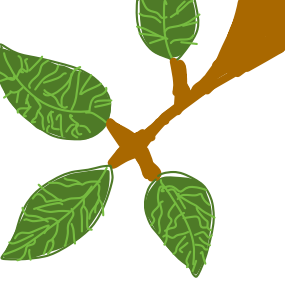
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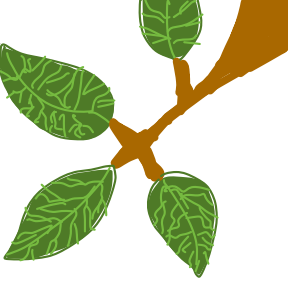
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Main idea:

$$\langle \bar{J}(z_1) \bar{J}(z_2) J(z_3) \cdots J(z_n) \rangle = \frac{z_{12}^4}{z_{12} z_{23} \cdots z_{n1}}$$

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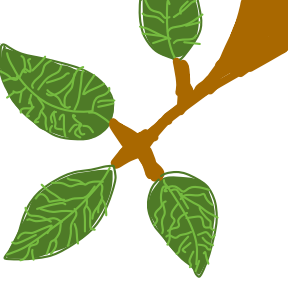
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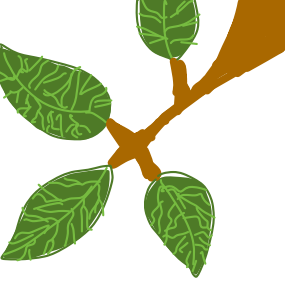
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[Melton, AS, Strominger, Wang '24]

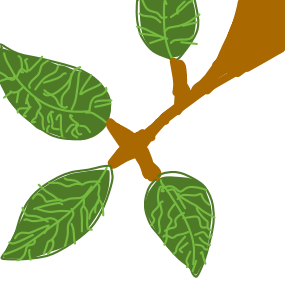
Recap: Liouville



- Liouville CFT describes constant scalar curvature metrics on S^2

$$g_{ab} = e^{\phi} \delta_{ab}$$

Recap: Liouville

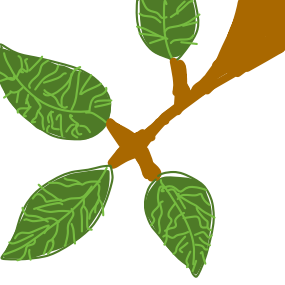


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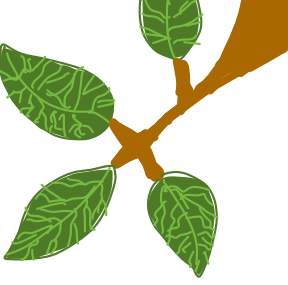
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Recap: Liouville

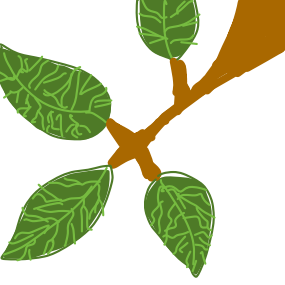


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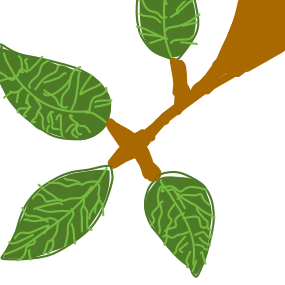
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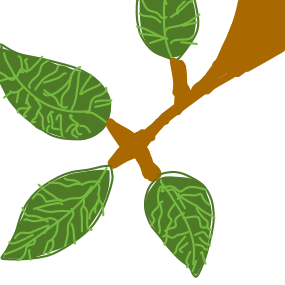
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Recap: Liouville

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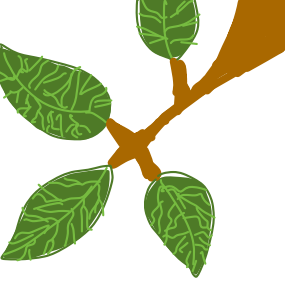
Recap: Liouville



- Semiclassical limit: $b \rightarrow 0$
- Heavy operators (change the classical saddle)

$$V(\alpha) = \exp\left(\frac{\eta\phi}{b^2}\right) \quad \alpha = \frac{\eta}{b} \sim O(b^{-1}) \text{ as } b \rightarrow 0$$

