

Billiards, heights and Hodge theory

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Hubert, Lanneau, Filip, Davis, Lelievre, Smillie, Ulcigrai, F. Calegari, ...*

Curves on a Hilbert modular surface

Assume K is a real quadratic field.

Let $X_K = \mathbb{H} \times \mathbb{H} / \mathrm{SL}_2(\mathcal{O} \oplus \mathcal{O}^\vee)$.

Theorem

The cusps of 'every' geodesic curve V on X_K
coincide with $\mathbb{P}^1(K)$.

Cor

There is 'no' compact geodesic curve V on a
Hilbert modular surface.

False for $\dim X_K > 2$.

Heights

Valuations

K = number field degree g over $\mathbb{Q}$

$\mathcal{O}_K$ = ring of integers in K

v = a place of K

$|x|_v$ = valuation of x in K

$v=2,3,5,\dots,\infty$ for $K=\mathbb{Q}$

$$|p^e|_p = 1/p^e$$

$$|a/b|_\infty = |a/b|$$

$$\prod_v |x|_v = 1$$

$$\prod_{v|p} |x|_v = |x|_p$$

Heights on K and $\mathcal{O}_K$

x in K

$$H(x) = H(1 : x) = \prod_v \max(1, |x|_v)$$

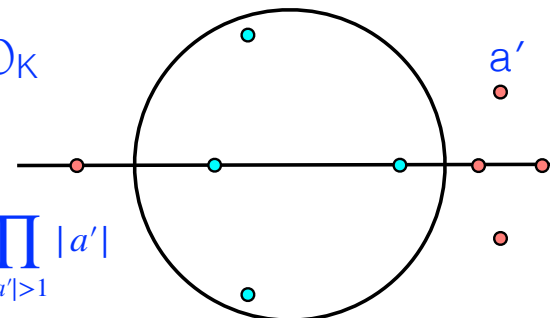
a in $\mathcal{O}_K$

$$H(a) = \prod_{v|\infty} \max(1, |a|_v)$$

Algebraic integers

a in $\mathcal{O}_K$

$$H(a)^g = \prod_{|a'|>1} |a'|$$



a' *Galois conjugates*

$$\text{Conjecture: } \prod_{|a'|>1} |a'| \geq \lambda_{10} = 1.17628\dots$$

(if not root of 1)

$$\text{Theorem (Dmitrov): } \max_{|a'|>1} |a'|^g > 2^{1/4}$$

Classical height on $\mathbb{P}^n(K)$

$$H(x) = H(x_0 : x_1 : \dots : x_n) = \prod_v \max_i |x_i|_v \geq 1$$

comparable to

$$\tilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v, \quad [a_0 : \dots : a_n] = [x], \quad a_i \text{ integers}$$

*only requires knowledge of integers $\mathcal{O}_K$
and infinite places of K*

Abelian varieties

Abelian varieties

$$A = \Omega(A)^*/H_1(A, \mathbb{Z}) \cong \mathbb{C}^g/L$$

Polarization = Hermitian inner product on
 $\Omega(A)^* \cong H_1(X, \mathbb{R})$ such that

$$[C, D] = -\operatorname{Im} \langle C, D \rangle$$

gives integral symplectic form on $H_1(X, \mathbb{Z})$.

Hodge norm:

$$\|C\|_A^2 = \langle C, C \rangle \text{ on } H_1(X, \mathbb{R}).$$

Example: The Jacobian

X compact Riemann surface, genus g

$\Omega(X)$ = space of holomorphic 1-forms, $\cong \mathbb{C}^g$.

$$A = \operatorname{Jac}(X) = \Omega(X)^*/H_1(X, \mathbb{Z}).$$

Polarization: $\langle \omega_1, \omega_2 \rangle = \frac{i}{2} \int_X \omega_1 \wedge \bar{\omega}_2.$

$[C, D]$ = usual intersection form on $H_1(X, \mathbb{Z})$

Hodge norm: $\|C\|_X = \sup \left\{ \left| \int_C \omega \right| : \langle \omega, \omega \rangle = 1 \right\}.$

Real multiplication

Real multiplication

A = a polarized abelian variety

K = totally real number field

$$\deg(K) = \dim(A) = g$$

A has **real multiplication** by K if we are given a map

$$T : K \longrightarrow \operatorname{End}(A) \otimes \mathbb{Q},$$

and T_k is self-adjoint for all k in K .

Get an orthonormal basis of eigenforms

$$\{\omega_v : v \mid \infty\} \subset \Omega(A)$$

Real multiplication

Examples:

- * Any elliptic curve A has RM by $K = \mathbb{Q}$.
- * If $U : X \rightarrow X$ is an automorphism of order n , then $T = U + U^{-1}$ is self-adjoint, and $A = \text{Jac}(X)$ has RM by $K = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

The projective line $\mathbb{P}_A^1(K)$

A has real multiplication by K

$$\implies H_1(A, \mathbb{Q}) \cong K^2$$

Definition

$\mathbb{P}_A^1(K)$ = space of K -lines in $H_1(X, \mathbb{Q})$

Hodge height

Hodge valuation at an infinite place of K

Diagonalize K acting on $H_1(A, \mathbb{R})$

$$H_1(A, \mathbb{R}) = \bigoplus_v S_v \xrightarrow{\pi_v} S_v$$

$$\|C\|_v = \|\pi_v(C)\|_A = \left| \int_C \omega_v \right|$$

$$|C|_v = \|C\|_v^{1/g} \quad \text{'Hodge valuation'}$$

Hodge height on $\mathbb{P}_A^1(K)$

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$$x \in \mathbb{P}_A^1(K)$$

$$C \in H_1(A, \mathbb{Z})$$

integral point

$$[x] = [C]$$

on given K line

Why a height?

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$$\tilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v$$

Theorem. Given a linear isomorphism

$$\iota : \mathbb{P}_A^1(K) \rightarrow \mathbb{P}^1(K)$$

we have $H(\iota(x)) \asymp H_A(x)$

Corollary

The Hodge height for a given A is bounded below:

$$\delta(A) = \inf_x H_A(x) > 0.$$

$\delta(A)$ measures how close A is to infinity in X_K .

