

On Gross and Zagier's Algebraicity Conjecture

(joint with Bruinier, Ehlen, and Yingkun Li)

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Outline of this talk:

- ▶ Gross and Zagier's algebraicity conjecture
- ▶ General set-up
- ▶ Regularized theta liftings and principal Higher Green functions
- ▶ Generalized algebraicity conjecture and main results
- ▶ Some ingredients of the proof

The classical higher Green function

$$G_s^{\Gamma_0(N)}(z_1, z_2) = -2 \sum_{\gamma \in \Gamma_0(N)} Q_{s-1} \left(1 + \frac{|z_1 - \gamma z_2|^2}{2\operatorname{Im}(z_1)\operatorname{Im}(\gamma z_2)} \right), \quad \Re(s) > 1,$$

where

$$Q_{s-1}(t) = \int_0^\infty (t + \sqrt{t^2 - 1} \cosh(u))^{-s} du$$

is the well-known Legendre function of second kind.

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$G_1^{\Gamma_0(N)}(z_1, z_2)$ is used in the calculation of the height pairing of Heegner points in Gross-Zagier formula.

The function $G_s^{\Gamma_0(N)}(z_1, z_2)$ is characterized by the following property.

- ▶ $\Gamma_0(N) \times \Gamma_0(N)$ -invariant
- ▶ smooth except for logarithmic singularity along the diagonal.

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When z_1, z_2 are CM points, $G_1(z_1, z_2) = -\log |j(z_1) - j(z_2)|^2 = \log |\beta|$ with $\beta \in \overline{\mathbb{Q}}$: Its CM values are algebraic in nature and has a beautiful 'factorization' formula: $j(z_1) - j(z_2)$ is a rational function on $X_0(1) \times X_0(1)$ and the CM point (z_1, z_2) is defined over number field .

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with

$$\gamma = \frac{(-2\alpha^2 + 1)^{50}(3\alpha^2 + \alpha - 5)^8}{(4\alpha^2 + 2\alpha - 3)^{40}(\alpha^2 - 3\alpha + 1)^{22}(-4\alpha^2 + 4\alpha + 1)^{23}} \approx -1.017 \times 10^{-10}.$$

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Problem (Zagier, ICM 1986): For distinct CM points

$z_1, z_2 \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ and $k \in \{3, 5, 7\}$, does there exist $\beta \in \bar{\mathbb{Q}} \subset \mathbb{C}$ such that

$$G_k(z_1, z_2) = \log |\beta|.$$

More generally, Let

$$G_s^{\Gamma_0(N),m}(z_1, z_2) = G_s^{\Gamma_0(N)}(z_1, z_2) \mid T_{m,z_1} = G_s^{\Gamma_0(N)}(z_1, z_2) \mid T_{m,z_2}.$$

For an **weakly holomorphic** modular form

$f = \sum_{m \gg -\infty} c(m)q^m \in M_{-2r}^{1,\infty}(\Gamma_0(N))$, consider the higher green function associated to f :

$$G_{r+1,f}^{\Gamma_0(N)}(z_1, z_2) := \sum_{m>0} c(-m)m^{r-1}G_{r+1}^{\Gamma_0(N),m}(z_1, z_2) \quad (1)$$

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$$G_{r+1,f}^{\Gamma_0(N)}(z_1, z_2) := \sum_{m \geq 0} c(-m)m^{r-1}G_{r+1}^{\Gamma_0(N),m}(z_1, z_2) \quad (1)$$

Ignoring the precise definition:

$$f \in M_{-2r}^{!,\infty}(\Gamma_0(N)) \rightarrow \text{Principal Higher Green function } G_{r+1,f}^{\Gamma_0(N)}(z_1, z_2).$$

Conjecture

(Gross-Zagier, 1986) Suppose $f \in M_{-2r}^{!,\infty}(\Gamma_0(N))$ has rational Fourier coefficients at the cusp infinity. Then for any CM point $(z_1, z_2) \in Y_0(N)^2$ with z_j having discriminant $d_j < 0$, there exists $\alpha = \alpha(z_1, z_2) \in \overline{\mathbb{Q}} \subset \mathbb{C}$ such that

$$G_{r+1,f}^{\Gamma_0(N)}(z_1, z_2) = |d_1 d_2|^{-r/2} \log |\alpha|.$$

Theorem (Bruinier-Li-Y., 2022): The above conjecture is true!

