

Implicit Renormalization Approach to the Problem of Cooper Instability

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In the vast majority of cases, superconducting transition takes place at exponentially low temperature T_c out of the Fermi liquid regime:

BCS (effective) theory.

At the level of (effective) weakly interacting theory, the (emergent) BCS physics is perfectly captured by Feynman diagrammatics.

The goal: To utilize Feynman diagrammatics to bridge the (emergent) long-wave BCS physics with (strongly correlated) ultraviolet microscopics.

Cooper instability via linear response: diagrammatic language

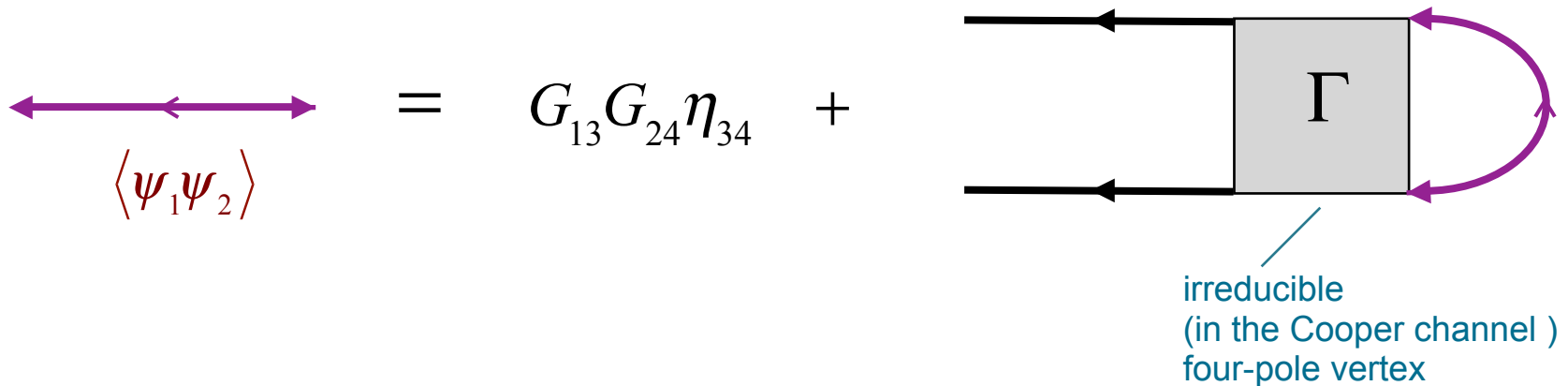
Modify the Hamiltonian:

$$H \rightarrow H + \left(\eta_{12}^* \psi_1 \psi_2 + \text{H.c.} \right)$$

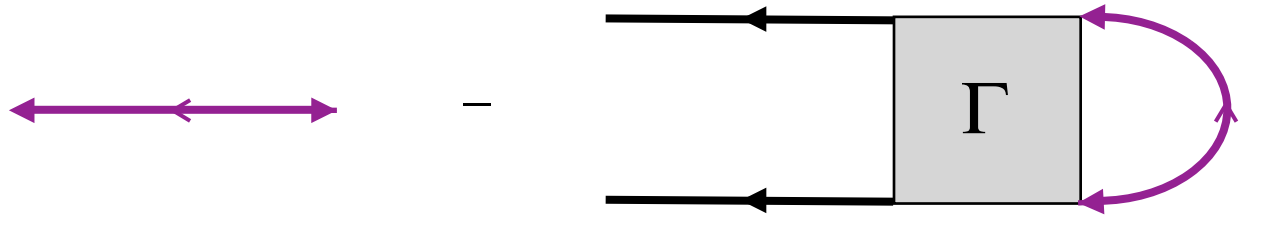
infinitesimally small
symmetry-breaking field

(infinitesimal) linear response: $\langle \psi_1 \psi_2 \rangle$

Diagrammatically:

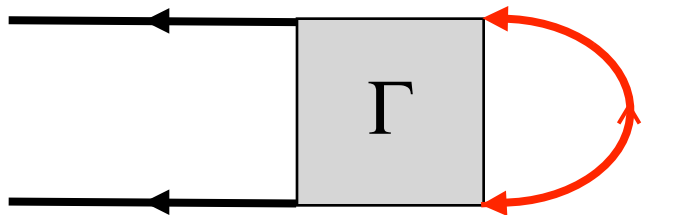


Singular response: eigenvector-eigenvalue problem


$$- = G_{13} G_{24} \eta_{34}$$

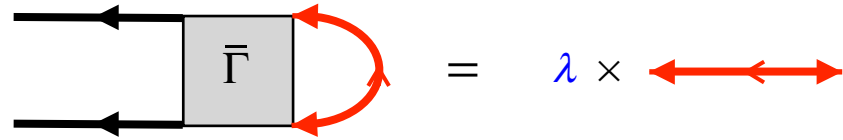
just some right-hand side...