Hilbert modular varieties

How to make A with RM by K ?

$$A = \mathbb{C}^g / \mathcal{O} \oplus \mathcal{O}^\vee \tau$$

$\mathcal{O}$ = ring of integers in K $\tau = (\tau_1, \dots, \tau_g) \in \mathbb{H}^g$

$$(a, b) \mapsto (a_i + b_i \tau_i) \quad T_k(z) = (k_i z_i)$$

Polarization: natural symplectic form on $\mathcal{O} \oplus \mathcal{O}^\vee$

Eigenforms: $\omega_i = dz_i$

Hilbert modular variety X_K

$$X_K = \mathbb{H}^g / \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) \longrightarrow \mathcal{A}_g$$

Sweeps out all Abelian varieties

$$A = \mathbb{C}^g / \mathcal{O} \oplus \mathcal{O}^\vee \tau$$

with real multiplication by K .

Hodge height, revisited

$$H_\tau(x) = \inf_{x=a/b} \left(\prod_i \frac{|a_i + b_i \tau_i|^2}{\mathrm{Im} \tau_i} \right)^{1/2g}$$

$$\delta(\tau) = \inf_x H_\tau(x) > 0$$

descends to a proper function on X_K .

Case of a torus

$$X_{\mathbb{Q}} = \mathbb{H} / \mathrm{SL}_2(\mathbb{Z}) \quad \tau \in \mathbb{H}$$

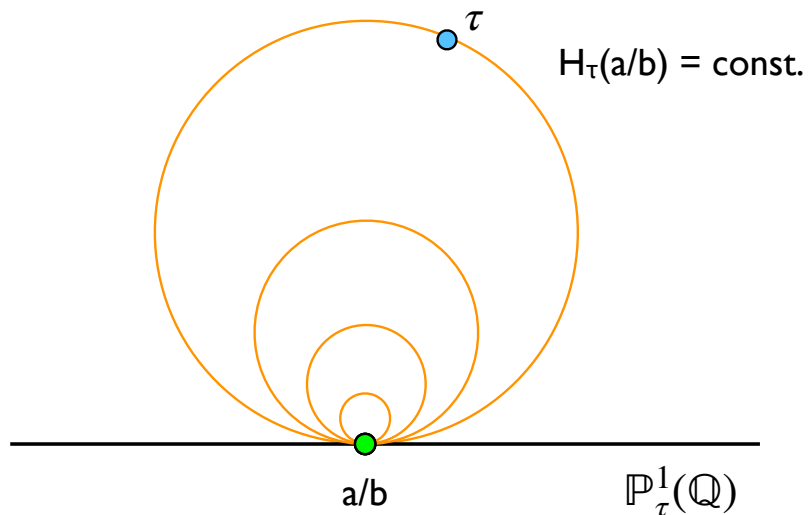
$$A = \mathbb{C} / \mathbb{Z} \oplus \mathbb{Z} \tau \quad H_1(A, \mathbb{Z}) \cong \mathbb{Z}^2$$

$$\|C\|_A^2 = \left| \int_C \omega \right|^2 = \frac{|a + b\tau|^2}{\mathrm{Im} \tau} \quad \text{Hodge norm}$$

$H_\tau(x)$ = length of geodesic on A with slope $x = a/b$

$\delta(\tau)$ = length of shortest geodesic on A

Level sets of height are horocycles in $\mathbb{H}$



Descent for $SL_2\mathbb{Z}$

Theorem

Every a/b is a cusp of $SL_2\mathbb{Z}$.

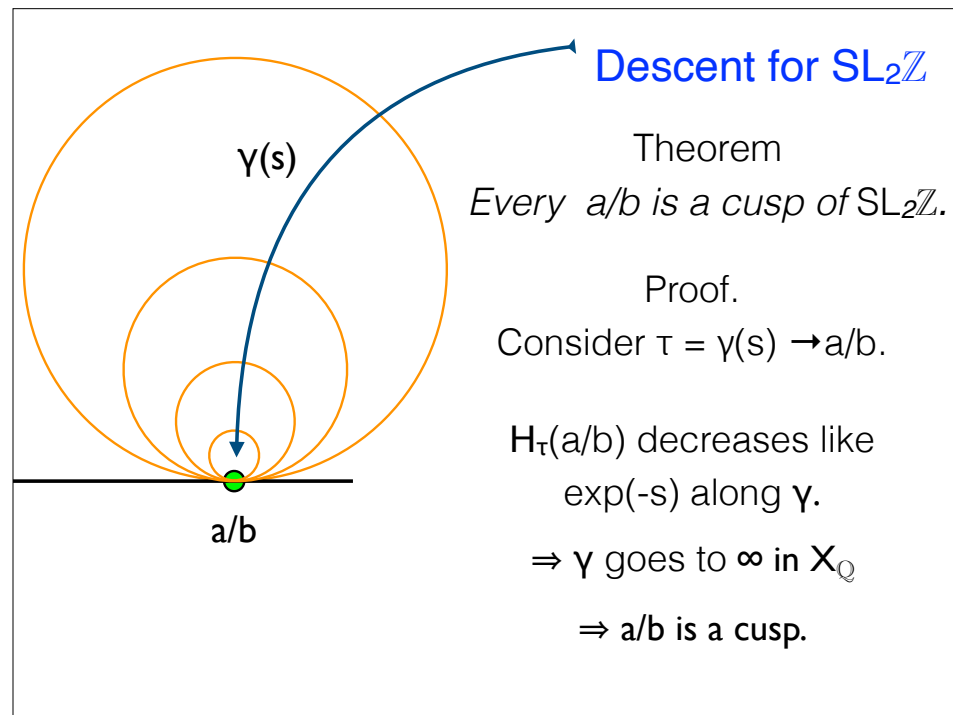
Proof.

Consider $\tau = \gamma(s) \rightarrow a/b$.

$H_\tau(a/b)$ decreases like $\exp(-s)$ along γ .

$\Rightarrow \gamma$ goes to ∞ in $X_{\mathbb{Q}}$

$\Rightarrow a/b$ is a cusp.



Geodesic curves

Curves on a Hilbert modular surface

Assuming K is quadratic:

Theorem

The cusps of 'every' geodesic curve $V = \mathbb{H}/\Delta$ on X_K coincide with $\mathbb{P}^1(K)$, and satisfy quadratic height bounds.