Recap: Liouville



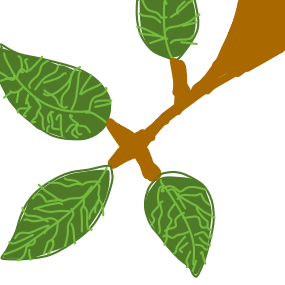
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- Light operators (sample the background saddle)

$$V(\alpha) = \exp(\sigma\phi) \quad \alpha = \sigma b \sim O(b) \text{ as } b \rightarrow 0$$

Recap: Liouville



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- Heavy operators (change the classical saddle)

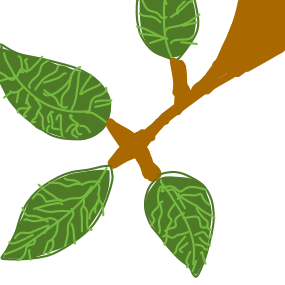
$$V(\alpha) = \exp\left(\frac{\eta\phi}{b^2}\right) \quad \alpha = \frac{\eta}{b} \sim O(b^{-1}) \quad \text{as } b \rightarrow 0$$

- Light operators (sample the background saddle)

$$V(\alpha) = \exp(\sigma\phi) \quad \alpha = \sigma b \sim O(b) \quad \text{as } b \rightarrow 0$$

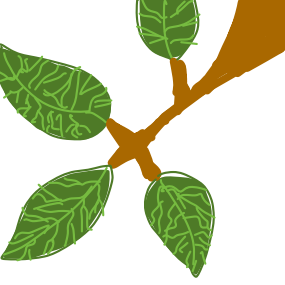
- We will set: $\mathcal{V}_{\bar{h}} \sim V(\alpha = b\bar{h})$

Light operator correlators



$$\mathcal{V}_{\bar{h}} = \Gamma(2\bar{h}) e^{-i\pi\epsilon\bar{h}} e^{\bar{h}\phi}$$

Light operator correlators

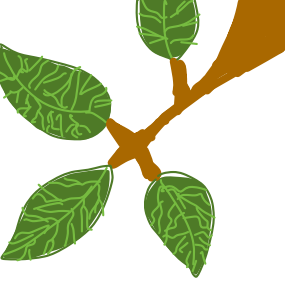


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We need the semiclassical correlator:

$$\left\langle \prod_{i=1}^n e^{\bar{h}_i \phi(z_i, \bar{z}_i)} \right\rangle = \sum_{\text{saddles}} e^{-S_L} \prod_{i=1}^n e^{\bar{h}_i \phi(z_i, \bar{z}_i)} \Big|_{\text{saddle}}$$

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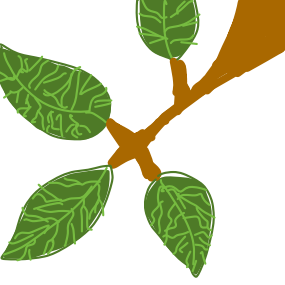
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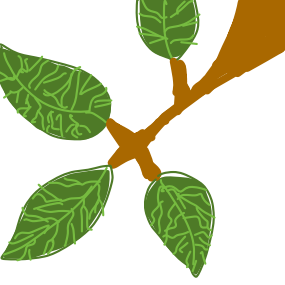
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What are the saddles of Liouville?

Saddles of Liouville



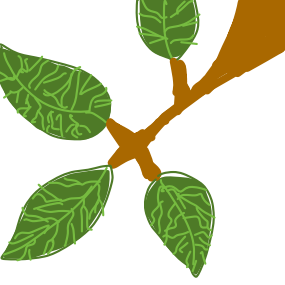
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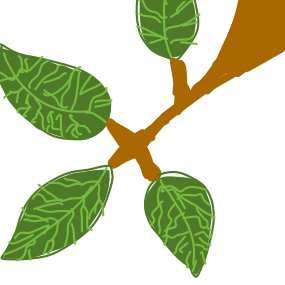
[Harlow, Maltz, Witten '11]



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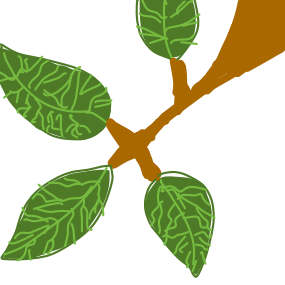
$$\phi = -2 \log(1 + |z|^2) - \log \Lambda + i\pi$$