Response diverges (i.e., the critical temperature is reached),
when the following **eigenvalue** becomes equal **1**.


$$= \lambda \times$$

Corresponding **eigenvector** defines the pairing channel.

Emergent BCS regime: long-wave effective theory



In this regime:

Green's function has a Fermi-liquid form (close to the Fermi surface):

$$G(\mathbf{k}, \omega_n) \approx \frac{z(\hat{k})}{i\omega_n - \mathbf{v}_F(\hat{k}) \cdot [\mathbf{k} - \mathbf{k}_F(\hat{k})]}$$

The effective four-pole vertex $\bar{\Gamma}$ is small and independent of temperature and frequency.

The eigenvector is temperature and frequency independent.

Temperature dependence of $\bar{\lambda}$ is essentially due to the Green's function factor:

$$\bar{\lambda}(T) = g \ln(\# / T) \quad \Rightarrow \quad T_c = \# e^{-1/g}, \quad g \ll 1$$

Can we deal with the fully microscopic description?

Eigenvalue-eigenvector problem for the gap function Δ



$$\lambda(T)\Delta_{\mathbf{k},\omega_n} = -T \sum_m \int \frac{d\mathbf{p}}{(2\pi)^d} \Gamma_{\mathbf{p},\omega_m}^{\mathbf{k},\omega_n} G_{\mathbf{p},m} G_{-\mathbf{p},-m} \Delta_{\mathbf{p},\omega_m}$$

A crucial obstacle for the (*otherwise straightforward*) DiagMC:
In the majority of interesting cases, the diagrammatic series for Γ
is well beyond the convergence radius at $T \sim T_c$.

How about extrapolating $\lambda(T)$ from $T \gg T_c$?

Works with $\bar{\lambda}(T)$ but not with $\lambda(T)$!

(Simple math—and deep physics—associated with this fact will be discussed later.)

Renormalization of the interaction in the Cooper channel

$$\boxed{-\bar{\Gamma}} = \boxed{-\Gamma} + \boxed{-\Gamma} \begin{array}{c} \text{---} \blackrightarrow \text{---} \\ \text{---} \blackleftarrow \text{---} \end{array} \boxed{-\bar{\Gamma}}$$

excluding the low-energy part

Not doable because of the “curse of multivariableness.”

However, $\bar{\lambda}(T)$ can be extracted from a **modified eigenvalue-eigenvector problem based on bare Γ** .

A few preliminary steps

matrix-vector notation:

$$\lambda \vec{\Delta} = -\hat{A} \vec{\Delta}, \quad \vec{\Delta} \equiv \Delta(\mathbf{p}, \omega)$$

Introduce low- and high-energy parts with respect to a certain characteristic energy scale Ω_c :

$$\begin{aligned} \vec{\Delta} &= \vec{\Delta}^{(1)} + \vec{\Delta}^{(2)} & \Delta^{(1)}(\mathbf{p}, \omega) &\equiv 0 & \text{at } \xi_p^2 + \omega^2 > \Omega_c^2 & \text{low-energy part} \\ & & \Delta^{(2)}(\mathbf{p}, \omega) &\equiv 0 & \text{at } \xi_p^2 + \omega^2 \leq \Omega_c^2 & \text{high-energy part} \end{aligned}$$

$$\hat{A} = \hat{A}^{(11)} + \hat{A}^{(22)} + \hat{A}^{(21)} + \hat{A}^{(12)}$$

$$\begin{cases} \lambda \vec{\Delta}^{(1)} = -\hat{A}^{(11)} \vec{\Delta}^{(1)} - \hat{A}^{(12)} \vec{\Delta}^{(2)} \\ \lambda \vec{\Delta}^{(2)} = -\hat{A}^{(22)} \vec{\Delta}^{(2)} - \hat{A}^{(21)} \vec{\Delta}^{(1)} \end{cases}$$

(So far, it is just an identical rewriting.)