Hilbert modular surfaces

K real quadratic field

$$X_K = \mathbb{H}^2 / \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

$$X_K \longrightarrow \mathcal{A}_2$$

Geodesic curves on X_K

$V = \mathbb{H} / \Delta \longrightarrow X_K$: a *Kobayashi geodesic curve*

$$\Delta \subset \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

specified by

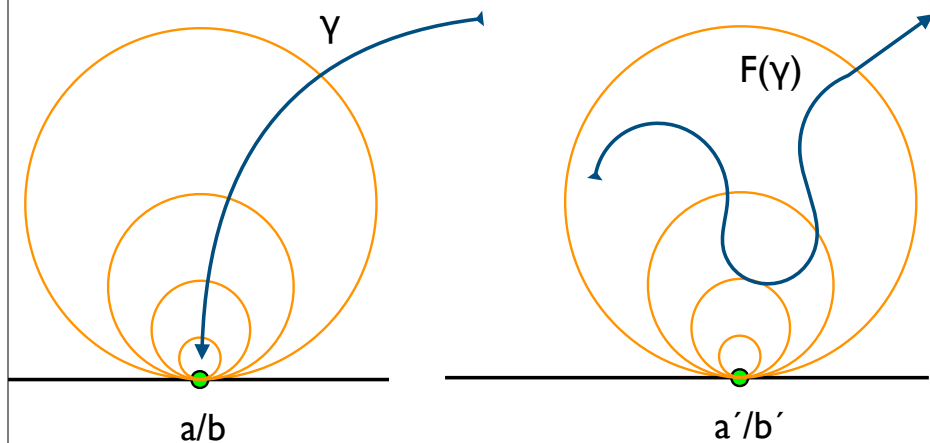
$$F : \mathbb{H} \rightarrow \mathbb{H}$$

$$F(g \cdot t) = g' \cdot F(t)$$

$$\tilde{V} = \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H} = \tilde{X}_K$$

$$t \mapsto \tau = (\tau_1, \tau_2) = (t, F(t))$$

Descent in $\mathbb{H} \times \mathbb{H}$



Descent in $\mathbb{H} \times \mathbb{H}$

$$H_\tau(x)^4 \leq \frac{|a + bt|^2}{\mathrm{Im} t} \cdot \frac{|a' + b'F(t)|^2}{\mathrm{Im} F(t)}$$

Can drive first term to zero like $\exp(-s)$
along a geodesic $\gamma \subset \mathbb{H}$

Second term grows slower than $\exp(s)$ along $F(\gamma) \subset \mathbb{H}$

Schwarz lemma

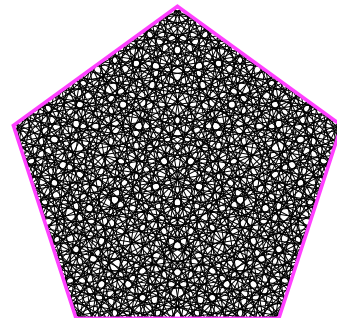
$$\implies H_\tau(x) \rightarrow 0 \implies \gamma \rightarrow \infty \text{ in } V \text{ and } X_K$$

Thus: *every x in $\mathbb{P}^1(K)$ is a cusp of V*
(with quadratic height bounds). QED

Billiards and triangle groups

Applications

Billiards in a regular pentagon

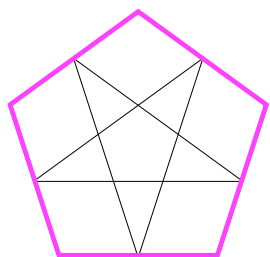


A dense set of slopes are periodic.

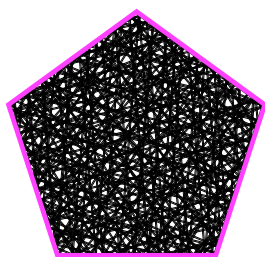
Which ones?

How do the periodic trajectories behave?

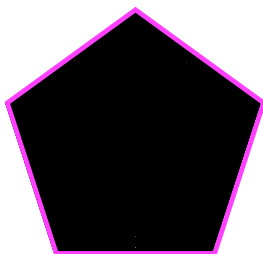
Lengths: Experiments



$$L(s) = 5$$



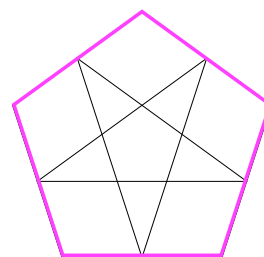
$$L(4s) = 469$$



$$L(20s) = 2338$$

$$L(6765s) = 1.734 \times 10^{25}$$

Lengths and heights



$$s = \tan(2\pi/5)$$

Theorem

The periodic slopes coincide with

$$\mathbb{Q}(\sqrt{5})s, \text{ and } \log L(xs) = O(h(x)^2).$$

$$h(n) = \log(n)$$

$$\text{can have } L(10^n) \sim 10^{n^2}$$

exponent 2 is sharp

Continued fractions

Every $s \in \mathbb{Q}(\gamma)$ can be expanded as a finite golden continued fraction,

$$s = [a_1, a_2, a_3, \dots, a_N] = a_1\gamma + \frac{1}{a_2\gamma + \frac{1}{a_3\gamma + \dots \frac{1}{a_N\gamma}}}$$

with $a_i \in \mathbb{Z}$.

$$\gamma = (1 + \sqrt{5})/2$$

Height bounds: length N and a_i are $O(1+h(s))$.

Golden Fractions

Corollary

Every x in $K = \mathbb{Q}(\sqrt{5})$ can be written uniquely as a 'golden fraction' $x = a/c$, up to sign.

a, c in $\mathbb{Z}[\gamma]$ relatively prime

(a, c) column of a matrix in Δ_5

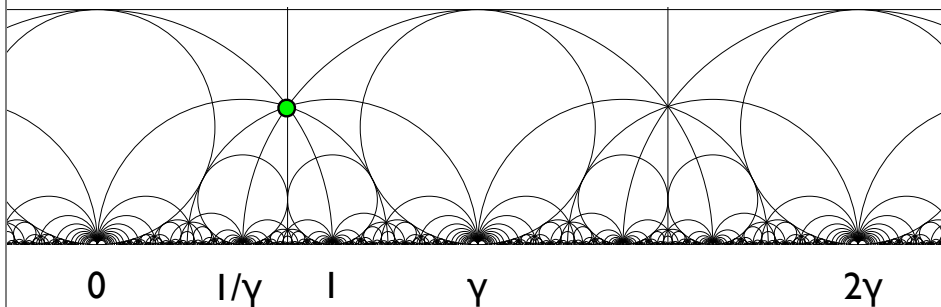
Quadratic height bounds: $h(a) + h(c) = O(1 + h(x)^2)$.

