

Counting rational and integral points on del Pezzo surfaces

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Conference on Arithmetic Geometry, Group Actions and Rationality Problems
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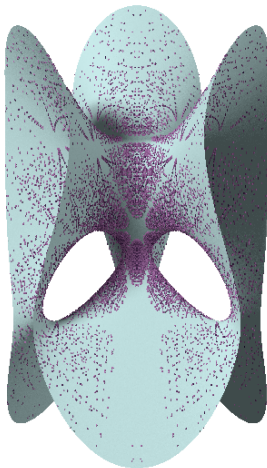
Rational points on del Pezzo surfaces

Clebsch's smooth cubic surface with 27 lines:



$$X = \{x_0^3 + \cdots + x_3^3 = (x_0 + \cdots + x_3)^3\} \subset \mathbb{P}^3$$

Clebsch's smooth cubic surface: rational points of bounded height outside the 27 lines.
(Challenge: prove Manin's conjecture for X .)



$$X(\mathbb{Q}) = \{(x_0 : \cdots : x_3) \mid x_0^3 + \cdots + x_3^3 = (x_0 + \cdots + x_3)^3\} \subset \mathbb{P}^3(\mathbb{Q})$$

Original Manin Conjecture (Franke–Manin–Tschinkel '89)

“Prompted by [Schanuel’s theorem for $\mathbb{P}_K^n$] and some numerical data about the distribution of rational points on $x_0^3 + 2x_1^3 + 3x_2^3 + 4x_3^3 = 0$ Yu. I. Manin conjectured that $N(B) \sim cB(\log B)^t$ [w.r.t. anticanonical height] should be typical for Fano manifolds X with dense $X(K)$, excluding some degenerate cases. [...] The available information is compatible with the further conjecture that $t = \text{rk}(\text{Pic } X) - 1$.”

Modern Manin Conjecture

Let X be a Fano variety over a number field K , with $X(K)$ Zariski dense.

Let H be an anticanonical height function.

Then there is a thin subset $Z \subset X(K)$ such that

$$N(B) := \#\{x \in X(K) \setminus Z \mid H(x) \leq B\} \sim c_{\text{Peyre}} B(\log B)^{\text{rk}(\text{Pic } X) - 1}.$$

- **Franke–Manin–Tschinkel '89:** flag varieties; numerics for a cubic surface
- **Tschinkel '92** (“*Arithmetic of algebraic surfaces*”, *PhD thesis, MIT*): ?
- **Manin–Tschinkel '93:** e.g.: $N(B) \ll B(\log B)^6$ for split smooth dP5 over $\mathbb{Q}$.
- **Salberger '93:** $N(B) \ll B(\log B)^4$ for split smooth dP5 over $\mathbb{Q}$.
- **Batyrev–Tschinkel '95–'98:** Manin’s conjecture for toric varieties over number fields (including smooth del Pezzo surfaces of degree ≥ 6).
- **Peyre '95:** Conjectural leading constant.
- **Batyrev–Tschinkel '96:** Counterexample to “classical” Manin conjecture (for a family of cubic surfaces).
- **Peyre–Tschinkel '01:** Numerical results for diagonal cubic surfaces.
- **Chambert-Loir–Tschinkel '02:** Manin’s conjecture for eq. compactifications of vector groups over number fields (incl. some singular dPs).
- **de la Bretèche '02:** Manin’s conjecture for a split smooth dP5 over $\mathbb{Q}$.
- **Heath-Brown '03:** $N(B) \ll B(\log B)^6$ for Cayley’s singular cubic surface.
- **Browning–de la Bretèche '07:** Manin’s conjecture for some singular dP4 over $\mathbb{Q}$.

Theorem (D., PhD thesis, '06)

Manin's conjecture holds for the split cubic surface with E_6 -singularity

$$X = \{x_0x_1^2 + x_1x_2^2 + x_3^3 = 0\} \subset \mathbb{P}^3 :$$

We have

$$N(B) \sim cB(\log B)^6.$$

With a power-saving error term: de la Bretèche–Browning–D. '07.