Implicit-renormalization formulation

Replace

$$\begin{cases} \lambda \vec{\Delta}^{(1)} = -\hat{A}^{(11)}\vec{\Delta}^{(1)} - \hat{A}^{(12)}\vec{\Delta}^{(2)} \\ \lambda \vec{\Delta}^{(2)} = -\hat{A}^{(22)}\vec{\Delta}^{(2)} - \hat{A}^{(21)}\vec{\Delta}^{(1)} \end{cases}$$

with:

$$\begin{cases} \bar{\lambda} \vec{\Delta}^{(1)} = -\hat{A}^{(11)}\vec{\Delta}^{(1)} - \hat{A}^{(12)}\vec{\Delta}^{(2)} \\ \vec{\Delta}^{(2)} = -\hat{A}^{(22)}\vec{\Delta}^{(2)} - \hat{A}^{(21)}\vec{\Delta}^{(1)} \end{cases}$$

Let us see that $\bar{\lambda}$ and (new) $\vec{\Delta}^{(1)}$ exactly correspond to the renormalized theory.

Substituting $\vec{\Delta}^{(2)} = -[\hat{I} + \hat{A}^{(22)}]^{-1} \hat{A}^{(21)}\vec{\Delta}^{(1)}$ (implied by the second equation) into the first equation, we get:

$$\bar{\lambda} \vec{\Delta}^{(1)} = -\hat{B}\vec{\Delta}^{(1)},$$

$$\hat{B} = \hat{A}^{(11)} - \hat{A}^{(12)}[\hat{I} + \hat{A}^{(22)}]^{-1} \hat{A}^{(21)}$$

This is exactly the kernel of the effective theory.

Illustrative examples: Models with momentum-independent vertices

$$\Delta_{\mathbf{p}, \omega_n} \equiv \Delta_{\omega_n}$$

$$G_{\mathbf{p}, \omega_n} = 1 / (i\omega_n - \xi_p) \quad \text{with linear dispersion}$$

Integrating over momentum:

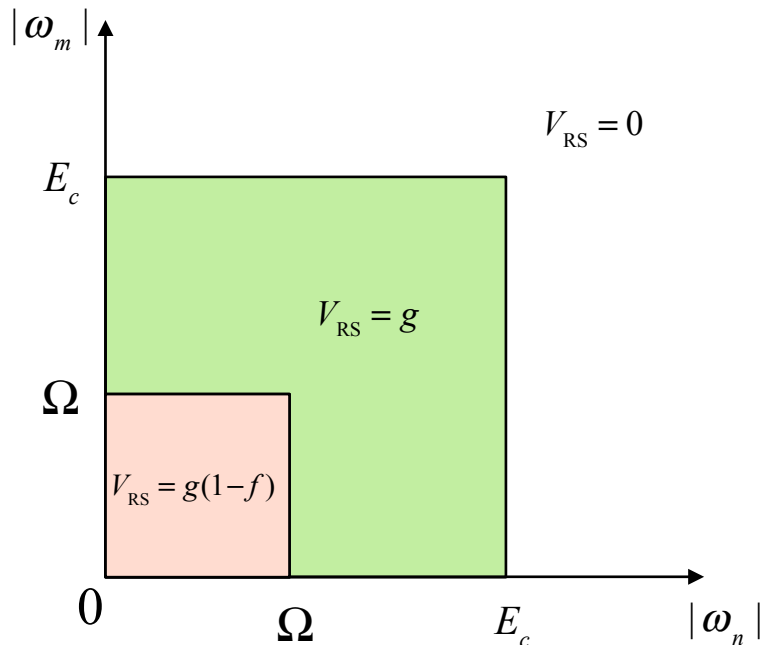
$$\lambda(T) \Delta_{\omega_n} = -\pi T \sum_m V(\omega_m, \omega_n) \frac{\Delta_{\omega_m}}{|\omega_m|}$$

Rietschel-Sham model

Most simple model capturing all the essential physics of BCS s-wave regime with all-repulsive bare interaction.

$$V_{\text{RS}} = \begin{cases} 0, & \text{if } |\omega_n| > E_c, \text{ or } |\omega_m| > E_c \\ g - gf\theta(\Omega - |\omega_n|)\theta(\Omega - |\omega_m|) & \text{otherwise} \end{cases}$$

$$g > 0, \quad 0 < f < 1 \quad \Rightarrow \quad V_{\text{RS}} \geq 0$$



two-parametric piecewise-constant eigenvector:

$$\Delta_{\omega_n} = [\textcolor{red}{a}\theta(|\omega_n| - \Omega) + \textcolor{red}{s}\theta(\Omega - |\omega_n|)]\theta(E_c - |\omega_n|)$$