History:

- ▶ Shou-Wu Zhang (1997, conditional) for $\mathbb{Q}(z_1) = \mathbb{Q}(z_2)$. Geometric explanation of this conjecture.
- ▶ M. Viazovska (2011) for $\mathbb{Q}(z_1) = \mathbb{Q}(z_2)$. Inspiration to our work.
- ▶ A. Mellit, for $z_1 = i, \frac{1+\sqrt{-3}}{2}$.
- ▶ Gross-Zagier (1986) on average
- ▶ Bruinier-Ehlen-Y (2021): Partial average on z_1 , or $Q(z_1)$ has class number being 2-power. Reformulate and generate the conjecture to orthogonal Shimura varieties.
- ▶ Li(2021, published in 2023): $N = 1$.

Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$ with an integral lattice L . Let L' be the dual of L .

- ▶ $H = \mathrm{GSpin}(V)$
- ▶ $\mathbb{D}$ —oriented negative two planes in $V_{\mathbb{R}}$.
- ▶ $K \subset H(\mathbb{A}_f)$ compact open, preserves L and acts on L'/L trivially.
- ▶ $X = X_K$ Shimura variety over $\mathbb{Q}$ with $X(\mathbb{C}) = H(\mathbb{Q}) \backslash \mathbb{D} \times H(\mathbb{A}_f) / K$.
- ▶ Given $m \in Q_{>0}$ and $\mu \in L'/L$ with $Q(\mu) \equiv m \pmod{1}$ we have special divisor $Z(m, \mu) \in Z^1(X)$.

CM points and CM cycles:

- ▶ F —a totally real number field with real embeddings $\sigma_1, \dots, \sigma_d$.
- ▶ (W, Q_F) —a quadratic space over F with signature $(0, 2), (2, 0), \dots, (2, 0)$. $E = F(\sqrt{-\det W})$ is a CM quadratic extension of F .
- ▶ $W_{\mathbb{Q}} = W$ with Q -quadratic form $Q(x) = \text{tr}_{F/\mathbb{Q}} Q_F(x)$ and signature $(2d - 2, 2)$. Assume that $V = W_{\mathbb{Q}} \oplus V_+$.
- ▶ $W_{\sigma_1} = W \otimes_{F, \sigma_1} \mathbb{R}$ is a negative 2-plane in $V_{\mathbb{R}}$ and gives two CM points $z_{\sigma_1}^{\pm}$.
- ▶ $Z(W, \sigma_1) = \{z_{\sigma_1}^{\pm}\} \times E^{\times} \backslash \mathbb{A}_{E, f}^{\times} / K \cap A_{E, f}^{\times}$ —CM cycle in X defined over F .
- ▶ One can use the above data to define $Z(W, \sigma_i)$ for all $i \geq 2$ in a way that $Z(W) = \sum Z(W, \sigma_i)$ is defined over $\mathbb{Q}$.

When $F = \mathbb{Q}$, W is simply a subspace of V of signature $(0, 2)$, the associated CM points and CM cycle is very simple and is called small CM points/cycle.

One example: let $V = M_2(\mathbb{Q})$ with $Q(x) = N \det x$ and

$$L = \left\{ x = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}.$$

In this case, $X = Y_0(N) \times Y_0(N)$. Special divisors are especially Hecke correspondences. Special points are (z_1, z_2) where z_i are Heegner points of discriminants $d_i < 0$.

Another example is $X_0(N)$, which is associated to $V_0 = M_0(\mathbb{Q})^{\text{tr}=0} \subset V$ and $L_0 = L \cap V_0$.

reductive dual pair and theta kernel:

The reductive dual pair $(\mathrm{SL}_2, O(V))$ gives rise to a Weil representation ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ on $S_L = \oplus \mathbb{C}\phi_\mu \cong \mathbb{C}[L'/L]$ with $\phi_\mu = \mathrm{Char}(\mu + \hat{L})$.

