

Partial Heights on Varieties Over Function Fields

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The goal of this talk is to introduce partial heights, a new notion generalizing the usual Weil heights, and consider application of partial heights to the geometric Bombieri–Lang conjecture.

This is joint work with Junyi Xie.

- 1 The geometric Bombieri–Lang conjecture
- 2 Partial heights on projective varieties
- 3 Canonical partial heights on abelian varieties

The Geometric Bombieri–Lang Conjecture

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This was conjectured by Louis Joel Mordell (1922), and proved by Gerd Faltings (1983).

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- If X is smooth, we say that X is of *general type* if the canonical bundle ω_X is big, i.e. $h^0(X, \omega_X^{\otimes m}) > cm^{\dim X}$ for some $c > 0$ and for all sufficiently large m . In general, say X is of *general type* if it is birational to a smooth projective variety of general type.

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- If $K = \mathbb{C}$, we say that X is *hyperbolic* (or more precisely Brody hyperbolic) if there is no non-constant holomorphic map $\mathbb{C} \rightarrow X(\mathbb{C})$. The notion extends to base field K endowed with an embedding $K \rightarrow \mathbb{C}$.

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- If $K = \mathbb{C}$, we say that X is *hyperbolic* (or more precisely Brody hyperbolic) if there is no non-constant holomorphic map $\mathbb{C} \rightarrow X(\mathbb{C})$. The notion extends to base field K endowed with an embedding $K \rightarrow \mathbb{C}$.
- We say that X is *algebraically hyperbolic* if there is no non-constant rational map $G \dashrightarrow X_{\overline{K}}$ from any group variety (or equivalently abelian variety) G over $\overline{K}$.

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Otherwise, the conjecture is widely open (over number fields).

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Conjecture

If X is algebraically hyperbolic, then $\mathrm{Sp}_{\mathrm{const}}(X)$ is Zariski closed in X .

The geometric Bombieri–Lang conjecture

Conjecture

Let k be an algebraically closed field of characteristic 0, and let K be a finitely generated field over k . Let X be an algebraically hyperbolic projective variety over K . Then the set $(X \setminus \mathrm{Sp}_{\mathrm{const}}(X))(K)$ is finite.

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- case that X has a finite morphism $X \rightarrow A$ to an abelian variety A over K by Xie–Yuan (2023).

Partial Heights on Projective Varieties

Weil height

In the geometric Bombieri–Lang conjecture, K/k is a finitely generated extension of characteristic 0. We can first reduce the problem to the case $k = \mathbb{C}$ and K/k is a function field of one variable, i.e., $K = \mathbb{C}(B)$ is the function field of a smooth projective curve B over $\mathbb{C}$.

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Let X be a projective variety over K , and let L be a line bundle on X . There is a Weil height function

$$h_L : X(\overline{K}) \longrightarrow \mathbb{R}$$

associated to L , which is unique up to bounded functions. To simplify the situation, we will only consider the restriction

$$h_L : X(K) \longrightarrow \mathbb{R}.$$

Weil height

Let $\pi : \mathcal{X} \rightarrow B$ be an integral model of X over B , which is assumed to be integral, projective and flat. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ extending L . Then we have a height function

$$h_{\mathcal{L}} : X(K) \longrightarrow \mathbb{R}, \quad x \longmapsto \deg(\mathcal{L}|_{\tilde{x}}),$$

where $\tilde{x}$ is the Zariski closure of x in $\mathcal{X}$. The function $h_{\mathcal{L}}$ is a Weil height function for L .

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Let $\omega_{\mathcal{L}}$ be a closed $(1,1)$ -form on $\mathcal{X}$ representing the cohomology class $c_1(\mathcal{L})$. Then we simply have

$$h_{\mathcal{L}}(x) = \int_{\tilde{x}} \omega_{\mathcal{L}}, \quad x \in X(K).$$

The integration is on the whole section $\tilde{x}$, but partial heights are defined by integration on a subset of $\tilde{x}$.

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Define the *partial height* of a point $x \in X(K)$ with respect to (D, ω) to be

$$h_{(D, \omega)}(x) = \int_{\tilde{x} \cap \mathcal{X}_D} \omega = \int_D x^* \omega.$$

Here $x : B \rightarrow \mathcal{X}$ denotes the section corresponding to $x \in X(K)$ by abuse of notations.