Cusps

The cusps of $\Delta(2, 5, \infty)$ coincide with $K = \mathbb{Q}(\sqrt{5}) \cup \{\infty\}$.

Leutbecher, 1970s

5 packing



Renormalization

Theorem (Veech)

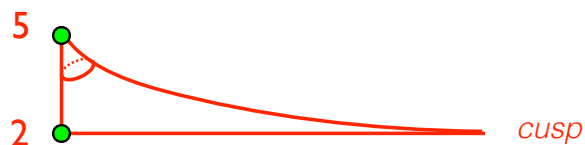
The periodic slopes for billiards in a regular pentagon correspond to the cusps of the triangle group $\Delta(2, 5, \infty)$.

Power of renormalization

(Up to renormalization:
There is only 1 type of
periodic billiard in a pentagon)



(unique cusp of $V = \mathbb{H}/\Delta(2,5,\infty)$)



Geodesic curve

Theorem

The quotient $V = \mathbb{H}/\Delta(2,5,\infty)$ naturally embeds as a geodesic curve on X_K , $K = \mathbb{Q}(\sqrt{5})$.

Corollary

The cusps of $\Delta(2,5,\infty)$ coincide with K , and satisfy quadratic height bounds.

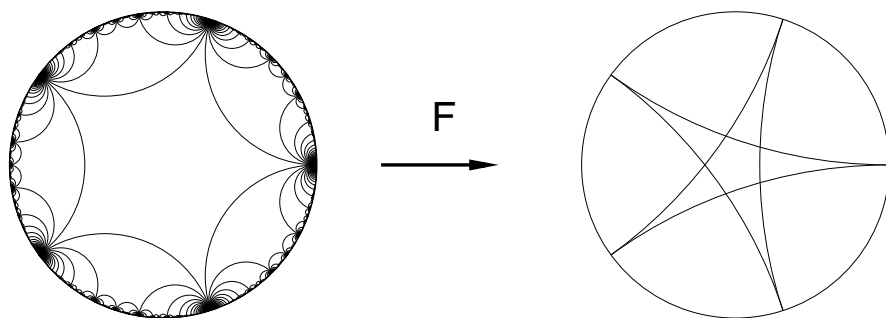
Corollary

Results on billiard and continued fractions follow.

Holomorphic pentagon-to-star map

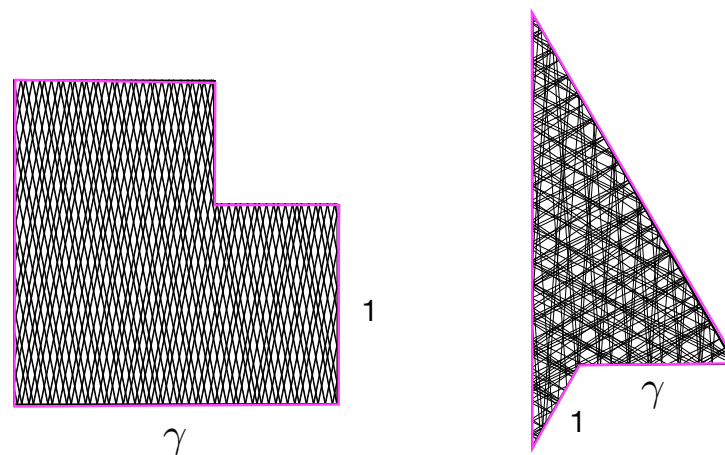
$\Delta = \Delta(2,5,\infty)$

Δ'



$V \longrightarrow X_K$ covered by $\mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{H}$
via $x \longrightarrow (x, F(x))$.

Applies to almost
all families of optimal billiards



...since these are also quadratic: Eskin - Filip - Wright

Heights and descent

Classical: To show the 'continued fraction' for x in $\mathbb{P}^1(K)$ terminates, show a suitable height $H(x)$ decreases at each step.

discrete, clever H

Modern : To show a geodesic γ in $V \subset X_K$ heads towards a cusp at x in $\mathbb{P}^1(K)$, show $H_A(x) \rightarrow 0$ as $A \in X_K$ moves along γ .

continuous, natural H

beyond quadratic fields... Undecidability?

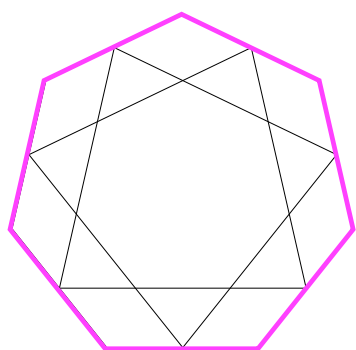
$\text{CUSP}(n)$ = Given $s = a/b$ in K , decide if s is a cusp of $\Delta(2, n, \infty)$

Question

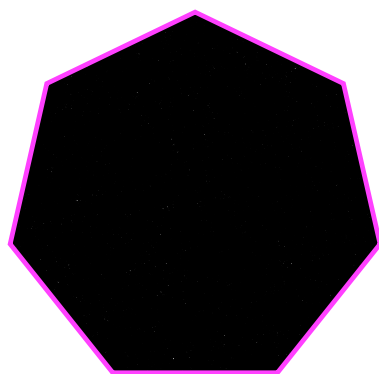
Is there an $n = 7, 9, 11, \dots$ such that $\text{CUSP}(n)$ is undecidable?

Open already for $n=7$

K = a cubic number field



$L(1)=7,$



$L(1+14\zeta_7) \approx 10^{40}.$

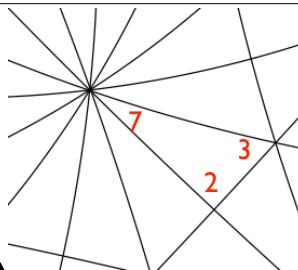
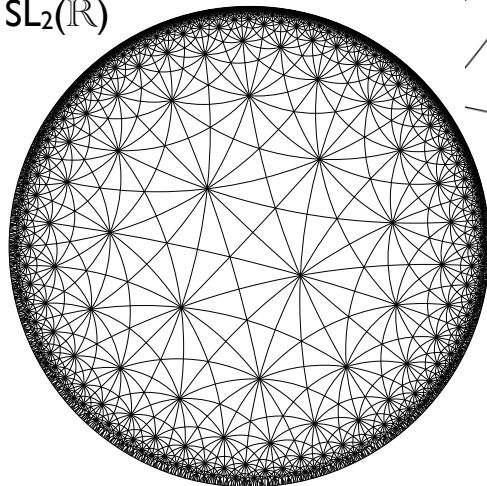
No known way to test for periodicity of billiards.
How long must we wait for continued fraction to terminate?