This gives rise to the theta kernel

$$\theta_L(\tau, z, h) = \sum \theta(\tau, z, h, \phi_\mu) \phi_\mu^\vee$$

with

$$\theta(\tau, z, h, \phi_\mu) = \nu \sum_{x \in V} e(Q(x_{z^\perp})\tau + Q(x_z)\bar{\tau}) \phi_\mu(h^{-1}x).$$

Here $x = x_z + x_{z^\perp}$ with respect to the decomposition $V_{\mathbb{R}} = z \oplus z^\perp$.

- ▶ smooth modular form of $\mathrm{SL}_2(\mathbb{Z})$ or weight $n/2 - 1$, valued in $S_L^\vee$ (variable τ)
- ▶ smooth function on X (variable (z, h)).

Differential Operators and modular forms

Let $\mathcal{A}_k$ be the space of smooth modular forms of weight k .

$$R = R_k = 2i \frac{\partial}{\partial \tau} + \frac{k}{v}, \quad \mathcal{A}_k \rightarrow \mathcal{A}_{k+2}, \text{ Maass weight raising operator;}$$

$$L = L_k = -2iv^2 \frac{\partial}{\partial \bar{\tau}}, \quad \mathcal{A}_k \rightarrow \mathcal{A}_{k-2}, \text{ Maass weight lowering operator;}$$

$$\xi_k = v^{k-2} \overline{L_k} = 2iv^k \overline{\frac{\partial}{\partial \bar{\tau}}}, \quad \mathcal{A}_k \rightarrow \mathcal{A}_{2-k}, \quad \xi - \text{operator.}$$

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WHFs—Weakly Holomorphic modular forms $M_{k,\rho_L}^!$ with values in S_L .

A holomorphic function $f : \mathbb{H} \rightarrow S_L$ is a WHF of weight k valued in S_L if

$$f(\gamma\tau) = (c\tau + d)^k \rho(\gamma) f(\tau), \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

and f is meromorphic at the cusp ∞ :

$$f(\tau) = \sum_{m \gg -\infty} c_f(m) q^m, \quad c(m) \in S_L.$$

HMFs—Harmonic Maass forms H_{k,ρ_L} with values in S_L .

Key properties:

- ▶ $f = f^+ + f^-$ with the holomorphic part

$$f^+(\tau) = \sum_{m \gg -\infty} c_f^+(m) q^m, \quad c(m) \in S_L.$$

and f^- exponentially decay when $v \rightarrow \infty$.

- ▶ An exact sequence (the last one is given by ξ)

$$0 \rightarrow M_{k,\rho_L}^! \rightarrow H_{k,\rho_L} \rightarrow S_{2-k,\overline{\rho_L}} \rightarrow 0.$$

Regularized theta lifting and Higher Green functions

Given an $f \in H_{1-\frac{n}{2}-2r, \rho_L}$, the regularized theta lifting

$$\Phi^r(z, h, f) = \frac{1}{(4\pi)^r} \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}}^{reg} \langle R^r f(\tau), \theta_L(\tau, z, h) \rangle d\mu(\tau).$$

is called a **higher Green function** (for $Z^r(f) = \sum_{m>0} c_f^+(-m)m^r Z(m)$, $Z(m) = \sum Z(m, \mu) \phi_\mu^\vee$).

When $f \in M_{1-\frac{n}{2}-2r, \rho_L}$, $\Phi^r(z, h, f)$ is called a **principal higher Green function**.

A little more precisely, for a smooth function g on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$,

$$\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}}^{reg} g(\tau) d\mu(\tau) = \mathrm{CT}_{s=0} \lim_{T \rightarrow \infty} \int_{\mathcal{F}_T} g(\tau) v^s d\mu(\tau)$$

with $\mathcal{F}_T = \{\tau = x + iy : |\tau| \geq 1, \quad 0 < y < T\}$.