Theorem (D.–Tschinkel, '07)

Manin's conjecture holds for the split quartic del Pezzo surface with D_4 -singularity

$$X = \{x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0\} \subset \mathbb{P}^4 :$$

We have

$$N(B) \sim cB(\log B)^5.$$

Many other singular del Pezzo surfaces: de la Bretèche, Browning, Le Boudec, ...

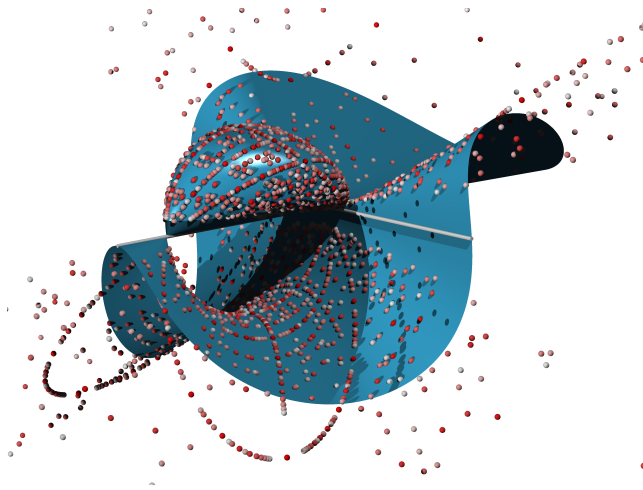
Cubic surface with 2 singularities ($A_1 + A_5$) and 2 lines:



$$X = \{x_0x_1x_2 + x_1^3 + x_2x_3^2 = 0\} \subset \mathbb{P}^3$$

Cubic surface X with 2 singularities ($A_1 + A_5$) and 2 lines.

Theorem (Baier-D.): *Manin's conjecture holds for X .*



$$X(\mathbb{Q}) = \{(x_0 : \cdots : x_3) \mid x_0 x_1 x_2 + x_1^3 + x_2 x_3^2 = 0\} \subset \mathbb{P}^3(\mathbb{Q})$$

Split smooth dP5 X : blow-up of $\mathbb{P}^2$ in $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$
 $N(B) := \#\{x \in U(K) \mid H(x) \leq B\}$, where $U := X \setminus \{\text{lines}\}$.

- *Manin–Tschinkel '93*: $N(B) \ll B(\log B)^6$ over $\mathbb{Q}$
- *Salberger '93*: $N(B) \ll B(\log B)^4$ over $\mathbb{Q}$
- *de la Bretèche '02*: $N(B) \sim cB(\log B)^4$ over $\mathbb{Q}$
- *Browning '22*: $N(B) \sim cB(\log B)^4$ over $\mathbb{Q}$
- *Heath-Brown–Loughran '24*: $N(B) \sim cB(\log B)^{t-1}$ over $\mathbb{Q}$ (generic nonsplit)

Question:

What about other number fields?

Arithmetic of Algebraic Surfaces

by

Yuri Tschinkel

Submitted to the
Department of Mathematics
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

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Abstract

Arithmetic of Algebraic Surfaces

YURI TSCHINKEL

Submitted to the Department of Mathematics
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We study the asymptotic behavior of the number of rational points of bounded height on Del Pezzo, K3 and Enriques surfaces.

We prove Manin's linear growth conjecture for Del Pezzo surfaces of degree 5 over arbitrary number fields k and establish the accumulating property for Del Pezzo surfaces of degree 4 and 3. Combinatorics of the graph of exceptional curves and the use of finite heights are the main ingredients of the proofs.

Thesis Advisor: Michael Artin
Title: Professor of Mathematics, M.I.T.

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Theorem (Tschinkel '92)

$N(B) \ll B^{1+\epsilon}$ for split smooth dP5 over any number field K .

Theorem (Glas–Hochfilzer '24)

$N(B) \ll B^{1+\epsilon}$ for smooth dP5 with conic bundle structure over number field K .