Complément

A spectral gap for
triangles

Triangle groups

$$\Delta(2,3,7) \subset \mathrm{SL}_2(\mathbb{R})$$



Moduli space of all triangles

$$A = (\mathbb{R}/2\mathbb{Z})^3 \simeq (S^1)^3$$

$a = (a_1, a_2, a_3)$ gives angles (πa_i)

Triangle $T(a)$ may be spherical, Euclidean or hyperbolic

Galois orbit: *When a in A is torsion of order n ,*

$$\mathrm{Gal}(a) = (\mathbb{Z}/n)^* \cdot a.$$

a_i roots of 1

Spectral gap

Ramification density:

$$\rho(a) = \frac{\#\{b \text{ in } \mathrm{Gal}(a) : T(b) \text{ is spherical}\}}{\#\mathrm{Gal}(a)}$$

Theorem

There exist constants $0 < \rho_H < \rho_S < 1$, such that

$$\rho(a) \in \{0, 1\} \cup [\rho_H, \rho_S].$$

- Probably $[\rho_H, \rho_S] = [1/12, 4/5]$.
- Usually $\rho(a) \approx 1/3$.
- Cases $\rho(a) = 0$ or 1 understood, modulo finite set

Proof of spectral gap

— *Equidistribution:*

$m(a)$ = uniform measure on $\mathrm{Gal}(a)$

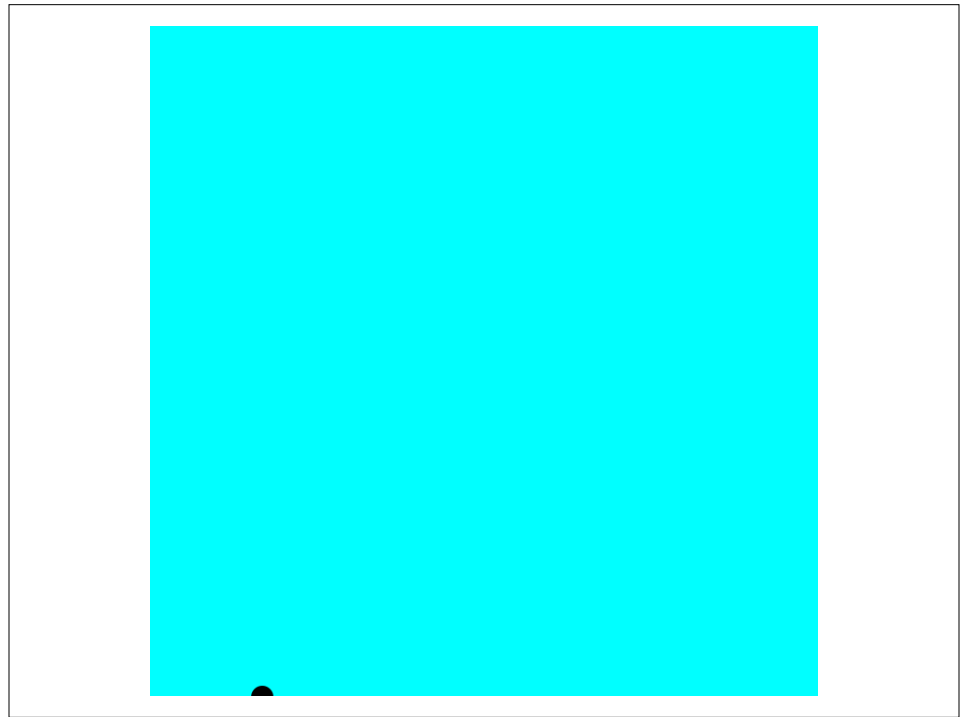
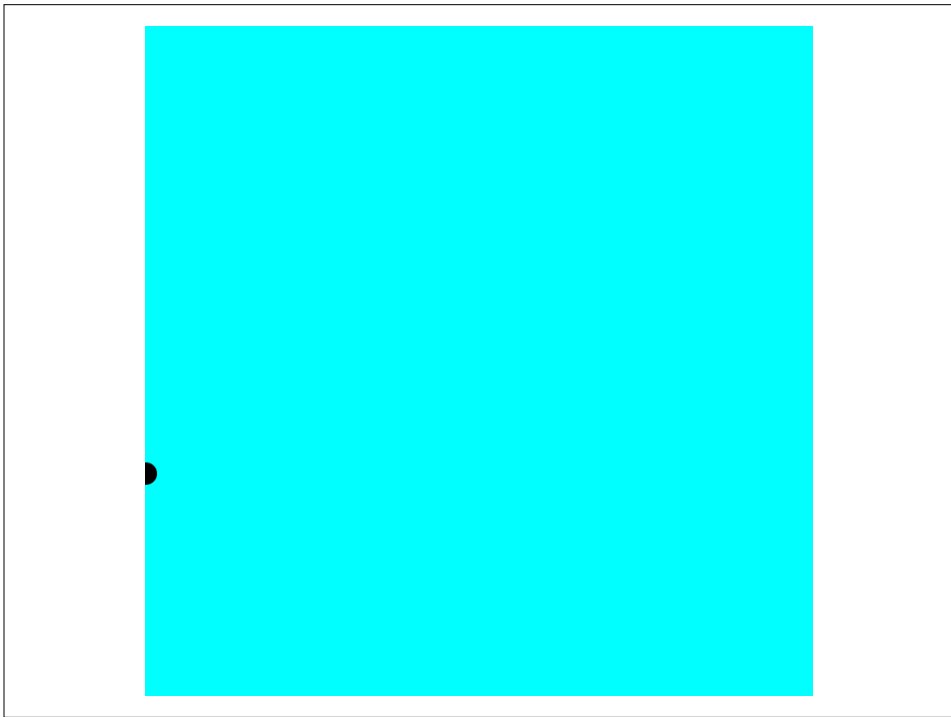
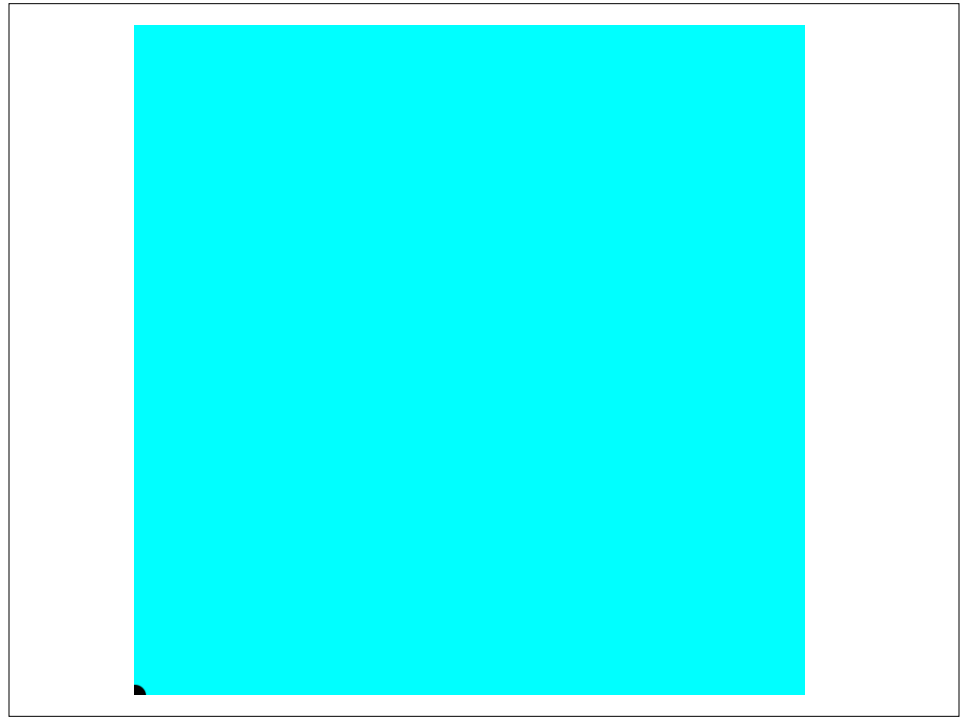
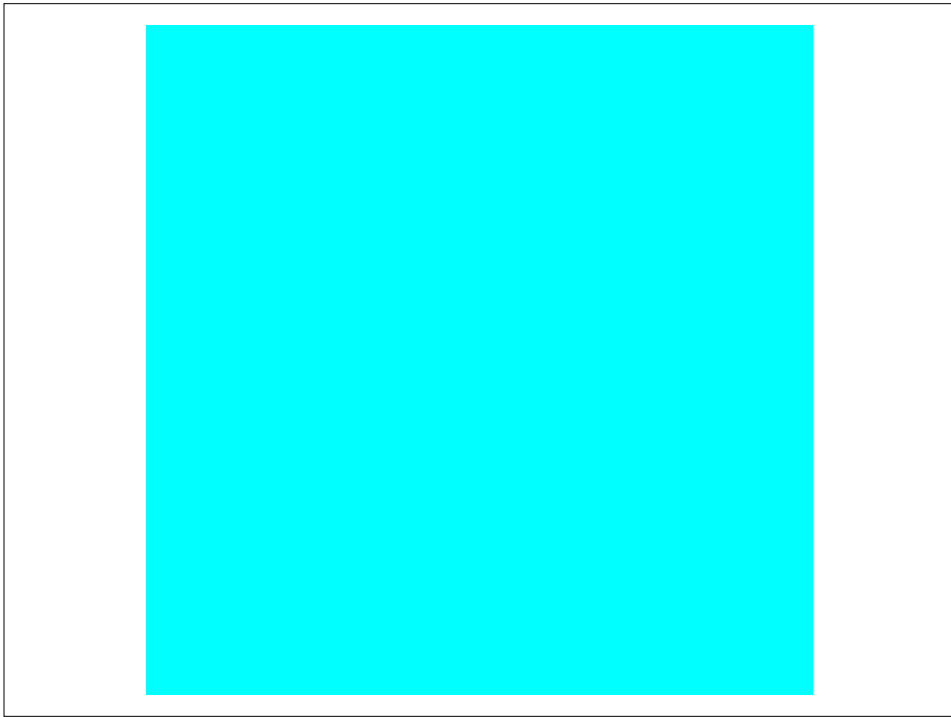
$m(a_n) \rightarrow m(B)$ = uniform measure on torus translates