- ▶ $SL_2(\mathbb{C})$ transformations and shifts by $2\pi i$ generate all complex saddles

$$\phi_{g,n} = 2\pi i n - 2 \log(|\alpha z + \beta|^2 + |\gamma z + \delta|^2) - \log \Lambda + i\pi$$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{C})$$

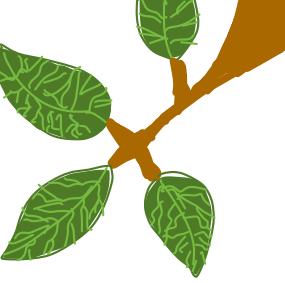
Light operator correlators



Semiclassical correlator:

$$\left\langle \prod_{i=1}^n e^{\bar{h}_i \phi(z_i, \bar{z}_i)} \right\rangle = \mathcal{N}(\Lambda, b) \int_{\mathrm{SL}_2(\mathbb{C})} dg \prod_{i=1}^n \frac{1}{(|\alpha z_i + \beta|^2 + |\gamma z_i + \delta|^2)^{2\bar{h}_i}}$$

Light operator correlators

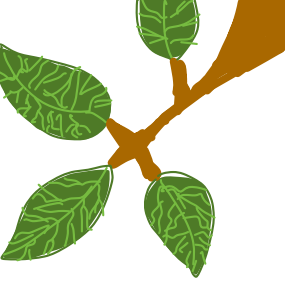


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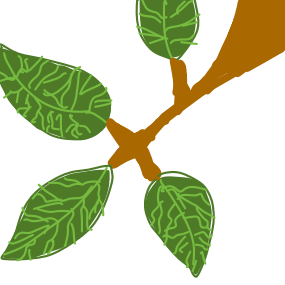
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Proposition (Freidel, Krasnov, Livine '10)

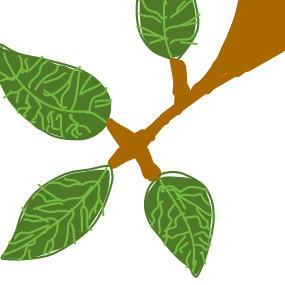
$$\int_{\mathrm{SL}_2(\mathbb{C})} dg \prod_{i=1}^n \frac{\Gamma(2\bar{h}_i) e^{-i\pi \varepsilon_i \bar{h}_i}}{(|\alpha z_i + \beta|^2 + |\gamma z_i + \delta|^2)^{2\bar{h}_i}} = 2\pi^2 D_{\bar{h}_1, \dots, \bar{h}_n}$$



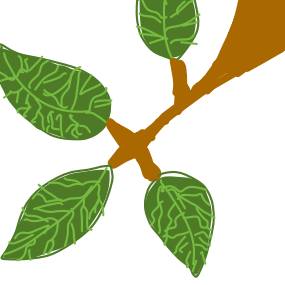
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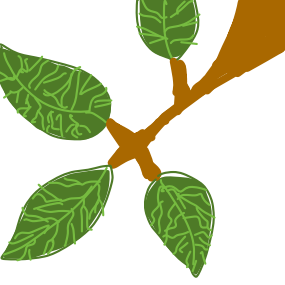
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► Hence

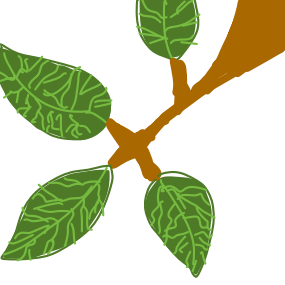
$$\int_{\mathrm{SL}_2(\mathbb{C})} \mathrm{d}g \prod_{i=1}^n \frac{\Gamma(2\bar{h}_i) e^{-i\pi\varepsilon_i \bar{h}_i}}{(|\alpha z_i + \beta|^2 + |\gamma z_i + \delta|^2)^{2\bar{h}_i}} = \int_{\mathrm{SL}_2(\mathbb{C})} \mathrm{d}g \prod_{i=1}^n \frac{\Gamma(2\bar{h}_i) e^{-i\pi\varepsilon_i \bar{h}_i}}{(\|g\lambda\|^2)^{2\bar{h}_i}}$$



