

Differential Calculus in Holomorphic Dynamics

(after Adam Epstein)

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The setting

- $\mathbb{S} := \mathbb{CP}^1$.
- $D \geq 2$ is an integer.
- Rat_D is the space of rational maps $f : \mathbb{S} \rightarrow \mathbb{S}$ of degree D .
- Rat_D identifies with a (Zariski) open subset of $\mathbb{CP}^{2d+1}$.
- $f \in \text{Rat}_D$.

Goal

Study the differentials at f of several dynamically relevant holomorphic maps defined in a neighborhood of f in Rat_D .

Rough statements

Non-dynamical statement.

Example

As we perturb f within Rat_D , the multiple critical points of f split generically.

Dynamical statements.

Example

As we perturb f within Rat_D , the multiple cycles of f split generically.

Example

As we perturb f , preserving all critical orbit relations, the multipliers of attracting cycles vary independently.

The tangent space $T_f \text{Rat}_D$

- $(\mathbb{C}, 0) \ni t \mapsto f_t \in (\text{Rat}_D, f)$ is a holomorphic curve.
- $\dot{f} := \left. \frac{df_t}{dt} \right|_{t=0} \in T_f \text{Rat}_D$ satisfies

$$\forall z \in \mathbb{S}, \quad \dot{f}(z) \in T_{f(z)} \mathbb{S}.$$

- $\mathcal{T}_f$ is the vector space of meromorphic vector fields τ on $\mathbb{S}$ such that

$$\forall z \in \mathbb{S}, \quad \text{ord}_z(\tau) \geq 1 - \deg_z(f).$$

Proposition

The linear map $\mathcal{T}_f \ni \tau \mapsto Df \circ \tau \in T_f \text{Rat}_D$ is an isomorphism.

Example

For $f_t(z) = z^2 + c + t$, the vector field $\tau = \frac{1}{2z} \frac{d}{dz} \in \mathcal{T}_f$ has a simple pole at $0 \in \mathcal{C}_f$ and a triple zero at $\infty \in \mathcal{C}_f$.

Proposition (Epstein-De Marco)

For all $n \geq 1$, the map $I_n : \text{Rat}_D \ni g \mapsto g^{\circ n} \in \text{Rat}_{D^n}$ is a **proper immersion**.

Lemma

The differential $D_f I_n : T_f \text{Rat}_D \rightarrow T_{f \circ n} \text{Rat}_{D^n}$ identifies to

$$\mathcal{T}_f \ni \tau \mapsto \sum_{k=0}^{n-1} (f^{\circ k})^* \tau \in \mathcal{T}_{f \circ n}.$$

*This linear map is **injective**.*

Proof.

- Let τ be in the kernel.
- $f^* \tau = \tau$.
- τ vanishes on the repelling cycles of f , thus $\tau = 0$.

Möbius conjugacy classes

- Θ is the space of globally holomorphic vector fields on $\mathbb{S}$.
- $\dim(\Theta) = 3$.
- The map $\Theta \ni \theta \mapsto \theta - f^*\theta \in \mathcal{T}_f$ is injective.
- $\mathcal{O}_f \subset \mathcal{T}_f$ is its image.

Proposition

The Möbius conjugacy class of f in Rat_D is a smooth submanifold of dimension 3.

Its tangent space at f identifies to $\mathcal{O}_f$.

Proof.

- Assume $\mu_t \circ f_t = f \circ \mu_t$, for some family of Möbius transformations $t \mapsto \mu_t$ with $\mu = \text{id}$.
- $\theta := \dot{\mu} \in \Theta$.
- $\dot{\mu} \circ f + \dot{f} = Df \circ \dot{\mu}$, i.e., $f^*\dot{\mu} + \tau = \dot{\mu}$.

- $\mathbf{x}$ is a cycle of f of period $p \geq 1$.
- $\lambda \in \mathbb{C}$ is the multiplier of the cycle.
- if $\lambda \neq 1$, the cycle may be followed holomorphically in a neighborhood of f in Rat_D .
- $\Lambda_{\mathbf{x}} : (\text{Rat}_D, f) \rightarrow (\mathbb{C}, \lambda)$ is the corresponding multiplier germ.
- The form $d_f \Lambda_{\mathbf{x}} : T_f \text{Rat}_D \rightarrow \mathbb{C}$ vanishes on the tangent space to the Möbius conjugacy class of f in Rat_D .

Theorem (Epstein)

If $\mathbf{X}$ is a set of cycles with multipliers in $\overline{\mathbb{D}} \setminus \{0, 1\}$, the forms $(d_f \Lambda_{\mathbf{x}} : T_f \text{Rat}_D \rightarrow \mathbb{C})_{\mathbf{x} \in \mathbf{X}}$ are linearly independent.