Theorem (Bernert-D. '24)

For split smooth dP5 over any number field K :

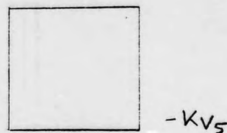
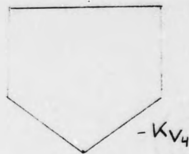
$$N(B) \sim \frac{1}{144} \left(\frac{2^{r_1} (2\pi)^{r_2} R_K h_K}{|\mu_K| \cdot |\Delta_K|^{1/2}} \right)^5 \frac{1}{|\Delta_K|} \prod_v \omega_v \cdot B(\log B)^4,$$

$$\text{with } \omega_v = \begin{cases} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^5 \left(1 + \frac{5}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right), & v = \mathfrak{p} \text{ prime,} \\ \frac{3}{2} \text{vol}\{y \in \mathbb{R}^3 : \max |y_i y_j (y_i - y_k)|_v \leq 1\}, & v \text{ real,} \\ \frac{12}{\pi} \text{vol}\{y \in \mathbb{C}^3 : \max |y_i y_j (y_i - y_k)|_v \leq 1\}, & v \text{ complex.} \end{cases}$$

Step 1: pass to universal torsor: 5 Plücker equations of $G(2, 5)$ (Skorobogatov '93):

$$\begin{aligned} a_4 a_{14} - a_3 a_{13} + a_2 a_{12} &= 0, & a_4 a_{24} - a_3 a_{23} + a_1 a_{12} &= 0, \\ a_4 a_{34} - a_2 a_{23} + a_1 a_{13} &= 0, & a_3 a_{34} - a_2 a_{24} + a_1 a_{14} &= 0, \\ a_{12} a_{34} - a_{13} a_{24} + a_{23} a_{14} &= 0, & \text{with lifted height } \prod_{v|\infty} |a_{ij} a_j a_{jk} a_k a_{kl}|_v &\leq B. \end{aligned}$$

LEMMA 7. On a Del Pezzo surface V_a of degree $9 - a$ ($a = 4, 5$) the anticanonical bundle has a basis of sections such that the zero divisors of the sections coincide with the following subgraphs of E_4 (resp. E_5):



PROOF:

We present a basis of section $\{s_i\}$ for V_4 and V_5 . Only these cases will be used later.

The anticanonical line bundle $-K_V$ on V_4 has section with zero divisors of the form:

$$(s_0)_0 = D_0 + D_{01} + D_1 + D_{13} + D_{02}$$

$$(s_1)_0 = D_{01} + D_1 + D_{13} + D_3 + D_{23}$$

Step 1: pass to universal torsor: 5 Plücker equations of $G(2, 5)$ (Skorobogatov '93):

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Step 2: $a_1, \dots, a_4, a_{12}, a_{23}, a_{34}$ determine a_{13}, a_{24}, a_{14} (under just two congruences!):

Ideally $|a_{ij}| \leq B_{ij} := \frac{|a_k a_l|^{1/3}}{|a_i a_j|^{2/3}} \cdot B^{1/3}$, hence (roughly)

$$\sum_{|a_1|, \dots, |a_4| \ll B} \sum_{|a_{12}| \leq B_{12}} \sum_{\substack{|a_{23}| \leq B_{23} \\ a_{23} \bmod a_4}} \sum_{\substack{|a_{34}| \leq B_{34} \\ a_{34} \bmod a_1}} 1 \asymp \sum_{|a_1|, \dots, |a_4| \ll B} \frac{B}{|a_1 \cdots a_4|} \asymp B(\log B)^4.$$

Actually maybe $|a_{12}| \leq B_{12}$, $|a_{23}| \leq B_{23}$, but $|a_{34}| \leq B_{34} \frac{B_{23}^{1/2}}{|a_{23}|^{1/2}} = \frac{|a_1|^{1/2}}{|a_3^2 a_4 a_{23}|^{1/2}} B^{1/2}$:
still ok with good order of summation. — **However**, over number field K :
any order of summation might be bad at some archimedean place.

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 still ok with good order of summation. — **However**, over number field K :
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Key step in our proof, allowing generalization to arbitrary number fields:

Proposition (D.–Bernert '24)

The number of points violating $|a_{ij}| \leq B_{ij} \cdot (\log \log B)^{1/4}$ is negligibly small.