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This gives a *partial height function*

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If $D = B$ and $\omega = \omega_{\mathcal{L}}$ represents $c_1(\mathcal{L})$, then we recover the original (full) height function. If $\omega_{\mathcal{L}}$ is semipositive, we further have

$$h_{(D,\omega_{\mathcal{L}})}(x) \leq h_{\mathcal{L}}(x)$$

by

$$\int_D x^* \omega_{\mathcal{L}} \leq \int_B x^* \omega_{\mathcal{L}}.$$

Non-degeneracy conjecture

Recall from the above $h_{(D, \omega_{\mathcal{L}})} \leq h_{\mathcal{L}}$. What about the other direction? Do we expect a constant $\epsilon > 0$ such that

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We say that the pair (D, ω) is *strictly positive* (resp. *semipositive*) if ω is strictly positive (resp. semipositive) on an open neighborhood of $\mathcal{X}_{\overline{D}} = \pi^{-1}(\overline{D})$ in $\mathcal{X}$. Here $\overline{D}$ denotes the closure of D in B .

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Conjecture (non-degeneracy)

Assume that X does not contain any (possible singular) rational curve. Let $h_L : X(K) \rightarrow \mathbb{R}$ be a Weil height function associated to an ample line bundle L on X . Let $h_{(D,\omega)} : X(K) \rightarrow \mathbb{R}$ be a partial height function associated to a strictly positive pair (D,ω) on $\mathcal{X}$. Let $\{x_n\}_{n \geq 1}$ be a sequence in $X(K)$. If $h_L(x_n)$ converges to infinity, then $h_{(D,\omega)}(x_n)$ converges to infinity.

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We will prove the conjecture for abelian varieties, and thus for varieties with finite morphisms to abelian varieties.

Non-degeneracy implies geometric Bombieri–Lang

The following theorem asserts that the non-degeneracy conjecture of the partial height, together with the Green–Griffiths–Lang conjecture, implies the geometric Bombieri–Lang conjecture.

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Theorem (Xie–Y.)

Let $K = \mathbb{C}(B)$ be the function field of a smooth projective curve B over $\mathbb{C}$. Let X be a projective variety over K , and let $\pi : \mathcal{X} \rightarrow B$ be an integral model of X over B . Let $h_{(D,\omega)} : X(K) \rightarrow \mathbb{R}$ be a partial height function associated to a pair (D, ω) on $\mathcal{X}$, where D is an open disc of B .

Assume that the fiber $\mathcal{X}_b$ is hyperbolic for every $b \in \overline{D}$. Assume that the non-degeneracy conjecture holds for $h_{(D,\omega)}$. Then the geometric Bombieri–Lang conjecture holds for $X/K/\mathbb{C}$.

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The key is to prove that $h_L : X(K) \rightarrow \mathbb{R}$ is bounded. Once this holds, the sections in $\mathcal{X}(B)$ form a bounded family of curves in $\mathcal{X}$. By Hilbert schemes or Chow varieties, every irreducible component of the Zariski closure of $\mathcal{X}(B)$ in $\mathcal{X}$ is dominated by birationally constant varieties. By the work of Noguchi on hyperbolicity, every irreducible component is actually birationally constant.

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Assume that $h_L : X(K) \rightarrow \mathbb{R}$ is not bounded (to get a contradiction). Take an infinite sequence $\{x_n\}$ in $\mathcal{X}(B)$ with $h_L(x_n) \rightarrow \infty$. By the non-degeneracy conjecture, we have the partial height $h_{(D,\omega)}(x_n) \rightarrow \infty$.

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This gives a sequence $x_n|_D : D \rightarrow \mathcal{X}$ of “discs” in $\mathcal{X}$. We will consider the “convergence” of these discs.

Idea of implication

Fix a Kähler metric μ_B on B . We have

$$h_{(D,\omega)}(x_n) = \int_D x_n^* \omega = \int_D \|dx_n\|^2 \mu_B = \int_D \|(dx_n)_b\|^2 \mu_B(b),$$

where at every point $b \in D$,

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As $h_{(D,\omega)}(x_n) \rightarrow \infty$, there is a sequence $b_n \in D$ such that

$$\|(dx_n)_{b_n}\| \longrightarrow \infty.$$

By the classical Brody lemma in complex geometry, we obtain a (non-constant) entire curve $\phi : \mathbb{C} \rightarrow \mathcal{X}$, which is a limit of a re-parametrization of $\{x_n : D \rightarrow \mathcal{X}\}_{n \geq 1}$. The entire curve is contained in a fiber of $\mathcal{X}_{\overline{D}} \rightarrow \overline{D}$. This contradicts to the hyperbolicity assumption.

Vojta's dictionary

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- an entire curve $\phi : \mathbb{C} \rightarrow Y$ in a complex projective variety Y .