Quadratic differentials

- A quadratic differential on $\mathbb{S}$ is a field of quadratic forms.
- The product of a quadratic differential $\mathbf{q} = qdz^2$ and a vector field $\tau = \tau \frac{d}{dz}$ is naturally the 1-form

$$\mathbf{q} \otimes \tau = q\tau dz.$$

- If $X \subset \mathbb{S}$ is a finite set and if $\mathbf{q}$ and τ are meromorphic near X , we define the pairing

$$\langle \mathbf{q}, \tau \rangle_X := \sum_{x \in X} \text{residue}(\mathbf{q} \otimes \tau, x).$$

- If τ is holomorphic near X , the residue only depends on the polar part $\langle \mathbf{q} \rangle_X$ of $\mathbf{q}$ along X .

The Thurston pushforward operator

- If $f : U \rightarrow V$ is a covering map of degree D and $\mathbf{q}$ is a quadratic differential on U .
- $f_*\mathbf{q}$ is the quadratic differential defined by

$$f_*\mathbf{q} := \sum_g g^*\mathbf{q}$$

where g ranges over the inverse branches of f .

- $\|\mathbf{q}\|_U = \int_U |\mathbf{q}|$.
- $\|f_*\mathbf{q}\|_V \leq \|\mathbf{q}\|_U$.
- If $\|\mathbf{q}\|_U < \infty$, then $\|f_*\mathbf{q}\|_V = \|\mathbf{q}\|_U$ if and only if $\mathbf{q} = \frac{1}{D}f^*f_*\mathbf{q}$.

The differential $d_f\Lambda_x$

Proposition

Assume x is a cycle of f with multiplier $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

- ① There exists a unique polar part of quadratic differential q_x along the cycle x , such that
 - $f^*q - q$ is holomorphic near x and
 - for any point x of the cycle and any local coordinate z vanishing at x , $q_x = \lambda \frac{dz^2}{z^2} + \mathcal{O}\left(\frac{dz^2}{z}\right)$.
- ② The differential $d_f\Lambda_x$ identifies with the form

$$\mathcal{T}_f \ni \tau \mapsto \langle q_x, \tau \rangle_x \in \mathbb{C}.$$

Example

If $f(z) = \lambda z + \mathcal{O}(z^3)$, then $q_{\{0\}} = \left[\lambda \frac{dz^2}{z^2} \right]_0$.

The spaces $\mathcal{E}_{f,A,B}$, $\Theta_{f,A,B}$ and $\mathcal{T}_{f,A,B}$

- If $C \subset \mathbb{S}$ is a finite set, Θ_C is the space of vector fields on C .
- $A \subset \mathbb{S}$ and $B \subset \mathbb{S}$ are finite sets such that
 - $\mathcal{V}_f \subset B$,
 - $A \subseteq B \cap f^{-1}(B)$ and
 - $\text{card}(A) \geq 3$.

Example

A is a union of repelling cycles of f and $B = A \cup \mathcal{V}_f$.

- $\mathcal{E}_{f,A,B} \subset \Theta_B \times \mathcal{T}_f$ is the set of (θ, τ) such that
 - $f^*\theta - \tau$ is holomorphic near $f^{-1}(B)$.
 - $f^*\theta - \tau = \theta$ on A .
- $\Theta_{f,A,B}$ is the image of $\mathcal{E}_{f,A,B}$ in Θ_B .
- $\mathcal{T}_{f,A,B}$ is the image of $\mathcal{E}_{f,A,B}$ in $\mathcal{T}_f$.

The dimension of $\mathcal{E}_{f,A,B}$

Lemma

The projection $\mathcal{E}_{f,A,B} \rightarrow \Theta_{f,A,B}$ is an isomorphism.

- If A is a union of repelling cycles of f and if $B = A \cup \mathcal{V}_f$, the projection $\mathcal{E}_{f,A,B} \rightarrow \mathcal{T}_{f,A,B}$ is an isomorphism.

Proposition

$$\dim(\mathcal{E}_{f,A,B}) = \text{card}(B \setminus A) + 3.$$

Proof.

- If $\theta \in \Theta_B$, there exists a $\tau \in \mathcal{T}_f$ such that $f^*\theta - \tau$ is holomorphic near $f^{-1}(B)$.
- The class of $\theta - f^*\theta + \tau$ in Θ_A/Θ only depends on the class of θ in Θ_B/Θ .
- $\iota_{f,A,B} : \Theta_B/\Theta \rightarrow \Theta_A/\Theta$ is the corresponding map.
- $\Theta_{f,A,B}/\Theta = \text{Ker}(\iota_{f,A,B})$.

Injectivity of ∇_f

- $\mathcal{Q}_C$ is the space of meromorphic quadratic differentials on $\mathbb{S}$ whose poles are simple and contained in C .
- $\mathcal{Q}_C \simeq (\Theta_C/\Theta)^*$ has dimension $\text{card}(C) - 3$.
- The transpose of $\iota_f : \Theta_B/\Theta \rightarrow \Theta_A/\Theta$ is

$$\nabla_f := \text{id} - f_* : \mathcal{Q}_A \rightarrow \mathcal{Q}_B.$$

- $\nabla_f(\mathcal{Q}_A) \simeq (\Theta_{f,A,B}/\Theta)^0$.