For $r = 0$,

1. (Borcherds, 98), When $f \in M_{1-\frac{n}{2}, \rho_L}$ with rational coefficients $c_f(m, \mu) = c_f(m)(\mu) \in \mathbb{Q}$ for $m \leq 0$, there is a meromorphic modular form $\Psi(z, h, f)$ on X of weight $\frac{1}{2}c_f(0, 0)$ with

$$\Phi^0(z, h, f) = -\log \|\Psi(z, h, f)\|_{Pet}^2$$

and $\text{Div}(\Psi) = Z(f) = \sum_{m>0} c_f(-m, \mu) Z(m, \mu)$. Moreover, at each cusp, Ψ has (Borcherds) product expansion.

2 (Bruinier 2002) When $f \in H_{1-\frac{n}{2}, \rho_L}$ with rational coefficients $c_f^+(m)$ for $m \leq 0$, $\Phi^0(Z, h, f)$ is a (automorphic) Green function for $Z(f)$. We have $(Z(f), \Phi^0(f)) \in \text{CH}^1(X)$.

3. Using it, Bruinier-Howard-Y (2015) proved a high dimensional analogue of the gross-Zagier formula for $U(n, 1)$ Shimura varieties.

In the case $V = M_2(\mathbb{Q})$ mentioned above,
 $f \in M_{-2r}^{!,\infty}(\Gamma_0(N)) \rightsquigarrow \vec{f} \in M_{-2r,\rho_L}^!$, and

$$\Phi_{r+1,f}^{\Gamma_0(N)}(z_1, z_2) = -\Phi^r(z_1, z_2, 1, \vec{f}).$$

(S-W. Zhang, 97): $\Phi_{r+1}^{\Gamma_0(N)}$ is related to some Green current of CM cycles on Kuga varieties over $X_0(N)$.

Conjecture

(Yingkun Li, 2023) Let notation be as above. Assume that $f \in M_{1-\frac{n}{2}-2r, \rho_L}^!$ with rational coefficients $c_f(m)$ for $m \leq 0$. Then there are $\lambda_j \in F$ and $\alpha_j \in E^{ab} \subset \mathbb{C}$ such that

$$\Phi^r(z_{\sigma_1}^{\pm}, h, f) \in \sum_{j=1}^d \lambda_j \log |\sigma_h(\alpha_j)|$$

for all $(z_{\sigma_1}^{\pm}, h) \in Z(W, \sigma_1)$. Here E^{ab} is the maximal abelian extension of E and $\sigma_h \in \text{Gal}(E^{an}/E)$ is given via Artin map.

Furthermore, for $n = d = 2$, $\lambda_j \in \mathbb{Q}(d_1 d_2)^{\frac{r}{2}}$ or $= 0$ and depending on whether $j \equiv r \pmod{2}$ or not.

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When $r = 0$ and $c_f(0, 0) = 0$, this is true by Borchers' work .

Theorem 1

(Bruinier-Ehlen-Y, 2021) Li's conjecture above is true for $d = 1$ (i.e., for small CM points).

This is a generalization of S-W Zhang's and Viazovska's work.

Theorem 2

(Bruinier-Li-Y, 2022) Li's conjecture above is true when E is biquadratic (a special case of $d = 2$). That is, there are Galois invariant maps $\alpha_1, \alpha_2 : \mathbb{A}_{E,f}^\times \rightarrow E^{ab}$ such that

$$\Phi(z, h, f) = \frac{1}{\kappa} (\log |\sigma_h(\alpha_1)| + \sqrt{d_F} \log |\sigma_h(\alpha_2)|)$$

Here z is a CM point in $\mathbb{D}$ associated to W . Moreover, when $n = 2$, $\alpha_j(h) = 1$ for $j \not\equiv r \pmod{2}$.

This proves Gross and Zagier's algebraicity conjecture if we take $V = M_2(\mathbb{Q})$ and $X = Y_0(N) \times Y_0(N)$.

Main Idea for the proof of Theorem 1: $V = W \oplus V_+$ and

$$\theta(z_W, 1) = \theta_{V^+}(\tau)\theta_W(\tau)$$

with θ_{V^+} holomorphic of weight $n/2$ and $\theta_W(\tau)$ anti-holomorphic of weight -1 .

Key Ingredient 1: (S. Ehlen, Yingkun Li, M. Viazovska independently)
Construct a Harmonic Maass form $g_W(\tau)$ of weight 1 such that the Fourier coefficients of $g_W^+(\tau)$ is algebraic in nature and well-behaved under ‘Galois’ action.