Integral points on del Pezzo surfaces

Integral points in the affine plane $\mathbb{A}^2$



Height ≤ 2

Integral points in the affine plane

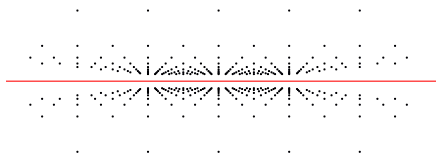
- $\mathbb{P}^2(\mathbb{Z}) = \mathbb{P}^2(\mathbb{Q})$ by clearing denominators.
Similarly for any projective variety.
- $\mathbb{A}^2(\mathbb{Z}) = \mathbb{Z}^2$. “Reason”: $\mathbb{A}^2 = \mathbb{P}^2 \setminus \{x_0 = 0\}$
 $\leadsto (x_0 : x_1 : x_2)$ is integral iff $x_0 \neq 0$ also modulo all primes p .
 $\leadsto$ precisely all $x = (1 : x_1 : x_2)$ are integral points.
- With height $H(x) = \max\{1, |x_1|, |x_2|\}$:
 $N_{\mathbb{A}^2, H}(B) \sim 4B^2$

Integral points on $\mathbb{A}^2 = \mathbb{P}^2 \setminus V(x_0)$ in the (x_0, x_1) -chart:



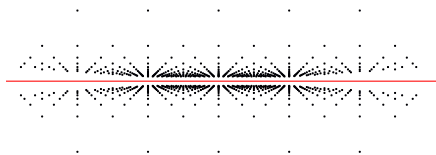
Height ≤ 5

Integral points on $\mathbb{A}^2 = \mathbb{P}^2 \setminus V(x_0)$ in the (x_0, x_1) -chart:



$$\text{Height} \leq 10$$

Integral points on $\mathbb{A}^2 = \mathbb{P}^2 \setminus V(x_0)$ in the (x_0, x_1) -chart:



$$\text{Height} \leq 15$$

Conjectural framework for integral points of bounded height (Chambert-Loir–Tschinkel)

The number of integral points of bounded log-anticanonical height on a suitable subset of a log Fano variety should grow asymptotically as $cB(\log B)^{t-1}$.

Analogue of Manin's conjecture for **integral points** is known for

- complete intersections of high dimension:
circle method (*Birch '62, Chambert-Loir–Tschinkel '10*)
- partial equivariant compactifications of vector groups/semi-simple groups:
harmonic analysis on adelic points (*CL–T, Takloo-Bighash–Tschinkel '13*)

What about integral points on del Pezzo surfaces?

My favorite singular dP4

- $X = \{x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0\} \subset \mathbb{P}^4$
- 3 lines: $L_1 = \{x_0 = x_2 = x_3 = 0\}$, $L_2 = \dots$, $L_3 = \dots$
- 2 singularities: $Q_1 = (0 : 1 : 0 : 0 : 0)$ (type A_1), $Q_2 = (0 : 0 : 0 : 0 : 1)$ (type A_3)

Integral points on our quartic del Pezzo surface

- $\mathcal{X} = \{x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0\} \subset \mathbb{P}_{\mathbb{Z}}^4$ integral model.
 $\rightsquigarrow \mathcal{X}(\mathbb{Z}) = X(\mathbb{Q})$.
- Choose **boundary** $Z = \{L_1\} = \{x_0 = x_2 = x_3 = 0\}$, for example.
 $\rightsquigarrow U = X \setminus Z$ with integral model $\mathcal{U} = \mathcal{X} \setminus \overline{Z}$.
 $\rightsquigarrow \mathcal{U}(\mathbb{Z}) = \{(x_0 : \cdots : x_4) \in X(\mathbb{Q}) \mid \gcd(x_0, x_2, x_3) = 1\}$

Counting integral points of bounded height

For $x \in \mathcal{U}(\mathbb{Z})$: height $H(x) = \max\{|x_0|, |x_2|, |x_3|\}$.

Counting function $N(B) := \#\{x \in \mathcal{U}(\mathbb{Z}) \cap V(\mathbb{Q}) \mid H(x) \leq B\}$.

Theorem (D.–Wilsch '24, over $\mathbb{Q}$ with boundary $Z = \{L_1\}$)

$N(B) \sim 2 \prod_p (1 - \frac{1}{p^2}) B(\log B)^2$, and similarly for $Z \in \{\{Q_1\}, \{Q_2\}, \{Q_1, Q_2\}, L_2, L_3\}$.