- $\mathrm{SL}_2(\mathbb{C}) \simeq H^3 \times \mathrm{SU}(2)$ through the Iwasawa decomposition

$$g = h \begin{pmatrix} \rho^{-1/2} & 0 \\ 0 & \rho^{1/2} \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}$$

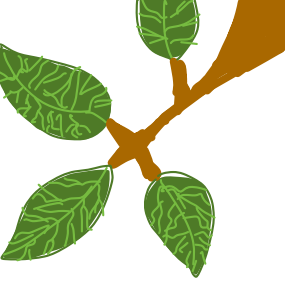
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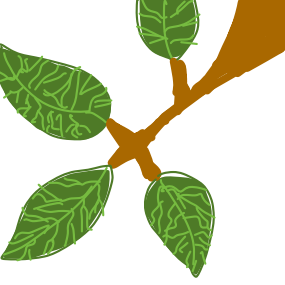
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- This reduces our integral to

$$\mathrm{vol}(\mathrm{SU}(2)) \int_{H^3} \frac{d\rho \, dy \, d\bar{y}}{\rho^3} \prod_{i=1}^n \Gamma(2\bar{h}_i) e^{-i\pi \varepsilon_i \bar{h}_i} \left(\frac{\rho}{\rho^2 + |y - z_i|^2} \right)^{2\bar{h}_i} = 2\pi^2 D_{\bar{h}_1, \dots, \bar{h}_n}$$



Summary of Leaf CFT



$$O_a^+ = J_a \mathcal{V}_{\bar{h}}$$

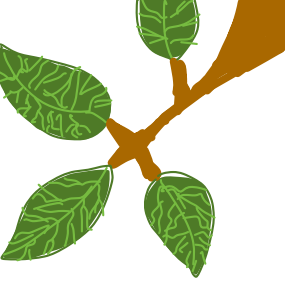
$$\mathcal{V}_{\bar{h}} = \Gamma(2\bar{h}) e^{-i\pi\epsilon\bar{h}} e^{\bar{h}\phi}$$

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$$\langle O_1^- O_2^- O_3^+ \cdots O_n^+ \rangle = \langle \eta\partial\eta(z_1) \eta\partial\eta(z_2) \rangle \langle J_1 J_2 \cdots J_n \rangle \langle \mathcal{V}_1 \mathcal{V}_2 \cdots \mathcal{V}_n \rangle$$

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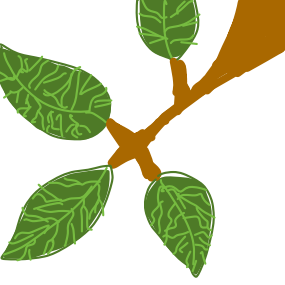
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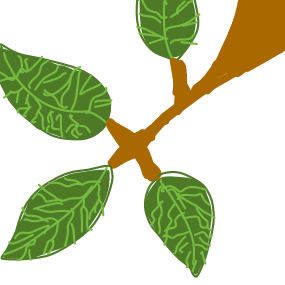
$$= 2\pi^2 \mathcal{N} L_{H_+^3}$$

Onward march...

- Complicated to compute dS_3 leaf amplitudes.