The problem reduces to an algebraic one:

$$\begin{cases} \textcolor{brown}{\lambda}\textcolor{red}{s} = -g\ell\textcolor{red}{a} - g(1-f)L\textcolor{red}{s} \\ \textcolor{brown}{\lambda}\textcolor{red}{a} = -g\ell\textcolor{red}{a} - gL\textcolor{red}{s} \end{cases}$$

$$L = \ln[\Omega / 0.882 T], \quad \ell = \ln[E_c / \Omega]$$

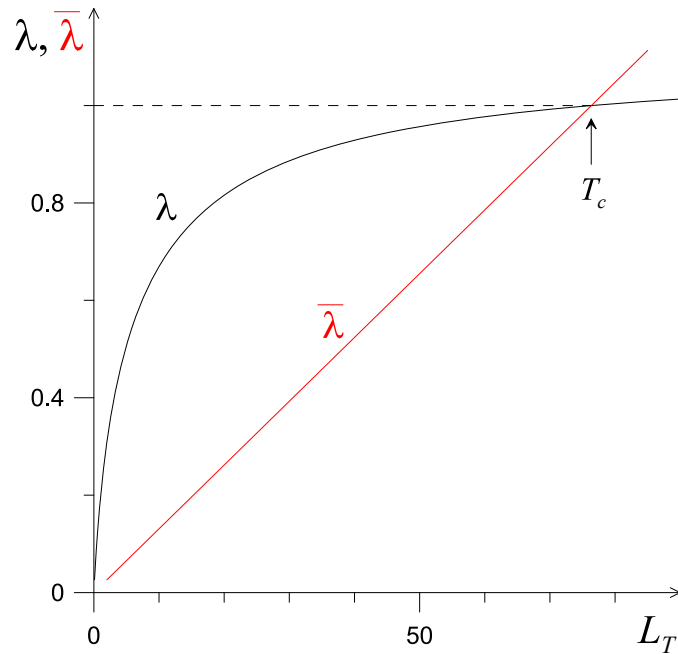
Non-linear dependence of λ on L !

Implicit-renormalization formulation:

$$\begin{cases} \bar{\lambda}\textcolor{red}{s} = -g\ell\textcolor{red}{a} - g(1-f)L\textcolor{red}{s} \\ \textcolor{red}{a} = -g\ell\textcolor{red}{a} - gL\textcolor{red}{s} \end{cases}$$

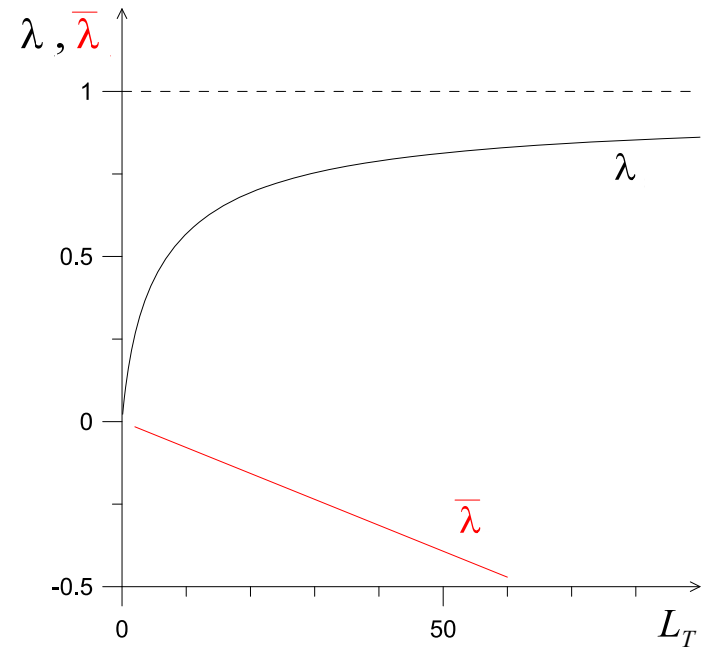
$$\bar{\lambda}\textcolor{red}{s} = g_{\text{eff}}L\textcolor{red}{s}, \quad g_{\text{eff}} = g \left[f - \frac{1}{1 + g\ell} \right]$$