critical: a_n is in B for all $n \gg 0$.

— *Geometry:*

Find possibilities for B

Moving tablecloth game



Spectral gap — *encore*

Theorem

For all but finitely many $\Delta(p,q,r)$,
spherical and # hyperbolic places are about the same.

Cor (Takeuchi)

There are only finitely many arithmetic triangle groups.

Cor (Waterman—Maclachlan)

There are only finitely many totally hyperbolic triangle groups.

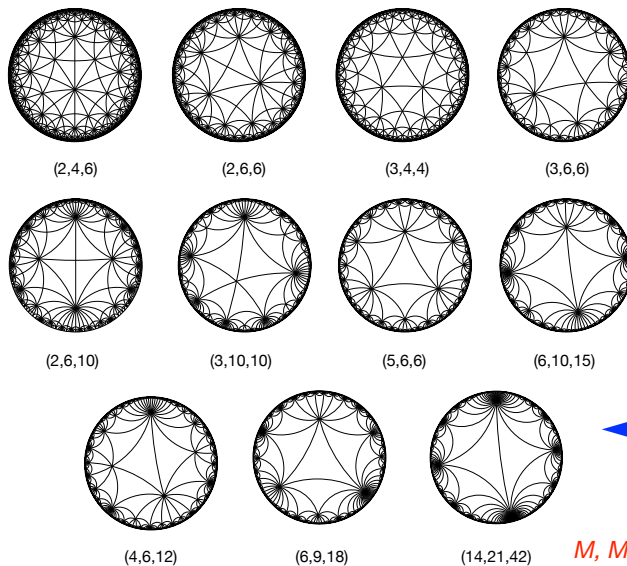
76 cocompact arithmetic triangle groups

	(e_1, e_2, e_3)	Field	Ram
1	$(2, 3, \infty), (2, 4, \infty), (2, 6, \infty), (2, \infty, \infty), (3, 3, \infty), (3, \infty, \infty), (4, 4, \infty), (6, 6, \infty), (\infty, \infty, \infty)$	$\mathbb{Q}$	$\emptyset$
2	$(2, 4, 6), (2, 6, 6), (3, 4, 4), (3, 6, 6)$	$\mathbb{Q}$	$2, 3$
3	$(2, 3, 8), (2, 4, 8), (2, 6, 8), (2, 8, 8), (3, 3, 4), (3, 8, 8), (4, 4, 4), (4, 6, 6), (4, 8, 8)$	$\mathbb{Q}(\sqrt{2})$	$\mathcal{P}_2$
4	$(2, 3, 12), (2, 6, 12), (3, 3, 6), (3, 4, 12), (3, 12, 12), (6, 6, 6)$	$\mathbb{Q}(\sqrt{3})$	$\mathcal{P}_2$
5	$(2, 4, 12), (2, 12, 12), (4, 4, 6), (6, 12, 12)$	$\mathbb{Q}(\sqrt{3})$	$\mathcal{P}_3$
6	$(2, 4, 5), (2, 4, 10), (2, 5, 5), (2, 10, 10), (4, 4, 5), (5, 10, 10)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_2$
7	$(2, 5, 6), (3, 5, 5)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_3$
8	$(2, 3, 10), (2, 5, 10), (3, 3, 5), (5, 5, 5)$	$\mathbb{Q}(\sqrt{5})$	$\mathcal{P}_5$
9	$(3, 4, 6)$	$\mathbb{Q}(\sqrt{6})$	$\mathcal{P}_2$
10	$(2, 3, 7), (2, 3, 14), (2, 4, 7), (2, 7, 7), (2, 7, 14), (3, 3, 7), (7, 7, 7)$	$\mathbb{Q}(\cos \pi/7)$	$\emptyset$
11	$(2, 3, 9), (2, 3, 18), (2, 9, 18), (3, 3, 9), (3, 6, 18), (9, 9, 9)$	$\mathbb{Q}(\cos \pi/9)$	$\emptyset$
12	$(2, 4, 18), (2, 18, 18), (4, 4, 9), (9, 18, 18)$	$\mathbb{Q}(\cos \pi/9)$	$\mathcal{P}_2, \mathcal{P}_3$
13	$(2, 3, 16), (2, 8, 16), (3, 3, 8), (4, 16, 16), (8, 8, 8)$	$\mathbb{Q}(\cos \pi/8)$	$\mathcal{P}_2$
14	$(2, 5, 20), (5, 5, 10)$	$\mathbb{Q}(\cos \pi/10)$	$\mathcal{P}_2$
15	$(2, 3, 24), (2, 12, 24), (3, 3, 12), (3, 8, 24), (6, 24, 24), (12, 12, 12)$	$\mathbb{Q}(\cos \pi/12)$	$\mathcal{P}_2$
16	$(2, 5, 30), (5, 5, 15)$	$\mathbb{Q}(\cos \pi/15)$	$\mathcal{P}_3$
17	$(2, 3, 30), (2, 15, 30), (3, 3, 15), (3, 10, 30), (15, 15, 15)$	$\mathbb{Q}(\cos \pi/15)$	$\mathcal{P}_5$
18	$(2, 5, 8), (4, 5, 5)$	$\mathbb{Q}(\sqrt{2}, \sqrt{5})$	$\mathcal{P}_2$
19	$(2, 3, 11)$	$\mathbb{Q}(\cos \pi/11)$	$\emptyset$

Takeuchi

Maclachlan-Reid

11 known totally hyperbolic triangle groups



$\rho(a) = 0$.

M, Maclachlan-Waterman

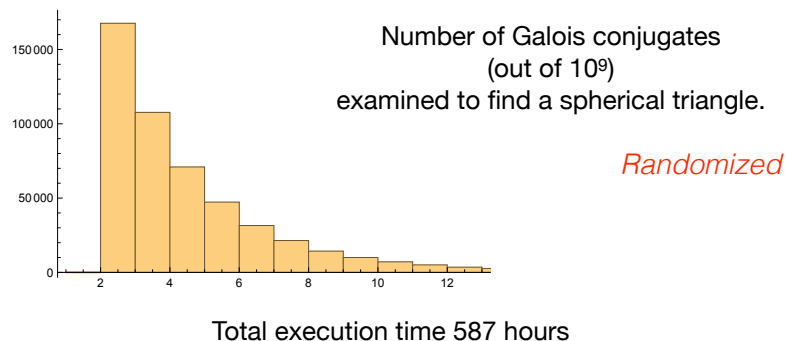
Problem

Are there more examples of totally hyperbolic triangle groups?

Experimental Evidence

There are no other purely hyperbolic triangle groups
with $(p,q,r) < 5000$.

Recently verified using 5,000 cores running in parallel
from 1-30 mins.



The importance of being (14,21,42)

Motivation

Theorem (Veech)

Every geodesic curve $V \rightarrow M_g$ has a cusp.

↑ **most cases**

Theorem

'Every' geodesic curve $V \rightarrow X_K$ has a cusp,
provided $\dim(X_K)=2$

What happens when $\dim(X_K) > 2$?

What happens if $\dim X_K > 2$?

Theorem

There exists a **compact** geodesic curve
 V on a 6D Hilbert modular variety,

$$V = \mathbb{H}/\Delta' \rightarrow X_K,$$

such that there is no compact Shimura variety with
 $V \subset S \subset X_K$.

(Δ' is Zariski dense in $SL_2(\mathcal{O}_K)$)

Matrix models

$$\Delta = \Delta(p,q,r) \subset \mathrm{SL}_2(\mathbb{R})$$

$$K = \mathbb{Q}(\text{traces of elements in } \Delta)$$

Δ can be realized as a subgroup of $\mathrm{SL}_2(K)$

Fallacy

Correction

$\Leftrightarrow$ quaternion algebra $B = \mathbb{Q}(\Delta)$ splits over K

$\Rightarrow \Delta$ is totally hyperbolic (B splits at all $v|\infty$)

Theorem

Among the 11 known totally hyperbolic cocompact triangle groups, only

$$\Delta(14,21,42)$$

is also split at all finite places.