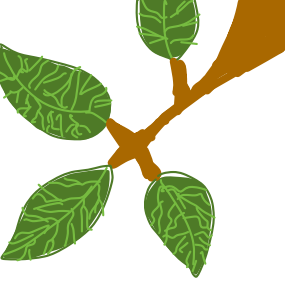


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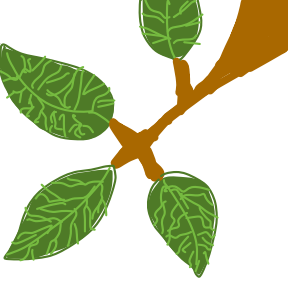
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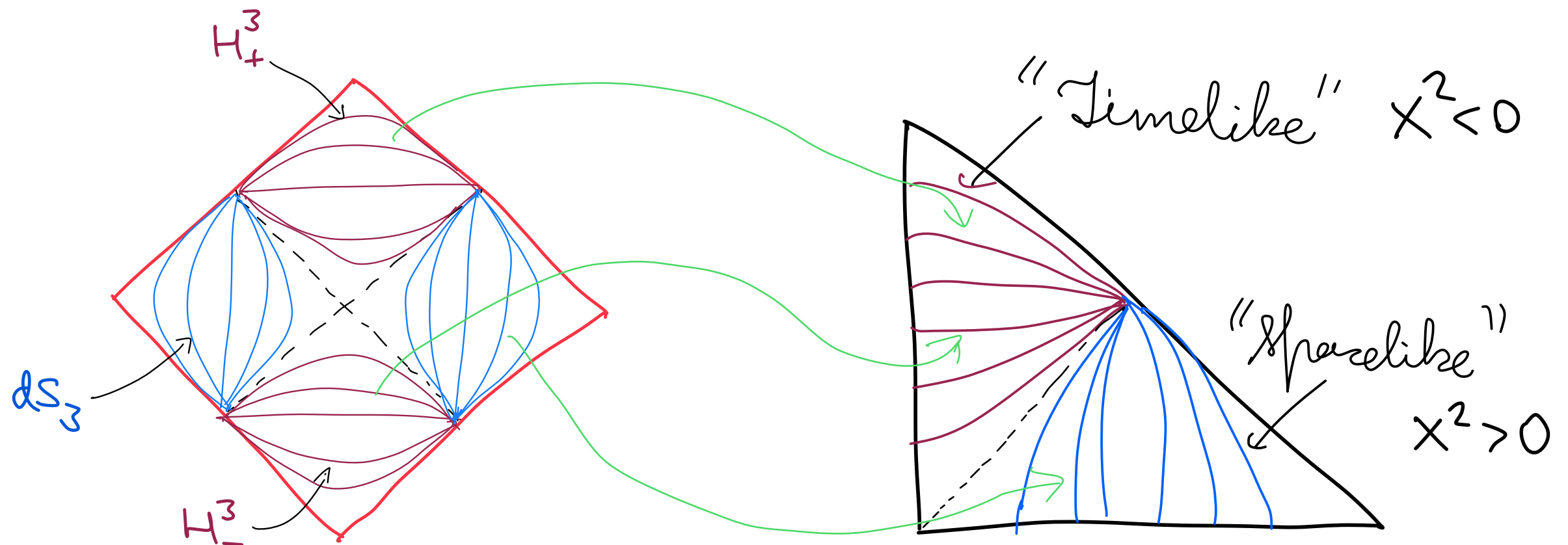
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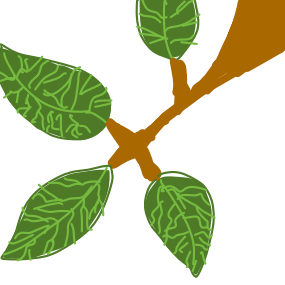
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- ▶ Foliated by Lorentzian $AdS_3/\mathbb{Z}$ slices.

[Atanasov, Ball, Melton, Raclariu, Strominger '23]

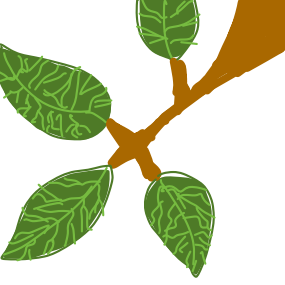




Kleinian leaf amplitudes

- $L_{H_+^3} + L_{H_-^3}$ analytically continue to a timelike leaf amplitude

$$L_{\text{AdS}_3/\mathbb{Z}}^+ = \frac{z_{12}^4}{z_{12}z_{23}\cdots z_{n1}} \int_{\text{AdS}_3/\mathbb{Z}} d^3\hat{x} \prod_{i=1}^n \frac{\Gamma(2\bar{h}_i)}{(\epsilon - i\hat{p}_i \cdot \hat{x})^{2\bar{h}_i}}$$

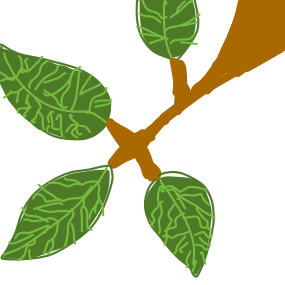


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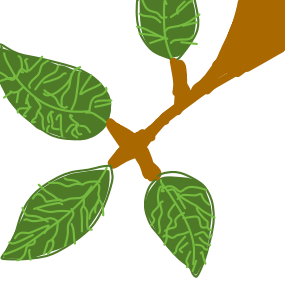