Theorem (Thurston)

If f is not a Lattès map, then $\nabla_f : \mathcal{Q}_A \rightarrow \mathcal{Q}_B$ is injective.

Proof.

- Relies on $\|f_*q\|_{\mathbb{S}} \leq \|q\|_{\mathbb{S}}$ and case of equality.
- Not algebraic.

Corollary

$$\dim(\Theta_{f,A,B}/\Theta) = \text{card}(B \setminus A).$$

Linear independance on $\mathcal{T}_{f,A,B}$

- $\mathbf{X}$ is a finite collection of cycles of f .
- $\mathcal{Q}_{A,\mathbf{X}}$ is the space of meromorphic quadratic differentials on $\mathbb{S}$ whose polar part is a linear combination of
 - simple polar parts along A and
 - the invariant polar parts $(\mathbf{q}_x)_{x \in \mathbf{X}}$.
- $\mathcal{Q}_{A,\mathbf{X}}$ has dimension $\text{card}(A) + \text{card}(\mathbf{X}) - 3$.
- The projection $\mathcal{Q}_{A,\mathbf{X}} \rightarrow \bigoplus_{x \in \mathbf{X}} \text{Span}(\mathbf{q}_x)$ is surjective.
- If $\mathbf{q} \in \mathcal{Q}_{A,\mathbf{X}}$ and $(\theta, \tau) \in \mathcal{E}_{f,A,B}$, then

$$\sum_{x \in \mathbf{X}} \langle \mathbf{q}, \tau \rangle_x = -\langle \mathbf{q}, \tau \rangle_{A \cup C_f} = -\langle \nabla_f \mathbf{q}, \theta \rangle_B.$$

Proposition (Epstein)

If the multipliers of the cycles in $\mathbf{X}$ are in $\overline{\mathbb{D}} \setminus \{0, 1\}$, then $\nabla_f := \text{id} - f_* : \mathcal{Q}_{A,\mathbf{X}} \rightarrow \mathcal{Q}_B$ is injective.

Cycles with multiplier 1

- x is a fixed point of f with multiplier 1 and multiplicity m .
- The polar part ω_x of the 1-form $\frac{dz}{z-f(z)}$ is well defined.
- The holomorphic index ρ is the residue of ω_x at x .
- As we perturb f in Rat_D , there is a splitting in m nearby fixed points, counting multiplicities.
- The sum of holomorphic indices at the perturbed fixed points defines a germ $R_x : (\text{Rat}_D, f) \rightarrow (\mathbb{C}, \rho)$.

Proposition

The form $dR_x : T_f \text{Rat}_D \rightarrow \mathbb{C}$ identifies with

$$\mathcal{T}_f \ni \tau \mapsto \langle \omega_x \otimes f^* \omega_x, \tau \rangle_x.$$

Goal

Study whether the splitting is generic.

Effective divisors and symmetric products

- Effective divisors of degree m identifies with elements of the symmetric product $\mathbb{S}_m$, i.e., the quotient of $\mathbb{S}^m$ modulo permutation of the coordinates.
- A tangent vector to $\mathbb{S}_m$ at the class of $\underbrace{[x, \dots, x]}_{m \text{ times}}$ identifies to a polar part of meromorphic function at x of order $\geq -m$.
- The tangent vector to the curve $t \mapsto (\varphi_t = 0)$ is the polar part of $\frac{\dot{\varphi}}{\varphi}$.

Example

- The tangent vector to the curve of divisors of fixed points of $f_t(z) = z + z^2 + t$ identifies to $\left[\frac{1}{z^2}\right]_0$.
- The tangent vector to the curve of divisors of fixed points of $f_t(z) = (1 + t)z + z^2$ identifies to $\left[\frac{1}{z}\right]_0$.

Splitting multiple fixed points

Proposition

The differential at f of the divisor of fixed points identifies with

$$\mathcal{T}_f \ni \tau \mapsto \sum_{x \in \text{Fix}(f)} [\omega_x(\tau)]_x \in T_{[\text{Fix}(f)]} \mathbb{S}_{D+1}.$$

Proposition

Within Rat_D , multiple fixed points (or multiple cycles) split generically.

Splitting multiple critical points

Proposition

The differential at f of the divisor of critical points identifies with

$$\mathcal{T}_f \ni \tau \mapsto \sum_{c \in \mathcal{C}_f} \left[\frac{d(\omega(\tau))}{\omega} \right]_c \in T_{[\mathcal{C}_f]} \mathbb{S}_{2D-2}$$

with $\omega = f^*dz$ and z a local coordinate near $\mathcal{V}_f$.

Example

- $f_t(z) = z^D + t$, $\tau = \frac{1}{Dz^{D-1}} \frac{d}{dz}$ and $\omega = Dz^{D-1}dz$.
- $\omega(\tau) = 1$ and $\left[\frac{d(\omega(\tau))}{\omega} \right]_0 = 0$.

Proposition

Within Rat_D , multiple critical points split generically.