Case1:
Cooper instability is present



$$L_T = \ln[\Omega / 0.882T]$$

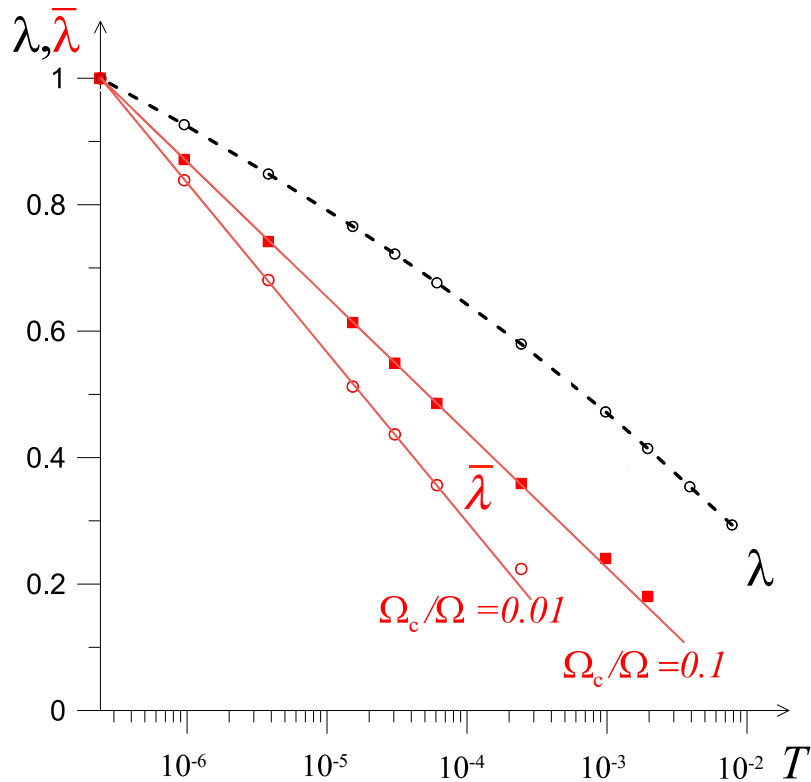
Case 2:
Cooper instability is absent



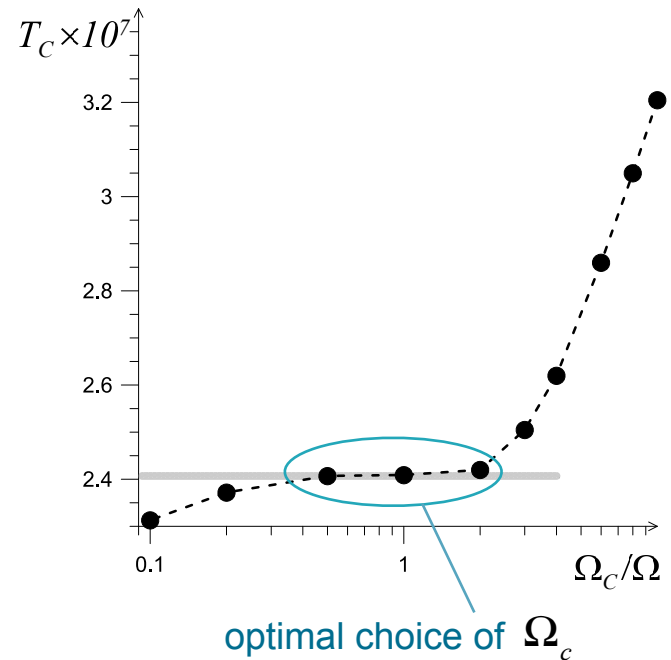
Behavior of λ at not very large L_T proves very misleading!

Less idealized model

$$V_{\text{eff}}(\omega) = g \left(1 - \frac{\Omega_a^2}{\omega^2 + \Omega^2} \right) = g \left(\frac{\omega^2 + \Omega_1^2}{\omega^2 + \Omega^2} \right), \quad \omega = \omega_m - \omega_n, \quad \Omega_a \leq \Omega, \quad \Omega_1^2 = \Omega^2 - \Omega_a^2 < \Omega^2$$



T_c extracted by linear extrapolation
at different values of Ω_c



Concluding remarks

- A protocol for extrapolating numerical data towards T_c from higher temperatures—applicable to first-principle description of real metals, as well as strongly interacting models—has to adequately capture the physics of the emergent weakly-interacting effective theory.
- Implicit renormalization protocol provides a simple, efficient, and unbiased way of solving the extrapolation problem. The scheme has a built-in tool of controlling the systematic error of extrapolation—the only systematics of the otherwise numerically exact method.
- The implicit renormalization approach is perfectly compatible with the DiagMC. One can solve the corresponding eigenvalue problem without invoking the matrix inversion or even explicitly calculating the four-point vertex function Γ .
- The implicit renormalization protocol also allows one to obtain the correct gap function immediately below T_c .