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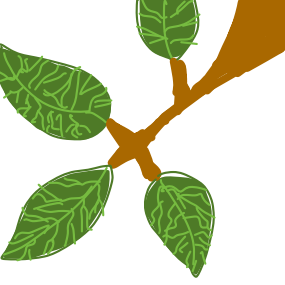


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Kleinian leaf amplitudes

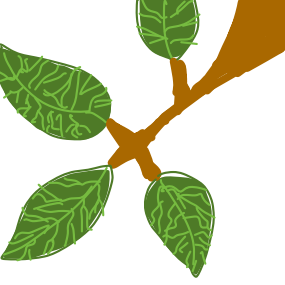
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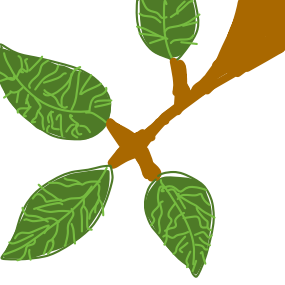
$$L_{\text{AdS}_3/\mathbb{Z}}^-(\epsilon_i, z_i, \bar{z}_i) = L_{\text{AdS}_3/\mathbb{Z}}^+(\epsilon_i, z_i, -\bar{z}_i)$$

Amplitudes at last!



$$A_{\text{MHV}} = \frac{\delta(\beta)}{(2\pi)^3} \left(L_{\text{AdS}_3/\mathbb{Z}}^+ + L_{\text{AdS}_3/\mathbb{Z}}^- \right)$$

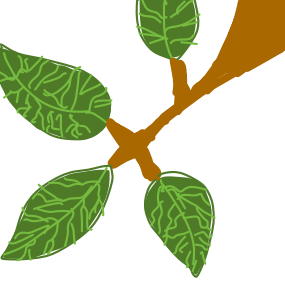
[Melton, AS, Strominger '23]



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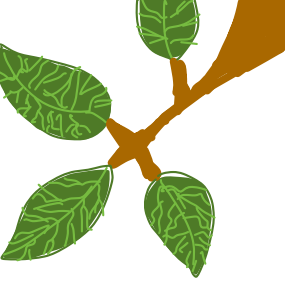
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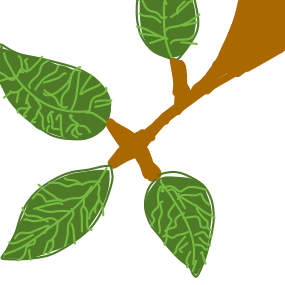


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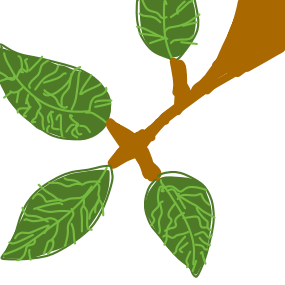
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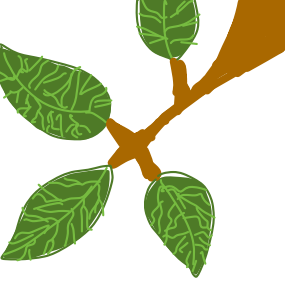
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- Reassemble the wedges into A_{MHV}



Future directions

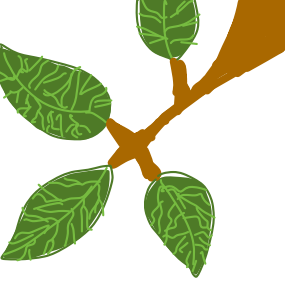
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[Melton, Niewinski, Strominger, Wang '23]



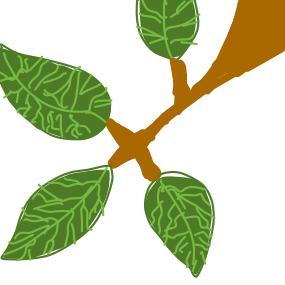
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[Melton, Niewinski, Strominger, Wang '23]
- Find a purely Minkowski description—requires understanding de Sitter leaves.



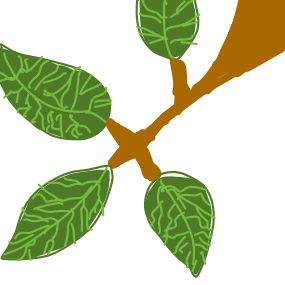
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- Find a string embedding? Or “understand” why Liouville?

Thanks!