Corollary

$\Delta(14,21,42)$ embeds in $\mathrm{SL}_2(K)$.

$$K = \mathbb{Q}(\cos \pi/21)$$

degree 6

Theorem (Cohen-Wolfart)

From the group theory:

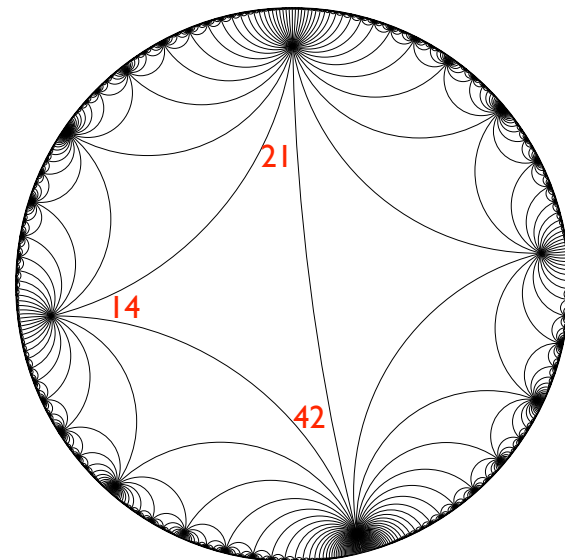
$$\Delta(14,21,42) \subset \mathrm{SL}_2(\mathcal{O}_K),$$

we obtain a geodesic curve

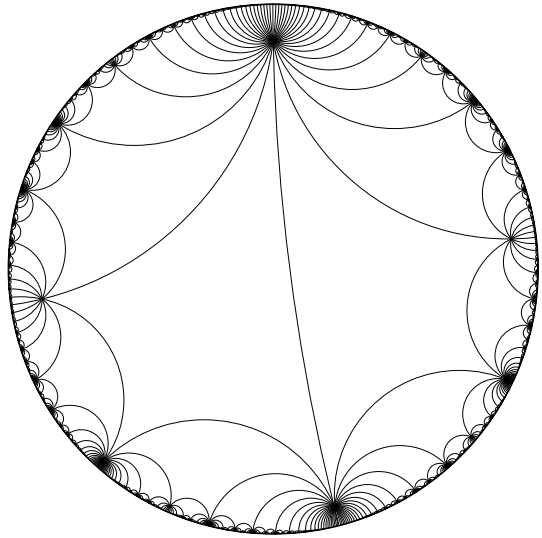
$$V = \mathbb{H}/\Delta' \rightarrow X_K.$$

Special to triangles!

Start with $\Delta(14,21,42)$



Pass to Δ' of index 2

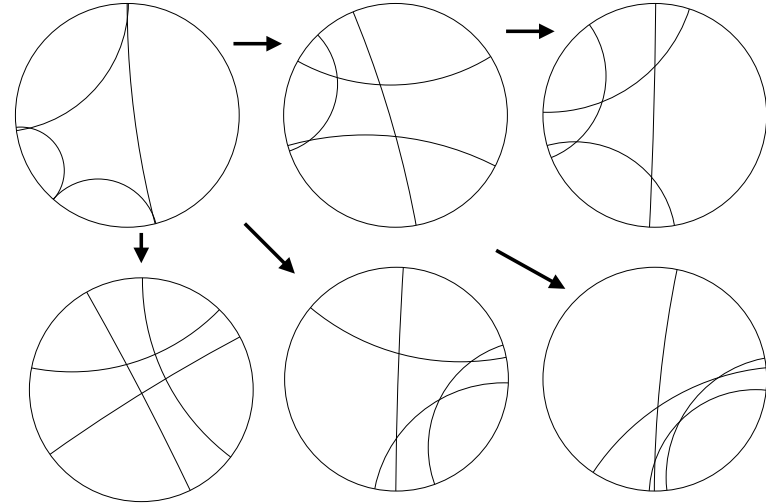


$$V = \mathbb{H}/\Delta'$$

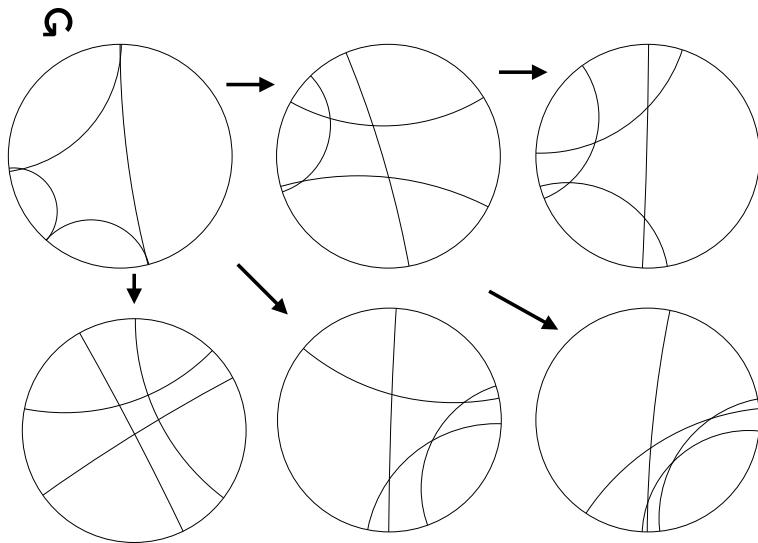
Construct 6 maps $\mathbb{H}$ to $\mathbb{H}$
from 6 real places of K

(uses Riemann mapping theorem)

$\Delta' \curvearrowright$



Resulting map $\mathbb{H} \rightarrow \mathbb{H}^6$ covers exotic $V \rightarrow X_K$



Conclusion

V gives an exotic compact geodesic curve on X_K , $\dim=6$.

(exotic because $\Delta' \subset \mathrm{SL}_2(\mathcal{O}_K)$ is Zariski dense,
so V is contained in no Shimura subvariety)

Conjecture

$\Delta(14,21,42)$ is the only triangle group whose invariant quaternion algebra splits.

Problem

*Are there more examples of exotic curves?
For example, with $\dim X_K = 3$?*

How to construct more geodesic curves?

merci viel mal!



References

Teichmüller dynamics and unique ergodicity ...

Modular symbols for Teichmüller curves

Billiards and the arithmetic of non-arithmetic groups

Galois orbits in the moduli space of all triangles

Triangle groups and Hilbert modular varieties

Triangle groups: Cusps, congruence and chaos

Billiards in regular polygons

www.math.harvard.edu/~ctm/papers