Key Ingredient 2: (Rankin-Cohen bracket operator)

$$R^r(\theta_{V^+}\xi(g_W)) = L([\theta_{V^+}, g_W]_r).$$

Key Ingredient 3: Stokes’ theorem.

In formula

$$\begin{aligned}
 \Phi^r(z_w, 1, f) &= \int^{reg} f \cdot R^r(\theta_{V^+} \xi(g_W) d\mu(\tau) \\
 &= \int^{reg} f \cdot L([\theta_{V^+}, g_W]_r) d\mu(\tau) \\
 &= \int^{reg} \bar{\partial}(f \cdot [\theta_{V^+}, g_W]_r) d\tau \\
 &= CT[f\theta_{V^+}g_W^+]
 \end{aligned}$$

as claimed.

To extend the above ideas to general CM points, we need

1. Construct a 'nice' harmonic Maass Hilbert form g_W over F of weight $(1, 1, \dots, 1)$ such that for all $1 \leq i \leq d$

$$\xi_{\tau_i} g_W(\tau_1, \dots, \tau_d) = \theta_{W, \sigma_i}(\tau_1, \dots, \tau_d).$$

This can be only done on average via Eisenstein series.

2. Extend the above formula involving R^r and Rankin-Cohen bracket operator to totally real number fields. This was done for real quadratic field.

Together, we obtain

Theorem 3

(Bruinier-Ehlen-Y, 2021) Let F be a real quadratic field, then

$$\Phi^r(Z(W, \sigma_1), f) + (-1)^r \Phi^r(Z(W, \sigma_2), f) = \frac{|d_F|^{\frac{r}{2}}}{\kappa_1} (\log |\alpha|)$$

for some $\alpha_i \in \mathbb{Q}^\times$ (can be written explicitly) and κ_1 being positive integer.

Then Yingkun Li found a creative way to prove the following theorem

Theorem 4

(Li, 2023) For any totally real number field F , and any W as above, his conjecture holds for the difference

$$\Phi^r(z, h_1, f) - \Phi^r(z, h_2, f)$$

for any $h_1, h_2 \in \mathrm{SO}(W)(\mathbb{A}_f)$, and the same CM point $z = z_{\sigma_j}^{\pm}$.

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Theorem 2 (our main result) follows from the following theorem

Theorem 5

(Bruinier-Li-Y. 2022) Assume the CM number field associated to W is a biquadratic quartic CM number field. Then

$$\Phi^r(Z(W, \sigma_1), f) - (-1)^r \Phi^r(Z(W, \sigma_2), f) = \frac{|d_F|^{\frac{r}{2}}}{\kappa_2} (\log \left| \frac{\sigma_1(\beta)}{\sigma_2(\beta)} \right|)$$

for some $\beta \in F^{\times}$

Rough Idea of Proof (for the case $r = 0$): A key is to find some Hilbert modular form (non-holomorphic) $\tilde{\mathcal{I}}(z_1, z_2)$ of weight $(1, 1)$ such that

$$(L_{z_1} + L_{z_2})\tilde{\mathcal{I}} = E(z_1, z_2, 0, wt(-1, 1)) - E(z_1, z_2, 0, wt(1, -1))$$

and understand the Fourier coefficients of $\tilde{\mathcal{I}}$.

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It is known

$$(L_{z_1} + L_{z_2})E'(z_1, z_2, 0, wt(1, 1)) = E(z_1, z_2, wt(-1, 1)) + E(z_1, z_2, wt(1, -1)),$$

which is used in the proof of Theorem 3.

The basic idea to prove Theorem 5 is similar to that of Theorem 3, but to find the analogue of the derivative of incoherent Eisenstein series' is much more subtle. In the end, we only find approximation which is good enough for us. In the process, we used

- ▶ usual theta liftings and seesaw principal
- ▶ Siegel-Weil formula on split $O(3,3)$ (instead of symplectic groups), here is one of the main reasons that we need E to be biquadratic.
- ▶ period integral
- ▶ subtle deformation of some integral

Natural question:

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Thanks!