Theorem (Ortmann '24, over any number field K with boundary $Z = \{L_1\}$)

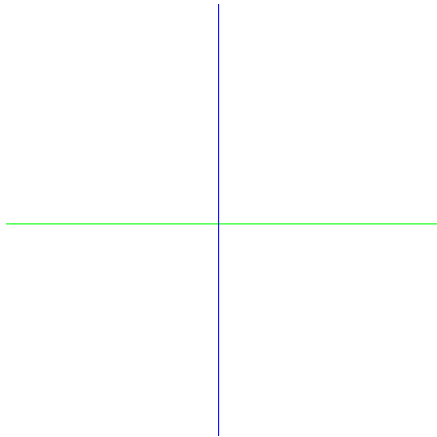
$N(B) \sim \frac{1}{q!^2 \cdot (q^2 + 3q + 2)} 2^{2r_1} (2\pi)^{2r_2} \frac{\rho_K}{|\Delta_K|} \prod_p (1 - \frac{1}{\mathfrak{N}p^2}) B(\log B)^{2q+2}$, with $q = \text{rk}(\mathcal{O}_K^\times)$.

Interpretation: $\pi: \tilde{X} \rightarrow X$ minimal desingularization.

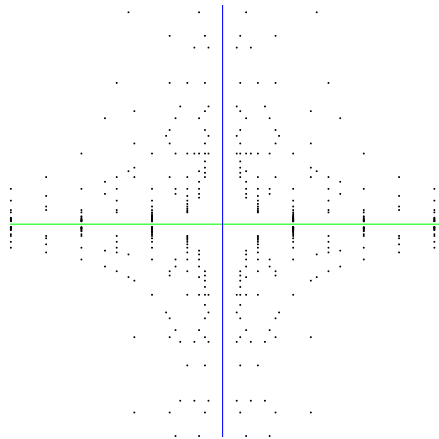
$D := \pi^{-1}(L_1)$ above the line L_1 through singularities Q_1, Q_2 has five components:



Integral points on $\tilde{X} \setminus D$: local chart around intersection of two components of D