This predicts the Bombieri–Lang conjecture: “hyperbolic” means no entire curves, and corresponds to “no infinite sequence of rational points”.

Our construction realizes the mechanism of Vojta's dictionary in the case that K/k is a function field of one variable. In fact, from an infinite sequence $x_n \in X(K)$ of un-bounded heights, we have constructed an entire curve in $X_\iota(\mathbb{C})$ for “most” embeddings $\iota : K \hookrightarrow \mathbb{C}$.

Canonical Partial Heights

Canonical partial height

Assume that $K = \mathbb{C}(B)$ for a smooth projective curve B over $\mathbb{C}$. Let A be an abelian variety over K , and L be a symmetric and ample line bundle on A . The canonical height $\hat{h}_L : A(K) \rightarrow \mathbb{R}$ is defined by Tate's limit

$$\hat{h}_L(x) = \lim_{m \rightarrow \infty} \frac{1}{4^m} h_L(2^m x), \quad x \in A(K).$$

By the Lang–Néron theorem, $\hat{h}_L$ defines a positive definite quadratic form on the finite-dimensional $\mathbb{R}$ -space

$$V(A) = (A(K)/A_0(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{R},$$

where $A_0/\mathbb{C}$ is the $K/\mathbb{C}$ -trace of A .

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We can also define the canonical partial height $\hat{h}_{(D,\omega)} : A(K) \rightarrow \mathbb{R}$ by

$$\hat{h}_{(D,\omega)}(x) = \lim_{m \rightarrow \infty} \frac{1}{4^m} h_{(D,\omega)}(2^m x), \quad x \in A(K).$$

We expect the canonical partial height has similar positivity and quadraticity properties.

Betti forms

To better understand the situation, we recall the notion of Betti form ω associated to the ample and symmetric line bundle L on A .

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Let U be an open subvariety of B such that A has a smooth integral model $\mathcal{A}_U \rightarrow U$ over U . The Betti form is a semipositive (smooth) $(1,1)$ -form on $\mathcal{A}_U$, which represents the cohomology classes of L on fibers of $\mathcal{A}_U \rightarrow U$ and satisfies the nice dynamical property

$$[m]^*\omega = m^2\omega, \quad m \in \mathbb{Z}.$$

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By result of Gauthier–Vigny (2019), we have a clean identity

$$\hat{h}_L(x) = \int_U x^*\omega = h_{(U, \omega)}(x)$$

for any $x \in A(K)$ viewed as a section $x : U \rightarrow \mathcal{A}_U$.

Canonical partial height

Take D to be an open disc in U , and take ω to be the Betti form on $\mathcal{A}_U$ associated to a symmetric and ample line bundle L on A . In this case, the partial height $h_{(D,\omega)} : A(K) \rightarrow \mathbb{R}$ given by

$$h_{(D,\omega)}(x) = \int_D x^* \omega$$

satisfies $h_{(D,\omega)}(2x) = 4h_{(D,\omega)}(x)$ by $[2]^* \omega = 4\omega$. So the *canonical partial height* $\hat{h}_{(D,\omega)} = h_{(D,\omega)}$.

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Similar to the Néron–Tate height, the canonical partial height is *quadratic* in the following sense.

Theorem (quadraticity, positivity)

The canonical partial height $h_{(D,\omega)} : A(K) \rightarrow \mathbb{R}$ is quadratic on $A(K)$ and trivial on $A_0(\mathbb{C})$. It induces a positive definite quadratic form on the vector space $V(A) = (A(K)/A_0(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{R}$.

Now we have two positive definite quadratic forms $\hat{h}_L$ and $h_{(D,\omega)}$ on the finite-dimensional vector space $V(A) = (A(K)/A_0(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{R}$.

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Theorem (non-degeneracy)

There is a constant $\epsilon > 0$ such that

$$h_{(D,\omega)}(x) \geq \epsilon \hat{h}_L(x), \quad \forall x \in A(K).$$

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This gives the non-degeneracy conjecture for abelian varieties.

Non-degeneracy

For any projective morphism $f : X' \rightarrow X$, we have a projection formula

$$h_{(D, f^* \alpha)}(x') = h_{(D, \alpha)}(f(x')), \quad x' \in X'(K).$$

This implies the following non-degeneracy theorem.

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Theorem (non-degeneracy)

Let X be a projective variety over K with a finite morphism $f : X \rightarrow A$ to an abelian variety A over K . Then the non-degeneracy conjecture of partial heights holds for X .

Thank you very much.