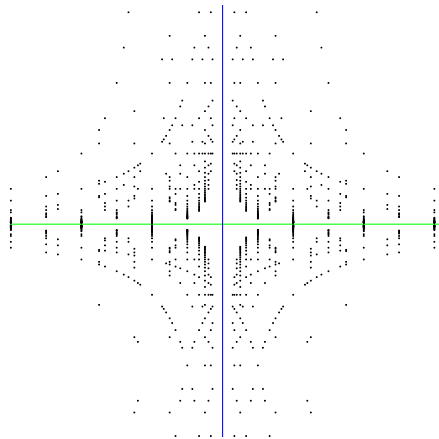


Integral points on $\tilde{X} \setminus D$



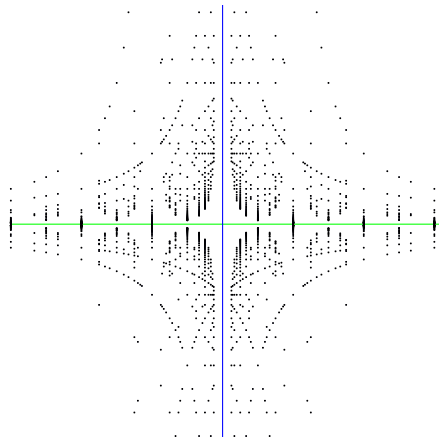
Height ≤ 25

Integral points on $\tilde{X} \setminus D$



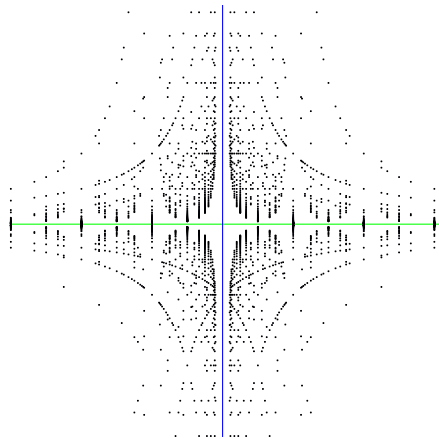
Height ≤ 50

Integral points on $\tilde{X} \setminus D$



Height ≤ 75

Integral points on $\tilde{X} \setminus D$



Height ≤ 100

Proof: *Analytic number theory on universal torsor*

Prototype: smooth hypersurface $X \subset \mathbb{P}_{\mathbb{Q}}^n$

Defined by $F(x_0, \dots, x_n) = 0$ of degree d

$$N_{U,H}(B) = \frac{1}{2} \# \left\{ (x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \left| \begin{array}{l} (x_0 : \dots : x_n) \in U \\ F(x_0, \dots, x_n) = 0 \\ \gcd(x_0, \dots, x_n) = 1 \\ \max\{|x_i|^{n+1-d}\} \leq B \end{array} \right. \right\}$$

Cox ring of $\tilde{X}$

$$\text{Cox } \tilde{X} = \bigoplus_{\text{Pic } X} H^0(X, \mathcal{O}(D)) \cong \mathbb{Q}[a_1, \dots, a_9] / (a_1 a_9 + a_2 a_8 + a_4 a_5^3 a_6^2 a_7)$$

where $a_1, \dots, a_9$ correspond to divisors $E_1, \dots, E_9$
 (for example: $D = E_3 + \dots + E_7$ above L_1).

Parameterization of rational points on V :

$$N_{V,H}(B) = \frac{1}{2^6} \# \left\{ (a_1, \dots, a_9) \in \mathbb{Z}^9 \left| \begin{array}{l} a_1 a_9 + a_2 a_8 + a_4 a_5^3 a_6^2 a_7 = 0, \\ |a_2 a_3 a_4 a_5 a_6 a_7 a_8|, |a_1^2 a_2^2 a_3^3 a_4^2 a_6|, \\ |a_1 a_2 a_3^2 a_4^2 a_5^2 a_6^2 a_7|, |a_3 a_4^2 a_5^4 a_6^3 a_7^2|, \\ |a_7 a_8 a_9| \leq B \\ \text{coprimality conditions} \\ a_1 \cdots a_7 \neq 0 \end{array} \right. \right\}$$

Integral points

Integral points on $X \setminus L_1$ or $\tilde{X} \setminus (E_3 + \cdots + E_7)$ have
Cox coordinates $a_3 = \cdots = a_7 = \pm 1$.

$\rightsquigarrow$ Parameterization of integral points on $\mathcal{U}$:

$$N_{\mathcal{U},H}(B) = \frac{1}{2^{6-5}} \# \left\{ (a_1, a_2, a_8, a_9) \in \mathbb{Z}^4 \left| \begin{array}{l} a_1 a_9 + a_2 a_8 + 1 = 0, \\ |a_2 a_8|, |a_1 a_2| \leq B \\ \text{coprimality conditions} \\ a_1 a_2 \neq 0 \end{array} \right. \right\}$$

Normally, integral points accumulate near the boundary divisor.

If several components of the boundary intersect, “most” points should be near this intersection.

Encoded in:

The Clemens complex $\mathcal{C}$

Dual graph of the strata of the irreducible components of D .

Example: $(\tilde{X}, D)$ as before

D



$\mathcal{C}$



Manin–Peyre conjecture for rational points

$$N_{V,H}(B) \sim \alpha \cdot \beta \cdot \omega_\infty \cdot \prod_p \omega_p \cdot B(\log B)^{\rho-1}$$

where $\rho = \text{rank Pic } \tilde{X} = 6$.

Heuristics for integral points (Chambert-Loir–Tschinkel)

$$N_{U,H}(B) \sim \sum_{A \subset \mathcal{C} \text{ maximal}} \alpha_A \cdot \beta \cdot \omega_{\infty,A} \cdot \prod_p \omega_p \cdot B(\log B)^{b-1}$$

where $b = \text{rank Pic } U - \text{rank } \mathbb{Q}[U]^\times / \mathbb{Q}^\times + (\dim \mathcal{C} + 1) = (6 - 5) - 0 + (1 + 1) = 3$.

A toric counterexample (Wilsch's thesis)

There is a toric variety with $\alpha_A = 0$ for all maximal $A \subset \mathcal{C}$.

$\rightsquigarrow$ Ignore these “obstructed” A , use maximal remaining $A \subset \mathcal{C}$.



Göttingen, 2006